Combinatorics related to continued fractions

Flemming Topsøe
University of Copenhagen
topsoe@math.ku.dk

Prepared for presentation at ICM2006
Support by
Danish Natural Science Research Council
(overheads used are posted on
www.math.ku.dk/~topsoe)

Continued fractions, a reminder

Finite continued fractions are expressions of the form

$$b_0 + \frac{a_1}{b_1+} \; \frac{a_2}{b_2+} \; \cdots \frac{a_n}{b_n}.$$

When possible, understood, e.g., for $n = 3$ as

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3}}}.$$

The a's and b's are the *elements* (partial numerators and partial denominators). Preferred notation:

$$K\begin{pmatrix} - & a_1 & . & . & . & a_n \\ b_0 & b_1 & . & . & . & b_n \end{pmatrix} = \frac{A_n}{B_n}.$$

Precisely, *(canonical) numerators* and *denominators*:

$$\frac{A_n}{B_n} = \frac{A_0^n}{B_0^n} = b_0 + \frac{a_1}{\frac{A_1^n}{B_1^n}} = \frac{b_0 A_1^n + a_1 B_1^n}{A_1^n}$$

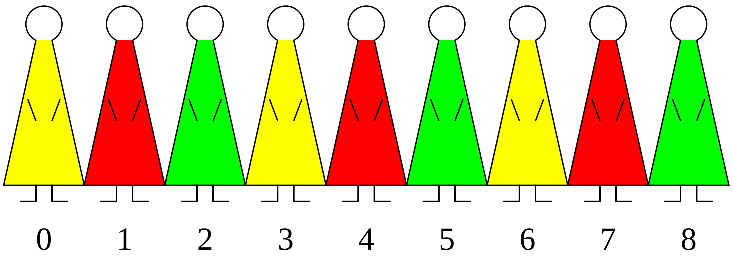
$\frac{A_n}{B_n} = C_n$, the n'th *approximant.* Infinite continued fraction *converges to C* if $\lim_{n\to\infty} C_n = C$.

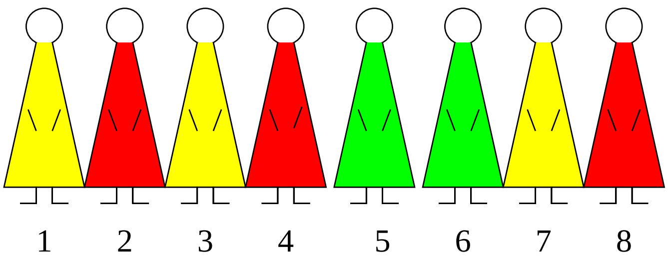
An observation

Problem: Formulas for A_n and B_n?
One possible solution: By recurrence relations!
A better solution, illustrated by an example:

$$A_8 = \cdots + a_1 a_4 a_7 b_2 b_5 b_8 + \cdots\ , \text{ (a sum of 55 terms),}$$
$$B_8 = \cdots + a_2 a_4 a_8 b_5 b_6 + \cdots\ , \text{ (a sum of 34 terms).}$$

... apparently connected with *families* of *mothers, daughters* and *free women*! Look here:





First family, say ω, has $M(\omega) = \{1, 4, 7\}$, $D(\omega) = \{0, 3, 6\}$ and $F(\omega) = \{2, 5, 8\}$.

Populations

$\Omega_i^j =$ *population* of all *families* over $[i, j]$ with

mothers (α's), daughters (γ's) and free women (β's)

as members according to the rules:

- a family $\omega \in \Omega_i^j$ contains one member of each age-class $i, \cdots, j$,
- every mother has only one daughter, and she is ...
- ... one age-class younger than the mother,
- youngest member (of age i) is not a mother.

Put $M(\omega)$, $D(\omega)$, $F(\omega)$ = sets of mothers, daughters and free women, respectively, identified by age.

A *cut-point* in ω allows split of the family in a "young" and an "old" part. Opposite process: *concatanation*. Special cases: $\Omega_i^{i-1} = \{\emptyset\}$, $\Omega_i^{i-2} = \emptyset$.

partition functions

Given $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 0}$, define, for an interval $[i,j]$, the *pointwise partition function* , $Z_i^j(\omega)$, and the *(accumulated) partition function* , Z_i^j, by:

Definitions:

$$Z_i^j(\omega) = \prod_{\nu\in M(\omega)} \alpha_\nu \prod_{\nu\in D(\omega)} \gamma_\nu \prod_{\nu\in F(\omega)} \beta_\nu \,,$$
$$Z_i^j = \sum_{\omega\in\Omega_i^j} Z_i^j(\omega)\,.$$

(special cases: $Z_i^{i-1} = 1, Z_i^{i-2} = 0$).

By mother↔daughter coupling, may assume $\gamma_\nu \equiv 1$.
Intention: $\alpha_n = a_n, \beta_n = b_n$ of continud fraction.

Example: If $\alpha_\nu \equiv 1$, $\beta_\nu \equiv 1$, then $Z_0^n = \#\Omega_0^n = F_{n+2}$, F for *Fibonacci*. Clearly, also $= \sum \binom{n-\nu}{\nu}$.

fundamental identities

basic identities:

$$Z_i^j = \alpha_\nu Z_i^{\nu-2} Z_{\nu+1}^j + Z_i^{\nu-1} Z_\nu^j \,, \qquad (1)$$
$$Z_i^j = \alpha_j Z_i^{j-2} + \beta_j Z_i^{j-1} \,, \qquad (2)$$
$$Z_i^j = \alpha_{i+1} Z_{i+2}^j + \beta_i Z_{i+1}^j \,. \qquad (3)$$

Proof. (1): Is there a mother of age ν or is there not? (2) and (3): Take $\nu = j$ or $\nu = i + 1$. □

Cor. For continued fractions: $A_n = Z_0^n, B_n = Z_1^n$.

Proof. Start is OK, induction (recursion) too. □

determinant identity With

$$\Delta_{i,j}^{k,l} = \begin{vmatrix} Z_i^k & Z_i^l \\ Z_j^k & Z_j^l \end{vmatrix} \,,$$

$$\Delta_{i,j}^{k,l} = (-1)^{k-j} \alpha_j^{k+1} Z_i^{j-2} Z_{k+2}^l \,.$$

the swopping lemma

Determinant identity follows by a simple combinatorial argument. Idea: two (or more) families *with a common cut-point* may split their families (in "young" and "old" parts) and swop parts. A general formulation:

swopping lemma

Given $\left(\Omega_{i_t}^{j_t}\right)_{t\in T}$ and permutation $t \curvearrowright \sigma(t)$ of T,

let $\mathbb{Z} = \sum_{(\omega_t)} \prod_t Z_{i_t}^{j_t}(\omega_t)$, summation being over all $(\omega_t)_{t\in T} \in \prod_t \Omega_{i_t}^{j_t}$ which have a common cut-point. Let $\mathbb{Z}^\sigma$ be the corresponding quantity related to the populations $\left(\Omega_{i_t}^{j_{\sigma(t)}}\right)_{t\in T}$. Then $\mathbb{Z} = \mathbb{Z}^\sigma$.

Proof. Swop! □

The determinant identity is derived as a corollary ...

the extended partition function diagram

(see separate OH)

Theorem Under the non-vanishing condition ($\alpha_n \neq 0, n \geq 1$), basic identities continue to hold in (i, j)-range $[0, \infty[\times [-2, \infty[$ with complementary definition

$$Z_i^j = \frac{(-1)^{i-j-1}}{a_{j+1}} Z_{j+2}^{i-2} \text{ for } i \geq j+2\,.$$

• "Usual" results known from continued fraction theory regarding linear dependence and relations rows$\leftrightarrow$columns continue to hold for the extended diagram.

• The determinant identity too extends in a natural way.

• Further, if α's and β's are all positive, quotients corresponding to a fixed pair of rows converges iff all such quotients converge iff the continued fraction converges.

further possibilities

• under the positivity assumption, α_n, β_n all > 0, natural probabilities can be assigned to families in Ω_i^j:

$$P_i^j(\omega) = \frac{Z_i^j(\omega)}{Z_i^j}\,.$$

Limit models for fixed i as $j \to \infty$ exist iff the continued fraction is convergent and if so defines (non-stationary) Markov chains (special issues: ergodicity ...)

• Relations to Padé tables and Frobenius identities can be established (study initiated)

• More speculatively: Can the γ's be useful, say as scalars whereas α's and β's could be operators ... e.g. in expressions like

$$b_0 + \gamma_0 \frac{a_1}{b_1 +} \;\; \gamma_1 \frac{a_2}{b_2 +} \;\; \cdots \gamma_{n-1} \frac{a_n}{b_n} \text{ ?}$$

In any case:

Conclusion (claims!)

- combinatorial structure is “just the right one”...
- and may be taken to lie behind fundamental constructs of basic continued fraction theory
- apparently, the combinatorial dimension has been overlooked since the times of the old masters.
- the new ideas mainly lead to well known results and as such contribute to a broadening of our understanding
- in addition, the combinatorial approach does offer new opportunities:
  - a close connection to certain non-stationary Markov chains
  - ???