# Preface

Ever since the birth of modern geometry more than 100 years ago topologists have studied the relations between different structures on a manifold: topological structures, piecewise linear structures and smooth structures. Which topological manifolds admit smooth structures and if they do, how many? In the course of the 1960s and '70s many problems were solved existence and uniqueness of many manifolds were proved, in particular of $\mathbb{R}^n$ for $n \neq 4$. However, in dimension 4 and partly in dimension 3 the answers to the questions above seemed to be concealed behind some impenetrable barrier. Even for the simplest four-manifold, $\mathbb{R}^4$ this question of uniqueness of smooth structures was not so easily settled. To phrase the problem in a popular way: for a four-manifold the dimension is large enough for strange phenomena to occur but too small to allow us enough room to tame them.

In the beginning of the '80s, however, something happened when Freedman published his article [8] in which he revealed a deep connection between four-manifolds and quadratic forms. More precisely, he showed that to a given quadratic form $Q$ of determinant $\pm 1$ there exists a simply-connected topological four-manifold whose intersection form equals $Q$. Moreover, if $Q$ is an even form, then there exists exactly one manifold (up to homeomorphism) which has $Q$ as its intersection form, and if $Q$ is not even, exactly 2 manifolds (up to homeomorphism) have $Q$ as their intersection form.

The ball was rolling and shortly thereafter the young mathematician, Simon Donaldson, published a pioneering article [5] showing that if $M$ is a smooth simply-connected manifold whose intersection form is positive definit, then the intersection form is trivial, i.e. diagonalizable with eigenvalues 1. There are many non-trivial intersection forms which means that many of the manifolds constructed by Freedman do not admit smooth structures. The astounding feature of Donaldson's article, however, was not so much the result (impressive enough as it was) but rather the procedure of his proof for which he utilized methods from gauge theories. Donaldson considered solutions (instantons) to certain SU(2) Yang-Mills equations, and by mod'ing out gauge equivalence, he obtained a moduli space, an orientable smooth 5-manifold with boundary and a number of singularities corresponding to solutions to the equation $Q(\alpha) = 1$. In 1990, following the path he himself had carved, he constructed what is now known as the Donaldson invariant. It is the integral of a certain form over the compactification of his moduli space. The Donaldson invariant became a valuable tool for topologists and geometers to distinguish 4-manifolds of the same homotopy type.

However, the technical difficulties involved in these constructions and later developments were immense, with one of the worst problems being that the moduli space was not compact and thus had to be compactified in a "smooth" way. The whole theory seemed to be drowning in a technical swamp when Edward Witten intervened in 1994, [23]. Recent work on supersymmetric $\mathcal{N} = 2$ Yang-Mills theories which he had carried out jointly with N. Seiberg had lead him to the realization that an entirely equivalent formulation of Donaldson's theory could be given within the frame of *abelian* gauge theories. The idea and procedure were the same as Donaldson's when he constructed his invariant 4 years earlier, but replacing the *non-abelian* gauge group SU(2) with the *abelian* gauge group U(1) lead to great simplifications in the theory. The detailed construction of Seiberg's and Witten's invariant is our scope.

The starting point of the thesis will be a compact oriented Riemannian 4-manifold $M$. A famous theorem attributed to Hirzebruch and Hopf states that any such manifold has a spin$^c$-structure (and often quite many). The Seiberg-Witten invariant, the definition of which is the goal of this thesis, is an invariant of the manifold and of the spin$^c$-structure.

In Chapter 1, we set out with the definition and elementary properties of spin and spin$^c$-structures. This material is supposed to be well-known so the exposition given here is in no way meant to be self-contained but is rather intended to give an overview and to introduce notation. Afterwards follows a more detailed treatment of the spinor bundles, of connections on spinor bundles and of the Dirac operator. Local expressions for the connections and for the Dirac operator are given and elementary properties of the Dirac operator such as formal self-adjointness, ellipticity and so on, are postulated. Next we will prove some very important identities named after Bochner, Weitzenböck and Lichnerowicz, which reveal a quite deep relationship between the Dirac operator, the spin connection and the curvature of the underlying Riemannian manifold.

This concludes the geometric preparations and we turn to analysis. Since Chapter 2 necessitates considering infinite-dimensional manifolds and Fredholm maps between them, an entire section is devoted to presenting some basic facts about Fredholm operators. The classic results, like invariance of the index under certain perturbations, are postulated, whereas some more exotic results are proved. We conclude this chapter with a treatment of fields of Fredholm operators, i.e. families of Fredholm operators parametrized by certain manifolds. The main result (and the only fact about such fields which is relevant for our purpose) is the existence of a certain line bundle, the determinant line bundle, associated with such a field. The section ends by mentioning the Smale-Sard Theorem, an extension of the classic Sard Theorem to the case of Fredholm maps between infinite-dimensional manifolds.

In Chapter 2, the starting point will be the Seiberg-Witten equations. No account of their physical/gauge theoretical origin is given, that would lead us far astray. The solutions to the Seiberg-Witten equations will be pairs $(\psi, \mathcal{A})$ of a positive spinor field $\psi$ (i.e. a section of the spinor bundle) and a connection $\mathcal{A}$ on the determinant line bundle for the spin$^c$-structure. We will define a gauge group and an action of this gauge group on the space of solutions to the Seiberg-Witten equations. The moduli space is defined to be the quotient of the solution set modulo the action of the gauge group. This moduli space happens to be as nice as one could possible imagine, namely a smooth, compact, oriented manifold of a certain finite dimension which depends only on the spin$^c$-structure and on the topology of the underlying manifold. To be able to prove all these nice features, however, we are forced to leave the safe haven of smooth structures and venture into the realm of infinite-dimensional Hilbert/Banach manifolds and Sobolev spaces. We give a very brief delineation of the theory of Sobolev spaces of sections as well as of elliptic differential operators among them. Then, by "Sobolev completing" the moduli space we obtain a space to which we can apply all our analytic tools: elliptic regularity, Fredholm theory, etc. Elliptic regularity is very important when showing that the moduli space is in fact compact (and also one of the curvature identities from Chapter 1 will be an indispensable asset in this regard). Having shown compactness, we undertake the next major task which is to give the moduli space a smooth structure. This is where the Fredholm theory will come into play, simply because we are able to show that the moduli space is a certain level set of a Fredholm map between two (infinite-dimensional) manifolds. After proving that the moduli space is actually a manifold, we show how we can return to the smooth setting by proving that all the "Sobolev" moduli spaces are diffeomorphic to the original smooth moduli space. As a final technical result we show that the moduli space is orientable. To this end, our theory of Fredholm fields will be utilized.

All this hard work leaves us with a compact, smooth, orientable, finite-dimensional manifold. We can integrate differential forms over such a manifold, and that is exactly what we have in mind. Upon introducing the so-called slant product we are able to define a certain cohomology class $\mu_\sigma(1)$ on the moduli space. The integral of $(1-\mu_\sigma(1))^{-1}$ over the moduli space is the famed Seiberg-Witten invariant. By a cobordism argument we finish Chapter 2 by proving that the Seiberg-Witten invariant depends on the spin$^c$-structure only.

Finally, in Chapter 3 we give an example by calculating the Seiberg-Witten invariant of Kähler manifolds. The result is due to Clifford Henry Taubes and appeared in [22] in 1994 as one of the first applications of Witten's new theory. In the first section of Chapter 3, we sketch an overview of the basics of complex and symplectic geometry. In the second section, we show how any almost Kähler manifold has a canonical spin$^c$-structure and how we may write Dirac operators in terms of so-called Cauchy-Riemann operators. Applying these formulas to the Seiberg-Witten equations for the canonical spin$^c$-structure allows us (after a considerable amount of calculations) to show that the moduli space consists of only one point and hence the Seiberg-Witten invariant must be ±1.

A few words on my sources are probably in order: In Chapter 1, primarily [13] and [12] have been used for Sections 1.1-1.3 although most of Section 1.2, proving that the spinor bundle is a Dirac bundle, is work of my own. For Sections 1.4 and 1.5 as well as large parts of Chapter 2 my main sources are [1], [18] and [17]. Section 3.1 is mostly inspired by [18] (Section 1.4) and [9] (Section 3.4) and partly by [15] and [11] and Section 3.2 by [7] (the article by Hutchings and Taubes).

Finally, thanks are due to my supervisor professor Bergfinnur Durhuus for his aid throughout the past six months. Thanks are also due to my good friend Mathias Dalhoff for proofreading parts of the manuscript.

Karlslunde, January 2009. *Thomas Hjortgaard Danielsen.*

## Preface to the Second Edition

In this second edition of the master's thesis, some more or less embarrassing typos have been corrected, the layout has been slightly modified and a paragraph on deformation complexes has been added.

Karlslunde, March 2009. *Thomas Hjortgaard Danielsen.*

# CONTENTS

# CHAPTER 1

# Preliminaries

## 1.1 Spin Structures

In this section let us recall/introduce the notions of spin and spin$^c$ structures and fix some notation.

**Definition 1.1 (Spin Structure).** Let $E \longrightarrow M$ be a real oriented Riemannian vector bundle of rank $n \geq 3$ and $\pi : P_{\mathrm{SO}}(E) \longrightarrow M$ its oriented orthonormal frame bundle. A *spin structure* on $E$ is a "lift" of $P_{\mathrm{SO}}(E)$ to a principal $\mathrm{Spin}(n)$-bundle. More precisely, a spin structure is a pair $(P_{\mathrm{Spin}}(E), \Phi)$ of a principal $\mathrm{Spin}(n)$-bundle $\widetilde{\pi} : P_{\mathrm{Spin}}(E) \longrightarrow M$ and a bundle map $\Phi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ such that

$$\Phi(p \cdot g) = \Phi(p) \cdot \Lambda(g)$$

where $\Lambda : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the double covering.

Often, the vector bundle in question will be the tangent bundle of some manifold $M$ (provided of course that this manifold is oriented and has been given a metric). Then we will write $P_{\mathrm{SO}}(M)$ and $P_{\mathrm{Spin}}(M)$ for the oriented frame bundle resp. the spin bundle. If $TM$ has a spin structure we say that $M$ is a *spin manifold*.

Let's try and see this from a local perspective. It is well-known that a principal $G$-bundle over a manifold $M$ can be (uniquely) described by the following data: a cover $(U_\alpha)_{\alpha\in A}$ of open sets and for each pair $(\alpha, \beta) \in A \times A$ a smooth map (the *transition function*) $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ (where $U_{\alpha\beta} := U_\alpha \cap U_\beta$) satisfying that $g_{\alpha\alpha}(x) = 1$ for all $x \in U_\alpha$ and satisfying the *cocycle condition*,

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = 1$$

for all triples $(\alpha, \beta, \gamma) \in A \times A \times A$ and for all $x \in U_{\alpha\beta\gamma}$. This collection of data is called a *gluing cocycle*. So if we consider our principal $\mathrm{SO}(n)$-bundle $P_{\mathrm{SO}}(E)$, then there exists a cover $(U_\alpha)$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{SO}(n)$ satisfying the cocycle condition. The existence of a spin structure is then equivalent to the existence of lifts $\widetilde{g}_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Spin}(n)$ over $\Lambda$ satisfying the cocycle condition. It is well-known that such a set of lifts exists if and only if the second Stiefel-Whitney class is zero. In fact this follows more or less by definition of the second Stiefel-Whitney class (in the setting of Cech cohomology). In the affirmative case the possible spin structures on $M$ are parametrized by elements of the first Cech cohomology group $\check{H}^1(M; \mathbb{Z}_2)$ which is, of course, isomorphic to the singular cohomology group $H^1(M;\mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$. For instance $S^n$ for $n \geq 3$ admits a spin structure, since $H^2(S^n;\mathbb{Z}_2) = 0$, so the second Stiefel-Whitney class can be nothing but 0. Since $H^1(S^n;\mathbb{Z}_2) = 0$ the spin structure must be unique.

The spin group $\mathrm{Spin}(n)$ has a distinguished complex representation, called the *spinor representation* $\kappa_n : \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(\Delta_n)$ where $\Delta_n = \mathbb{C}^{2^k}$ and where $k = \lfloor \frac{n}{2} \rfloor$, the integer part of $\frac{n}{2}$. This is a faithful representation (hence does *not* descend to a representation of $\mathrm{SO}(n)$) and when $n$ is odd it is irreducible. For $n$ even, it decomposes into a direct sum of two irreducible representations $\kappa_n = \kappa_n^+ \oplus \kappa_n^-$ and the corresponding representation spaces are denoted $\Delta_n^+$ and $\Delta_n^-$ respectively.

Given a principal $\mathrm{Spin}(n)$-bundle $\pi : Q \longrightarrow M$ (originating from a spin structure, say) we can form the associated complex vector bundle w.r.t. the spinor representation, namely $S := Q \times_{\kappa_n} \Delta_n$ which is the quotient of the direct product $Q \times \Delta_n$ under the equivalence relation $(p, v) = (p \cdot g, \kappa_n(g^{-1})v)$ and with projection $q : S \longrightarrow M$ given by $q([p, v]) = \pi(p)$. This is called the *spinor bundle* and sections of this bundle are called *spinors* or *Dirac spinors*. If $n$ is even this bundle splits, in the same way as the representation, into two bundles $S = S^+ \oplus S^-$ where in fact $S^\pm$ is the associated bundle $Q \times_{\kappa_n^\pm} \Delta_n^\pm$. Sections of these vector bundles are called positive resp. negative *Weyl spinors* or even resp. odd *chiral spinors*. We will discuss these vector bundles and some of their properties in more detail in the next section when we define the Dirac operator.

Next, recall how the Lie group $\mathrm{Spin}^c(n)$ is defined: It is the group inside $\mathrm{Cl}_{0,n} \otimes \mathbb{C}$ generated by $\mathrm{Spin}(n) \otimes 1$ and $1 \otimes \mathrm{U}(1)$. Equivalently, $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\pm 1} \mathrm{U}(1)$, the quotient where we collapse the subgroup $\{\pm(1, 1)\}$. Thus, in $\mathrm{Spin}(n) \times \mathrm{U}(1)$ we identify $(g, z)$ with $(-g, -z)$. The equivalence class containing $(g, z)$ will be denoted $[g, z]$. This is usually how we will view $\mathrm{Spin}^c(n)$.

We can define a Lie group homomorphism $\rho^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n)$ by $[g, z] \longmapsto \Lambda(g)$ (again, $\Lambda : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the double covering). This is well-defined, since $\Lambda(-g) = \Lambda(g)$, however, contrary to $\Lambda$, this is no longer a covering map, since its fibers are not discrete. Instead we may view this map as an $n$-dimensional representation of $\mathrm{Spin}^c(n)$.

Similarly, we define a Lie group homomorphism $\lambda : \mathrm{Spin}^c(n) \longrightarrow \mathrm{U}(1)$ by $[g, z] \longmapsto z^2$. Again, this is well-defined since $(-z)^2 = z^2$. We may view $\lambda$ as a 1-dimensional unitary representation of $\mathrm{Spin}^c(n)$.

Finally, $\mathrm{Spin}^c(n)$ *is* a covering space, not of $\mathrm{SO}(n)$ as we saw above, but of $\mathrm{SO}(n) \times \mathrm{U}(1)$. We simply define the homomorphism $\Lambda^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1)$ by $\Lambda^c([g, z]) = (\Lambda(g), z^2)$. One can then check that this is a smooth double covering of $\mathrm{SO}(n) \times \mathrm{U}(1)$.

**Definition 1.2** (**Spin$^c$-structure**). Let $E \longrightarrow M$ be a real oriented Riemannian vector bundle of rank $n \geq 3$ with oriented orthonormal frame bundle $\pi : P_{\mathrm{SO}}(E) \longrightarrow M$. A *spin$^c$-structure* on $E$ is a principal $\mathrm{Spin}^c(n)$-bundle $\widetilde{\pi} : P^c_{\mathrm{Spin}}(E) \longrightarrow M$ and a bundle map $\Phi^c : P^c_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ such that $\Phi^c(p \cdot g) = \Phi^c(p) \cdot \rho^c(g)$ where $\rho^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n)$ is the map defined above.

Two spin$^c$-structures $(P^c_{\mathrm{Spin}}(E)^1, \Phi^c_1)$ and $(P^c_{\mathrm{Spin}}(E)^2, \Phi^c_2)$ are said to *isomorphic* if there exists a bundle isomorphism $P^c_{\mathrm{Spin}}(E)^1 \longrightarrow P^c_{\mathrm{Spin}}(E)^2$ making the following diagram commutative:

$$\begin{tikzcd}
P^c_{\mathrm{Spin}}(E)^1 \arrow[rr] \arrow[dr, "\Phi^c_1"'] & & P^c_{\mathrm{Spin}}(E)^2 \arrow[dl, "\Phi^c_2"] \\
& P_{\mathrm{SO}}(E) &
\end{tikzcd}$$

The set of isomorphism classes of spin$^c$-structures on $E$ is denoted $\mathrm{Spin}^c(E)$. In the case where $E$ happens to be the tangent bundle of an oriented Riemannian manifold a manifold equipped with a spin$^c$-structure is called a *spin$^c$-manifold*. The set of isomorphism classes of spin$^c$-structures on $M$ is denoted $\mathrm{Spin}^c(M)$.

Assume that $E$ is an oriented Riemannian vector bundle with a spin structure $\pi : P_{\mathrm{Spin}}(E) \longrightarrow M$ and bundle map $\Phi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$. Then $E$ has a canonical spin$^c$-structure given in the following way: Define

$$P^c_{\mathrm{Spin}}(E) := P_{\mathrm{Spin}}(E) \times_{\pm 1} \mathrm{U}(1)$$

more precisely we take the product $P_{\mathrm{Spin}}(E) \times \mathrm{U}(1)$ and mod out by the equivalence relation $\sim$ given by $(p, z) \sim (p', z')$ iff $\pi(p) = \pi(p')$ and $(p', z') = \pm(p, z)$. The space $P^c_{\mathrm{Spin}}(E)$ can then be equipped with a right $\mathrm{Spin}^c(n)$ action

$$[p, z] \cdot [g, z'] = [p \cdot g, zz']$$

which in combination with the projection map $\widetilde{\pi} : P^c_{\mathrm{Spin}}(E) \longrightarrow M$ given by $\widetilde{\pi}([p, z]) = \pi(p)$ turns $P^c_{\mathrm{Spin}}(E)$ into a principal $\mathrm{Spin}^c(n)$-bundle. Finally, define $\Phi^c : P^c_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ by $\Phi^c([p, z]) = \Phi(p)$. We see that

$$\begin{aligned}\Phi^c([p, z] \cdot [g, z']) &= \Phi^c([p \cdot g, zz']) = \Phi(p \cdot g) = \Phi(p) \cdot \Lambda(g)\\ &= \Phi^c([p, z]) \cdot \rho^c([g, z'])\end{aligned}$$

and thus that $(P^c_{\mathrm{Spin}}(E), \Phi^c)$ is a spin$^c$-structure on $E$.

This shows that the concept of a spin$^c$-structure is more general than that of a spin structure. I will not go into a topological discussion of when spin$^c$-structures exist, except mentioning the following result

**Theorem 1.3** (**Hirzebruch-Hopf**). *Any oriented Riemannian 4-manifold is a spin$^c$-manifold.*

A proof of this statement can be found in [17], Lemma 3.1.2.

**Definition 1.4** (**Determinant Line Bundle**). Given a principal $\mathrm{Spin}^c(n)$-bundle $Q \longrightarrow M$ we can form the complex line bundle $L := Q \times_\lambda \mathbb{C}$ associated to the 1-dimensional representation $\lambda$ defined above. This bundle is called the *determinant line bundle*, $\det(Q)$ for the $\mathrm{Spin}^c(n)$-bundle $Q$. If the $\mathrm{Spin}^c(n)$-bundle originates from a spin$^c$-structure $\sigma$, we will often write $\det(\sigma)$ for the determinant line bundle.

The determinant line bundle can be given a hermitian fiber metric by defining $\langle [p, w], [p, w'] \rangle := \langle w, w' \rangle_{\mathbb{C}} = w\overline{w}'$, we simply transfer the usual inner product on $\mathbb{C}$ to $L$. This is well-defined since

$$\langle [p \cdot g, \lambda(g)w], [p \cdot g, \lambda(g)w'] \rangle = \langle \lambda(g)w, \lambda(g)w' \rangle_{\mathbb{C}} = \langle w, w' \rangle_{\mathbb{C}}$$

since $\lambda(g) \in \mathrm{U}(1)$ and we may therefore form the unitary frame bundle $L^0 := P_{\mathrm{U}}(L)$, a principal $\mathrm{U}(1)$-bundle over $M$.

**Lemma 1.5.** *There is a bundle isomorphism $L^0 \cong P^c_{\mathrm{Spin}}(E) \times_\lambda \mathrm{U}(1)$.*

Proof. We define the bundle map $P^c_{\mathrm{Spin}}(E) \times_\lambda \mathrm{U}(1) \longrightarrow L^0$ in the following way: the element $[p, z] \in P^c_{\mathrm{Spin}}(E) \times_\lambda \mathrm{U}(1)$ should be mapped to the frame (i.e. the isometric isomorphism $\mathbb{C} \longrightarrow L_{\widetilde{\pi}(p)}$) given by $w \longmapsto [p, zw]$. Since $z \in \mathrm{U}(1)$, this is an isometric isomorphism, i.e. a frame at $\widetilde{\pi}(p)$. Since any frame at $\widetilde{\pi}(p)$ must be of the form $w \longmapsto w[p, z] = [p, zw]$ for a unit vector $[p, z] \in L_{\widetilde{\pi}(p)}$, i.e. for $z \in \mathrm{U}(1)$, we see that the bundle map is surjective. Its not hard to see that it is injective also and hence a bundle isomorphism. $\square$

Let's put a local perspective on this as we did for the spin structures. Our starting point is again a principal $\mathrm{SO}(n)$-bundle which is given in terms of a cover $(U_\alpha)$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{SO}(n)$. Picking a spin$^c$-structure (if it exists) is then equivalent to picking lifts $\widetilde{g}_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Spin}^c(n)$ along $\rho^c$ such that the cocycle condition is satisfied. We see that $\widetilde{g}_{\alpha\beta}$ must be of the form $[h_{\alpha\beta}, z_{\alpha\beta}]$ where $h_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Spin}(n)$ and $z_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{U}(1)$ are such that $\Lambda \circ h_{\alpha\beta} = g_{\alpha\beta}$ and such that the pair

$$(h_{\alpha\beta}(x)h_{\beta\gamma}(x)h_{\gamma\alpha}(x), z_{\alpha\beta}(x)z_{\beta\gamma}(x)z_{\gamma\alpha}(x))$$

is either $(-1,-1)$ or $(1,1)$, thus $(h_{\alpha\beta})$ and $(z_{\alpha\beta})$ need only satisfy the cocycle condition up to a sign. However if the bundle admits a spin structure, we may pick $h_{\alpha\beta}$ such that it does satisfy the cocycle condition, and then we can pick $z_{\alpha\beta} = 1$, this is the canonical spin$^c$-structure of a spin structure.

Given the families of maps $(h_{\alpha\beta})$ and $(z_{\alpha\beta})$ we can define $\lambda_{\alpha\beta} := z_{\alpha\beta}^2$ which maps $U_{\alpha\beta}$ into $\mathrm{U}(1)$. This family of maps satisfies the cocycle condition and thus represents a principal $\mathrm{U}(1)$-bundle. This bundle is nothing but the unitary frame bundle of the determinant line bundle. By the remarks above we then conclude that the determinant line bundle of the canonical spin$^c$-structure induced by a spin structure has trivial determinant line bundle.

Let $M$ be a spin$^c$-manifold. How many different spin$^c$-structures does this manifold have? To shed some light on this question, let $\mathrm{Pic}^\infty(M)$ denote the set of complex Riemannian line bundles (i.e. bundles carrying a sesquilinear, conjugate symmetric, positive definit 2-form). This is a group under tensor product, known as the *Picard group*. This group is in 1-1 correspondence with the set of principal $\mathrm{U}(1)$-bundles over $M$ - the map from $\mathrm{Pic}^\infty(M)$ to the set of principal $\mathrm{U}(1)$-bundles is simply given by forming the unitary frame bundle. Recall that the first Chern class is a bijection

$$c_1 : \mathrm{Pic}^\infty(M) \longrightarrow H^2(M;\mathbb{Z}). \tag{1.1}$$

We can let $\mathrm{Pic}^\infty(M)$ act on $\mathrm{Spin}^c(M)$ in the following way: If $\sigma \in \mathrm{Spin}^c(M)$ is a spin$^c$-structure given by the gluing cocycle $[h_{\alpha\beta}, z_{\alpha\beta}]$ and if $\mathcal{L} \in \mathrm{Pic}^\infty(M)$ is given by the gluing cocycle $(\zeta_{\alpha\beta})$ then we define the spin$^c$-structure $\sigma \otimes \mathcal{L}$ by the gluing cocycle $[h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}]$. Since $(z_{\alpha\beta}\zeta_{\alpha\beta})^2 = \lambda_{\alpha\beta}\zeta_{\alpha\beta}^2$ we see that

$$\det(\sigma \otimes \mathcal{L}) = \det(\sigma) \otimes \mathcal{L}^{\otimes 2}.$$

**Proposition 1.6.** *The action of* $\mathrm{Pic}^\infty(M)$ *on* $\mathrm{Spin}^c(M)$ *is free and transitive. Thus for a fixed spin*$^c$*-structure* $\sigma_0$ *the map* $\mathcal{L} \longmapsto \sigma_0 \otimes \mathcal{L}$ *is a bijection* $\mathrm{Pic}^\infty(M) \longrightarrow \mathrm{Spin}^c(M)$*. Composing with* (1.1) *we obtain a bijection*

$$\mathrm{Spin}^c(M) \xrightarrow{\sim} H^2(M;\mathbb{Z}).$$

*Moreover, at most finitely many spin*$^c$*-structures have the same determinant line bundle.*

Proof. First, the action is free: if $\sigma \otimes \mathcal{L} = \sigma$, i.e. if $[h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}] = [h_{\alpha\beta}, z_{\alpha\beta}]$, then we must have $z_{\alpha\beta}\zeta_{\alpha\beta} = z_{\alpha\beta}$, i.e. $\zeta_{\alpha\beta} = 1$ and since this is the gluing cocycle for $\mathcal{L}$, this bundle must be trivial.

The action is transitive: Assume we have two spin$^c$-structures $\sigma_1$ and $\sigma_2$ given by gluing cocycles $[h^{(i)}_{\alpha\beta}, z^{(i)}_{\alpha\beta}]$. Since $\Lambda(h^{(1)}_{\alpha\beta}) = \Lambda(h^{(2)}_{\alpha\beta}) = g_{\alpha\beta}$ we must have $h^{(1)}_{\alpha\beta} = \pm h^{(2)}_{\alpha\beta}$. By a change of sign if necessary we can thus assume $h^{(1)}_{\alpha\beta} = h^{(2)}_{\alpha\beta}$. Now put $\zeta_{\alpha\beta} := z^{(2)}_{\alpha\beta}/z^{(1)}_{\alpha\beta}$. Clearly $\zeta_{\alpha\beta}$ maps into $\mathrm{U}(1)$ and we see that

$$[h^{(2)}_{\alpha\beta}, z^{(2)}_{\alpha\beta}] = \Big[h^{(1)}_{\alpha\beta}, z^{(1)}_{\alpha\beta}\frac{z^{(2)}_{\alpha\beta}}{z^{(1)}_{\alpha\beta}}\Big] = [h^{(1)}_{\alpha\beta}, z^{(1)}_{\alpha\beta}\zeta_{\alpha\beta}]$$

and hence that $\sigma_2 = \sigma_1 \otimes \mathcal{L}$.

At last, assume $\sigma_1$ and $\sigma_2$ are two spin$^c$-structures having the same determinant line bundle. By the first part of the proof, there exists a unique line bundle $\mathcal{L}$ such that $\sigma_2 = \sigma_1 \otimes \mathcal{L}$. If $\mathcal{L}$ is given by the gluing cocycle $(\zeta_{\alpha\beta})$, then the requirement $\det(\sigma_1) = \det(\sigma_2)$ implies that $\zeta^2_{\alpha\beta} = 1$, i.e. $\zeta_{\alpha\beta}$ maps into $\mathbb{Z}_2$. Thus $(\zeta_{\alpha\beta})$ determines an element of the Cech cohomology group $\check{H}^1(M; \mathbb{Z}_2)$ which is isomorphic to the singular cohomology group $H^1(M; \mathbb{Z}_2)$. Since this is finite, $\mathcal{L}$ belongs to a finite set, hence the conclusion. ☐

In the same way we defined the spinor bundles associated to a principal $\mathrm{Spin}(n)$-bundle, we can form spinor bundles associated to a $\mathrm{Spin}^c(n)$-bundle. $\mathrm{Spin}^c(n)$ sits inside $\mathrm{Cl}^{\mathbb{C}}_n$ and the fundamental representation of this algebra on the space $\Delta_n$ of Dirac spinors restricts to a group representation $\kappa^c_n : \mathrm{Spin}^c(n) \longrightarrow \mathrm{Aut}(\Delta_n)$. Thus if $E$ is a real vector bundle carrying a spin$^c$-structure $P^c_{\mathrm{Spin}}(E)$ we define the *complex spinor bundle*:

$$S^c(E) := P^c_{\mathrm{Spin}}(E) \times_{\kappa^c_n} \Delta_n.$$

If the principal $\mathrm{Spin}^c(n)$-bundle is given by the gluing cocycle $[h_{\alpha\beta}, z_{\alpha\beta}]$ then $S^c(E)$ is given by the gluing cocycle $\kappa^c_n([h_{\alpha\beta}, z_{\alpha\beta}])$. If we change spin$^c$-structure from $\sigma$ to $\sigma \otimes \mathcal{L}$ where $\mathcal{L}$ is a line bundle given by the gluing cocycle $(\zeta_{\alpha\beta})$ then the spinor bundle is given by the gluing cocycle

$$\kappa^c_n([h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}] = \kappa^c_n([h_{\alpha\beta}, z_{\alpha\beta}])\zeta_{\alpha\beta}$$

and hence the "new" spinor bundle is just $S^c(E) \otimes \mathcal{L}$.

As observed above $k := \dim_{\mathbb{C}} \Delta_n = \mathrm{rank}_{\mathbb{C}}\, S^c(E)$ is an even number. In the representation theory of spin groups it is shown that the two representations $\kappa^{\wedge k}_n = \kappa_n \wedge \cdots \wedge \kappa_n$ and $\lambda^{\otimes k/2}$ are equivalent. This means that the associated vector bundles are isomorphic, giving us the following relations between the spinor bundles and determinant line bundle $L$:

$$\Lambda^k S^c(E) \cong L^{\otimes k/2}. \tag{1.2}$$

The same result holds for $S^c(E)^\pm$:

$$\Lambda^{k/2}(S^c(E)^\pm) \cong L^{\otimes k/4}. \tag{1.3}$$

The formula (1.2) also explains the name determinant line bundle, since the top exterior product of a vector space or a vector bundle traditionally is called the *determinant*.

We also need to discuss the notion of *connections* on vector bundles and principal bundles. First we recall the definitions

**Definition 1.7 (Connection on a Vector Bundle).** Let $E$ be a smooth $\mathbb{K}$-vector bundle ($\mathbb{K}$ being either $\mathbb{R}$ or $\mathbb{C}$) over $M$ and let $\Omega^1(M, E)$ denote the $E$-valued 1-forms, i.e. sections of $T^*M \otimes_{\mathbb{K}} E$. By a *connection* on $E$ we understand a $\mathbb{K}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Omega^1(M, E)$$

satisfying the "Leibniz rule"

$$\nabla(fs) = df \otimes s + f\nabla s \tag{1.4}$$

for $f \in C^\infty(M)$ and $s \in \Gamma(E)$. $\nabla s$ is called the *covariant derivative* of $s$.

If $E$ is a Riemannian vector bundle, we say that a connection $\nabla$ is *compatible with the metric* (or just *metric*) if

$$X\langle s,s'\rangle = \langle \nabla_X s, s'\rangle + \langle s, \nabla_X s'\rangle$$

for all vector fields $X$ and all sections $s, s' \in \Gamma(E)$.

One can show that any vector bundle can be equipped with a connection, and that the space of connections is an affine space modeled on $\Omega^1(M, \mathrm{End}(E))$, i.e. any two connections differ by an element in $\Omega^1(M, \mathrm{End}(E))$.

Given a connection on $TM$ we define its *torsion* by

$$\tau_\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

This is an anti-symmetric 2-tensor. The Fundamental Theorem of Riemannian Geometry states that on a Riemannian manifold, there exists a unique metric connection whose torsion tensor vanishes identically. This is called the *Levi-Civita connection* on $M$.

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a smooth principal $G$-bundle over $M$. For each $p \in P$ we have the so-called *vertical subspace* $V_pP \subseteq T_pP$, namely the kernel of the differential $d\pi_p : T_pP \longrightarrow T_{\pi(p)}M$. A connection on $P$ is then loosely speaking a smooth choice of an algebraic complement over each point. Formally

**Definition 1.8** (**Connection on a Principal Bundle**). For the principal $G$-bundle $G \hookrightarrow P \longrightarrow M$ a *connection* is a smooth tangent distribution $HP$ on $P$ such that for each $p \in P$ we have $T_pP = H_pP \oplus V_pP$ ($H_pP$ is called the *horizontal subspace*) and such that $H_{p\cdot g}P = (d\sigma_g)_pH_pP$ where $\sigma_g : P \longrightarrow P$ is the map $p \longmapsto p \cdot g$.

There are several equivalent definitions of a connection on a principal $G$-bundle. One of them is given in terms of a connection 1-form: If $\mathfrak{g}$ denotes the Lie algebra of $G$, then a *connection 1-form* is a smooth $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the following two axioms: $(\sigma_g)^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ and $\omega_p(A^\sharp(p)) = A$ for all $A \in \mathfrak{g}$ where $A^\sharp$ is the *fundamental vector field* on $P$ determined by $A$. Letting $\sigma_p$ be the map $G \longrightarrow P$, $g \longmapsto p \cdot g$, then $A^\sharp$ is given by

$$A^\sharp(p) = d\sigma_p(A) = \frac{d}{dt}\bigg|_{t=0} (p \cdot \exp(tA)$$

so the second axiom could be phrased as $\omega_p \circ d\sigma_p = \mathrm{id}$.

A connection 1-form induces a connection on the principal bundle, simply by $H_pP = \ker \omega_p$, and vice versa.

Yet a third definition of a connection is via local gauge potentials satisfying a compatibility conditions on the overlap of their local domains.

Now assume that $\pi : P \longrightarrow M$ is a principal $\mathrm{SO}(n)$-bundle over $M$ and assume it has a connection. Assume furthermore that the bundle lifts to a spin-bundle $\widetilde{\pi} : S(P) \longrightarrow M$. The map $\Phi : S(P) \longrightarrow P$ is in fact a double covering map: it is not hard to see that we can pick local trivializations of $S(P)$ and $P$ on a common neighborhood $U \subseteq M$ such that $\Phi$ locally takes the form $(x,g) \longmapsto (x, \Lambda(g))$. Since $\Lambda$ is a double covering map, $\Phi|_U$ is a double covering map, and since being a covering map is a local property, $\Phi$ itself is a double covering map. In particular it is a local diffeomorphism and its differential $d\Phi_p : T_pS(P) \longrightarrow T_{\Phi(p)}P$ is an isomorphism for all $p \in S(P)$. But then we can lift the connection on $P$ to a connection on $S(P)$, simply by defining the horizontal subspace $H_pS(P) := (d\Phi_p)^{-1}(H_{\Phi(p)}P)$. Defined in this way from a local diffeomorphism, it is obviously a smooth tangent distribution. The additional requirement is also satisfied, as can be seen directly as follows:

$$\begin{aligned}H_{p\cdot g}S(P) &= (d\Phi_p)^{-1}(H_{\Phi(p\cdot g)}P) = (d\Phi_p)^{-1}(H_{\Phi(p)\cdot\Lambda(g)}P)\\ &= (d\Phi_p)^{-1}\left(d\sigma_{\Lambda(g)}H_{\Phi(p)}P\right) = d\widetilde{\sigma}_g(d\Phi_p)^{-1}(H_{\Phi(p)}P)\\ &= d\widetilde{\sigma}_g H_p S(P),\end{aligned}$$

the fourth identity follows from the requirement $\Phi\circ(\widetilde{\sigma}_g) = \sigma_{\Lambda(g)}\circ\Phi$. Thus, the connection on $P$ lifts to a connection on $S(P)$. If $P$ is the oriented orthonormal frame bundle and the connection is the Levi-Civita connection, the lifted connection is called the *spin connection*.

The situation for spin$^c$-structures is somewhat more complicated. Assume again $P_{\mathrm{SO}}(E)$ is the frame bundle of a vector bundle $E$ and assume it carries a spin$^c$-structure as well as a connection. Since the map $\Phi^c : P^c_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ is *not* a covering map (because $\rho^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n)$ is not a covering map), we cannot simply lift a connection from $P_{\mathrm{SO}}(E)$ to $P^c_{\mathrm{Spin}}(E)$ as we did before. To fix a connection on the $\mathrm{Spin}^c(n)$-bundle we need not only a connection on the $\mathrm{SO}(n)$-bundle but also a connection $\mathcal{A}$ on the frame bundle $L^0$ of the determinant line bundle $L$. Let's spend a few moments to describe how this works out. Consider the product bundle $\pi\times\pi^0 : P_{\mathrm{SO}}(E)\times L^0 \longrightarrow M\times M$. This is an $\mathrm{SO}(n)\times\mathrm{U}(1)$-principal bundle over $M\times M$. The connections on $P_{\mathrm{SO}}(E)$ and $L^0$ give a natural connection on $P_{\mathrm{SO}}(E)\times L^0$: namely choose in the tangent space $T_{(p,q)}(P_{\mathrm{SO}}(E)\times L^0)\cong T_pP_{\mathrm{SO}}(E)\times T_qL^0$ the horizontal subspace $H_pP_{\mathrm{SO}}(E)\times H_qL^0$. Then we get a decomposition

$$T_{(p,q)}(P_{\mathrm{SO}}(E)\times L^0) = (H_pP_{\mathrm{SO}}(E)\times H_qL^0)\oplus(V_pP_{\mathrm{SO}}(E)\times V_qL^0).$$

It shouldn't be hard to check that this is a connection on the product bundle.

Let $\Delta : M \longrightarrow M\times M$ be the diagonal map $x\longmapsto(x,x)$ and consider the pullback bundle $Q := \Delta^*(P_{\mathrm{SO}}(E)\times L^0)$ (i.e. the restriction to $M$ viewed as the diagonal in $M\times M$). $Q$ is what we will call the *fibered product* or the *spliced bundle* of $P$ and $L^0$. Restrict the connection on $P_{\mathrm{SO}}(E)\times L^0$ to this new bundle (i.e. pull the connection back along $\Delta$). Thus we have a principal $\mathrm{SO}(n)\times\mathrm{U}(1)$-bundle $Q\longrightarrow M$ carrying a connection determined by the connections on $P_{\mathrm{SO}}(E)$ and $L^0$. If $\pi_1 : Q\longrightarrow P_{\mathrm{SO}}(E)$ and $\pi_2 : Q\longrightarrow L^0$ denote the obvious projection maps, and if $\omega$ is the connection 1-form for the connection on $P$ and $\mathcal{A}$ is the connection form for the connection on $L^0$ one can show that the connection on $Q$ is given by the connection 1-form

$$\omega^{\mathcal{A}} := (\pi_1^*\omega)\oplus(\pi_2^*\mathcal{A}) \tag{1.5}$$

which takes values in the Lie algebra $\mathfrak{spin}^c(n)\cong\mathfrak{spin}(n)\oplus i\mathbb{R}$.

In order to lift this connection to the $\mathrm{Spin}^c(n)$-bundle we need a covering of $Q$. We know that $\Lambda^c : \mathrm{Spin}^c(n)\longrightarrow\mathrm{SO}(n)\times\mathrm{U}(1)$ is a double covering, and so inspired by this we seek a bundle map $P^c_{\mathrm{Spin}}(E)\longrightarrow Q$ which locally looks like $\Lambda^c$. Our candidate: $\Xi(p) := (\Phi^c(p),[p,1])$. To see that it locally looks like $\Lambda^c$, pick trivializations $\Psi^c$ and $\Psi$ for $P^c_{\mathrm{Spin}}(E)$ resp. $P_{\mathrm{SO}}(E)$ over a common domain $U\subseteq M$ such that $\Psi\circ\Phi^c\circ(\Psi^c)^{-1}(x,[g,z]) = (x,\Lambda(g))$ (remember that $[g,z]\in\mathrm{Spin}^c(n)$ for $g\in\mathrm{Spin}(n)$). The trivialization $\Psi^c$ for $P_{\mathrm{Spin}}(E)$ gives a trivialization $\overline{\Psi}$ of $L^0$ over $U$ by (using the isomorphism from Lemma 1.5)

$$\overline{\Psi}([s(x),z]) = (x,z)$$

(where $s(x) := (\Psi^c)^{-1}(x,e)$ is the local section of $P^c_{\mathrm{Spin}}(E)$ corresponding to the trivialization $\Psi$ and $e\in\mathrm{Spin}^c(n)$ is the neutral element), and further $\Psi$ and $\overline{\Psi}$

give a trivialization of $Q$ over $U$, denoted $\Psi\times\overline{\Psi}$ (with a slight abuse of notation, since the trivialization is only a "fiberwise" product). We want to show that

$$(\Psi \times \overline{\Psi}) \circ \Xi \circ (\Psi^c)^{-1}(x,[g,z]) = (x,\Lambda(g),z^2), \tag{1.6}$$

and to see this put $p := (\Psi^c)^{-1}(x,[g,z])$. This is mapped to $(\Phi^c(p),[p,1])$ by $\Xi$ and $\Psi$ maps the first component to $(x,\Lambda(g))$ as it should. Note that

$$[p,1] = [s(x)\cdot[g,z],1] = [s(x),\lambda([g,z])1] = [s(x),z^2]$$

which by $\overline{\Psi}$ is mapped to $(x,z^2)$, thus we have verified (1.6).

Now we have a double covering map $\Xi : P^c_{\mathrm{Spin}}(E) \longrightarrow Q$ and thus we can repeat what we did before, lifting the connection on $Q$ to a connection on $P^c_{\mathrm{Spin}}(E)$. If $E$ is the tangent bundle for a spin$^c$-manifold $M$, the connection is called the *spin$^c$-connection.*

## 1.2 The Dirac Operator

In this section we let $M$ denote an oriented Riemannian manifold. No compactness condition is imposed unless specified.

Let $E$ be a real oriented Riemannian vector bundle over $M$ of rank $n$ (often we will take it to be the tangent bundle). Thus we can construct its oriented frame bundle $P_{\mathrm{SO}}(E)$. We want to construct the so-called *Clifford bundle* over $E$, i.e. an algebra bundle over $M$ whose fiber at $x$ is isomorphic to the Clifford algebra $\mathrm{Cl}(E_x)$. The construction is accomplished as an associated bundle in the following way: Consider $\mathrm{Cl}_{0,n}$, the Clifford algebra over $\mathbb{R}^n$ with the usual negative definit inner product. We have a representation $\rho$ of $\mathrm{SO}(n)$ on $\mathrm{Cl}_{0,n}$: any $A \in \mathrm{SO}(n)$ viewed as a linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserves the inner product, and hence induces an algebra homomorphism $\widetilde{A} : \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Cl}_{0,n}$, so the representation is given as $\rho(A) = \widetilde{A}$.

**Definition 1.9 (Clifford Bundle).** The *Clifford bundle* of $E$ is the associated bundle

$$\mathrm{Cl}(E) = P_{\mathrm{SO}}(E) \times_\rho \mathrm{Cl}_{0,n}\,.$$

Elements in $\mathrm{Cl}(E)$ are equivalence classes $[p,\xi]$, where $p \in P_{\mathrm{SO}}(E)$ and $\xi \in \mathrm{Cl}_{0,n}$ and the equivalence relation on $P_{\mathrm{SO}}(E)\times\mathrm{Cl}_{0,n}$ is $(p,\xi)\sim(p\cdot A,\rho(A^{-1})\xi)$. The projection map is $\widetilde{\pi} : \mathrm{Cl}(E) \longrightarrow M$, $[p,\xi] \longmapsto \pi(p)$ where $\pi : P_{\mathrm{SO}}(E) \longrightarrow M$ is the projection in the frame bundle. The vector space structure on the fibers is given by

$$a[p,\xi] + b[p,\xi'] = [p, a\xi + b\xi']$$

(note, by transitivity of the right $\mathrm{SO}(n)$-action on each fiber in $P_{\mathrm{SO}}(E)$ we can always assume the $p$'s to be equal). Similarly, the algebra structure is given by

$$[p,\xi]\cdot[p,\xi'] = [p,\xi\xi'],$$

and the identity element is $[p,1]$. It is easy to check that these operations are well-defined. Since $\xi^2 = -\|\xi\|^2 \cdot 1$, we get

$$[p,\xi][p,\xi] = [p,\xi\cdot\xi] = [p,-\|\xi\|^2\cdot 1] = -\|\xi\|^2[p,1],$$

thus each fiber is indeed a Clifford algebra of type $(0,n)$. Thus $\mathrm{Cl}(E)_x$ (the fiber in the Clifford bundle) is isomorphic to $\mathrm{Cl}(E_x)$ (the Clifford algebra of the vector space $E_x$).

Observe that we have $\mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$ and that $\mathbb{R}^n$ is a $\rho$-invariant subspace and that $\rho(A)|_{\mathbb{R}^n} = A$, so $\rho$ restricted to this invariant subspace is just the defining representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$. We write it as id. But this means that we have the subbundle $P_{\mathrm{SO}}(E) \times_{\mathrm{id}} \mathbb{R}^n \cong E$ sitting inside $\mathrm{Cl}(E)$. Elements in $E \subseteq \mathrm{Cl}(E)$ are characterized by being of the form $[p, v]$ where $v \in \mathbb{R}^n$. In particular we may view $\Gamma(E)$ as sitting inside $\Gamma(\mathrm{Cl}(E))$.

For the purpose of studying spinor bundles, as we will do later in this section, we need another description of the Clifford bundle. Assume that the oriented Riemannian vector bundle $E$ has a spin structure $\overline{\pi} : P_{\mathrm{Spin}}(E) \longrightarrow M$ with double covering bundle map $\Phi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$. Consider the representation $\mathrm{Ad} : \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(\mathrm{Cl}_{0,n})$ given by

$$\mathrm{Ad}(g)\xi = g\xi g^{-1}$$

(recall that $\mathrm{Spin}(n)$ sits inside $\mathrm{Cl}_{0,n}$, so multiplication makes sense), then the following diagram commutes (simply because $\mathrm{Ad}(g)$ is the unique extension of $\Lambda(g)$ to $\mathrm{Cl}_{0,n}$):

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r, "\mathrm{Ad}"] \arrow[d, "\Lambda"'] & \mathrm{Aut}(\mathrm{Cl}_{0,n}) \\
\mathrm{SO}(n) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

>From the principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}}(E)$ and the representation Ad, we can form the associated bundle $P_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_{0,n}$.

Analogously, define $\mathrm{Ad}^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{Aut}(\mathrm{Cl}_{0,n})$ by $\mathrm{Ad}^c([g, z]) = \mathrm{Ad}(g)$. This is well-defined since $\mathrm{Ad}(-g) = \mathrm{Ad}(g)$.

**Lemma 1.10.** *The map* $\Psi : P_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Cl}(E) = P_{\mathrm{SO}}(E) \times_{\rho} \mathrm{Cl}_{0,n}$ *given by* $[p, \xi] \longmapsto [\Phi(p), \xi]$ *is a well-defined smooth algebra bundle isomorphism.*

*Similarly, the map* $\Psi^c : P^c_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}^c} \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Cl}(E)$ *given by* $[p, \xi] \longmapsto [\Phi^c(p), \xi]$ *is a well-defined smooth algebra bundle isomorphism.*

Proof. It is well-defined, since

$$\begin{aligned}\Psi([p \cdot g^{-1}, \mathrm{Ad}(g)\xi]) &= [\Phi(p \cdot g^{-1}), \mathrm{Ad}(g)\xi] = [\Phi(p) \cdot \Lambda(g)^{-1}, \rho(\Lambda(g))\xi] \\ &= [\Phi(p), \xi] = \Psi([p, \xi]),\end{aligned}$$

the third identity is a consequence of equivariance of $\Psi$ and of the commuting diagram just above. Restricted to the fiber over $x$, the map is an algebra homomorphism (we skip checking linearity):

$$\begin{aligned}\Psi_x([p, \xi][p, \xi']) &= \Psi_x([p, \xi \cdot \xi']) = [\Phi(p), \xi \cdot \xi'] = [\Phi(p), \xi] \cdot [\Phi(p), \xi'] \\ &= \Psi_x([p, \xi])\Psi_x([p, \xi']).\end{aligned}$$

It is injective, for if $0 = \Psi_x([p, \xi]) = [\Phi(p), \xi]$, then $\xi$ must be 0, hence $[p, \xi] = 0$. Moreover $\Psi_x$ is surjective, since $\Phi$ is. Thus $\Psi$ is an algebra bundle isomorphism. The verification that $\Psi^c$ is an algebra bundle isomorphism is completely similar and so we skip it. □

**Definition 1.11** (**Dirac Bundle**)**.** Let $E$ be a real Riemannian vector bundle over $M$ and $\nabla$ a metric connection. A complex vector bundle $S$ over $M$, is called a $\mathrm{Cl}(E)$-*module* or a left *Clifford module* if for each $x \in M$ there is a representation of the algebra $\mathrm{Cl}(E)_x$ on $S_x$.

A left $\mathrm{Cl}(E)$-module is called a *Dirac bundle* over $E$, provided it is equipped with a fiber metric $\langle\ ,\ \rangle$ and a compatible connection $\widetilde{\nabla}$ satisfying the two additional conditions:

1) Clifford multiplication is skew-adjoint, i.e. for each $x \in M$ and each $V_x \in E_x$ and $\psi_1, \psi_2 \in S_x$:

$$\langle V_x \cdot \psi_1, \psi_2\rangle + \langle \psi_1, V_x \cdot \psi_2\rangle = 0. \tag{1.7}$$

2) The connection on $S$ is compatible with the connection on $E$ in the following sense:

$$\widetilde{\nabla}_X(V \cdot \psi) = (\nabla_X V) \cdot \psi + V \cdot (\widetilde{\nabla}_X \psi) \tag{1.8}$$

for $X \in \mathfrak{X}(M)$, $V \in \Gamma(E)$ and $\psi \in \Gamma(S)$.

The single most important example of a Dirac bundle is the *spinor bundle*

$$S(E) := P_{\mathrm{Spin}}(E) \times_{\kappa_n} \Delta_n$$

as defined in the previous section. To show that it is a Dirac bundle we first equip it with an action of the Clifford bundle $\mathrm{Cl}(E) = P_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_{0,n}$. Consider on the threefold product $P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n$ the equivalence relation $\sim$

$$(p, \xi, v) \sim (p \cdot g^{-1}, \mathrm{Ad}(g)\xi, \kappa_n(g)v)$$

for any $g \in \mathrm{Spin}(n)$. Elements in the quotient space $P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n / \sim$ are denoted $[p, \xi, v]$. As in the proof of Lemma 1.10 one can show that the map

$$\mathrm{Cl}(E) \times S(E) \longrightarrow P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n / \sim$$

given by $([p, \xi], [p, v]) \longmapsto [p, \xi, v]$ is a well-defined bundle isomorphism. This allows us to define the Clifford action in the following way: Define

$$\widetilde{\mu} : P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n \longrightarrow P_{\mathrm{Spin}}(E) \times \Delta_n$$

by $\widetilde{\mu}(q, \xi, v) := (q, \rho_n(\xi)v)$ (where $\rho_n : \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Aut}(\Delta_n)$ is the *spin representation* of the Clifford algebra) and note that the following diagram is commutative

$$\begin{tikzcd}
P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n \arrow[r, "\widetilde{\mu}"] \arrow[d, "\cdot g"'] & P_{\mathrm{Spin}}(E) \times \Delta_n \arrow[d, "\cdot g"] \\
P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n \arrow[r, "\widetilde{\mu}"'] & P_{\mathrm{Spin}}(E) \times \Delta_n
\end{tikzcd}$$

where the first vertical map is $(p, \xi, v) \longmapsto (p \cdot g^{-1}, \mathrm{Ad}(g)\xi, \kappa_n(g)v)$ and the second is $(p, v) \longmapsto (p \cdot g^{-1}, \kappa_n(g)v)$. Thus $\widetilde{\mu}$ induces a map $\mu : \mathrm{Cl}(E) \times S(E) \longrightarrow S(E)$ given explicitly by the formula

$$[p, \xi] \cdot [p, v] := \mu([p, \xi], [p, v]) = [p, \rho_n(\xi)v].$$

This is the desired Clifford action, turning $S(E)$ into a left $\mathrm{Cl}(E)$-module.

Next we want to give $S(E)$ a metric. Inside $\mathrm{Cl}_{0,n}$ we have the finite group

$$G_n := \{e_{i_1} \cdots e_{i_k} \mid 1 \leq k \leq n,\ 1 \leq i_1 < \cdots < i_k \leq n\}$$

(where $\{e_1, \ldots, e_n\}$ is some orthonormal basis for $\mathbb{R}^n$). Restricting the spin representation $\rho_n$ of $\mathrm{Cl}_{0,n}$ to $G_n$ gives a representation of $G_n$ on $\Delta_n$, also denoted $\rho_n$. By a well-known result from representation theory, there exists an inner product $\langle\, , \rangle_{\Delta_n}$ on $\Delta_n$ relative to which $\rho_n$ is a unitary representation, i.e.

$$\langle \rho_n(e_{i_1} \cdots e_{i_k})v, \rho_n(e_{i_1} \cdots e_{i_k})w\rangle_{\Delta_n} = \langle v, w\rangle_{\Delta_n}.$$

(To make the notation in the following less cumbersome, we will simply write the action of $\rho_n(\xi)$ on $v$ as $\xi \cdot v$.) If $\xi = \sum_{i=1}^n a_i e_i$ is a unit vector in $\mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$ then $\rho_n(e)$ is a unitary operator as well: First we observe

$$\begin{aligned}\langle e_i \cdot v, e_j \cdot w\rangle_{\Delta_n} &= \langle e_j \cdot (e_i \cdot v), e_j^2 \cdot w\rangle_{\Delta_n} = -\langle (e_j e_i)\cdot v, w\rangle_{\Delta_n} \\ &= \langle (e_i e_j)\cdot v, w\rangle_{\Delta_n} = \langle (e_i^2 e_j)\cdot v, e_i \cdot w\rangle_{\Delta_n} \\ &= -\langle e_j \cdot v, e_i \cdot w\rangle_{\Delta_n},\end{aligned}$$

and from this we get

$$\begin{aligned}\langle \xi \cdot v, \xi \cdot w\rangle_{\Delta_n} &= \Big\langle \sum_{i=1}^n a_i e_i \cdot v, \sum_{j=1}^n a_j e_j \cdot w \Big\rangle_{\Delta_n} \\ &= \sum_{i=1}^n a_i^2 \langle e_i \cdot v, e_i \cdot w\rangle_{\Delta_n} + \sum_{i\neq j} a_i a_j \langle e_i \cdot v, e_j \cdot w\rangle_{\Delta_n} \\ &= \sum_{i=1}^n a_i^2 \langle v, w\rangle_{\Delta_n} + \sum_{i<j} a_i a_j \big(\langle e_i \cdot v, e_j \cdot w\rangle_{\Delta_n} + \langle e_j \cdot v, e_i \cdot w\rangle_{\Delta_n}\big) \\ &= \langle v, w\rangle_{\Delta_n}.\end{aligned}$$

Since $\mathrm{Spin}(n)$ is generated by unit vectors, we see immediately that $\kappa_n$ is a unitary representation w.r.t. this inner product.

Furthermore, for any $\xi \in \mathbb{R}^n$, unit vector or not, $\rho_n(\xi)$ is a skew-adjoint map:

$$\langle \xi \cdot v, w\rangle_{\Delta_n} = \langle \tfrac{\xi}{\|\xi\|} \cdot (\xi \cdot v), \tfrac{\xi}{\|\xi\|} \cdot w\rangle_{\Delta_n} = \frac{1}{\|\xi\|^2}\langle \xi^2 \cdot v, \xi \cdot w\rangle_{\Delta_n} = -\langle v, \xi \cdot w\rangle_{\Delta_n}.$$

Note that this implies that $\rho_n(\xi)$ is skew-adjoint, when $\xi$ is in $\mathrm{Cl}^1_{0,n}$ (the odd part of $\mathrm{Cl}_{0,n}$) and that $\rho_n(\xi)$ is self-adjoint when $\xi \in \mathrm{Cl}^0_{0,n}$ (the even part of $\mathrm{Cl}_{0,n}$). In particular, $\kappa_n = \rho_n|_{\mathrm{Spin}(n)}$ is self-adjoint.

We can readily extend this inner product to a fiber metric on $S(E)$, simply by defining

$$\langle [p, v], [p, w]\rangle := \langle v, w\rangle_{\Delta_n}.$$

This is well-defined by unitarity of $\kappa_n(g)$:

$$\begin{aligned}\langle [p \cdot g^{-1}, \kappa_n(g) v], [p \cdot g^{-1}, \kappa_n(g) w]\rangle &= \langle \kappa_n(g) v, \kappa_n(g) w\rangle_{\Delta_n} = \langle v, w\rangle_{\Delta_n} \\ &= \langle [p, v], [p, w]\rangle.\end{aligned}$$

Checking condition 1 in the definition of a Dirac bundle is not hard: Let $V_x \in E_x \subseteq \mathrm{Cl}(E)_x$ and $\psi_1, \psi_2 \in S_x(E)$. We have presentations $V_x = [p, v]$ and $\psi_i = [p, w_i]$ where $v \in \mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$, $p \in P_{\mathrm{Spin}}(E)$ and $w_i \in \Delta_n$, and hence:

$$\begin{aligned}\langle V_x \cdot \psi_1, \psi_2\rangle &= \langle [p, v] \cdot [p, w_1], [p, w_2]\rangle = \langle [p, v \cdot w_1], [p, w_2]\rangle \\ &= \langle v \cdot w_1, w_2\rangle_{\Delta_n} = -\langle w_1, v \cdot w_2\rangle_{\Delta_n} = -\langle [p, w], [p, v] \cdot [p, w_2]\rangle \\ &= -\langle \psi_1, V_x \cdot \psi_2\rangle.\end{aligned}$$

In the previous section we equipped $S(E)$ with a connection, namely the lift of some connection on $P_{\mathrm{SO}}(E)$. If $\omega$ denotes the connection 1-form for the connection on $P_{\mathrm{SO}}(E)$, the connection 1-form of the lifted connection can be constructed as follows: $\widetilde{\omega} := (d\Lambda)^{-1} \circ \Phi^* \omega$. This is a $\mathfrak{spin}(n)$-valued 1-form, and to see that it is a connection form, we only have to check that the two axioms are satisfied. The first one:

$$\begin{aligned}\widetilde{\sigma}_g^* \widetilde{\omega} &= (d\Lambda)^{-1} \sigma_g^* \Phi^* \omega = (d\Lambda)^{-1} \Phi^* \sigma_{\Lambda(g)}^* \omega \\ &= (d\Lambda)^{-1} \Phi^* \operatorname{Ad}(\Lambda(g^{-1})) \circ \omega = (d\Lambda)^{-1} \circ \operatorname{Ad}(\Lambda(g^{-1}))(\Phi^* \omega) \\ &= \operatorname{Ad}(g^{-1}) \circ (d\Lambda)^{-1} \Phi^* \omega = \operatorname{Ad}(g^{-1}) \circ \widetilde{\omega}.\end{aligned}$$

For the second one let $p \in P_{\mathrm{Spin}}(E)$ and recall $\widetilde{\sigma}_p : \mathrm{Spin}(n) \longrightarrow P_{\mathrm{Spin}}(E)$ given by $g \longmapsto p \cdot g$ and note that $\Phi \circ \widetilde{\sigma}_p = \sigma_{\Phi(p)} \circ \Lambda$, where $\sigma_{\Phi(p)}$ is the similar map on the bundle $P_{\mathrm{SO}}(E)$. Then:

$$\begin{aligned}\widetilde{\omega}_p \circ (\widetilde{\sigma}_p)_* &= (d\Lambda)^{-1} \circ (\Phi^*\omega)_p \circ d\widetilde{\sigma}_p = (d\Lambda)^{-1}\omega_{\Phi(p)} \circ d\Phi \circ d\widetilde{\sigma}_p \\ &= (d\Lambda)^{-1}\omega_{\Phi(p)} \circ d\sigma_{\Phi(p)} \circ d\Lambda = (d\Lambda)^{-1} \circ d\Lambda = \mathrm{id}_{\mathfrak{spin}(n)} .\end{aligned}$$

This is the connection 1-form of the lifted connection, since one can easily check that $\ker(\widetilde{\omega})_p = H_p P_{\mathrm{Spin}}(E)$.

There is a standard procedure for transforming connections on principal bundles to connections on the associated bundles. In general let $\pi : P \longrightarrow M$ be a principal $G$-bundle, $\rho : G \longrightarrow \mathrm{Aut}(V)$ a finite-dimensional representation of $G$ on $V$ and $E := P \times_\rho V$ the associated vector bundle. Recall that there is a 1-1 correspondence between sections of $E$ and functions $f : P \longrightarrow V$ satisfying $f(p \cdot g) = \rho(g^{-1})f(p)$ (the so-called *equivariant functions*). If $\psi \in \Gamma(E)$ we write $\widehat{\psi}$ for the associated equivariant function.

The map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ given by $(X, \psi) \longmapsto \nabla_X\psi$ where $\nabla_X\psi$ is the section of $E$ corresponding to the equivariant function $p \longmapsto \overline{X}_p(\widehat{\psi})$ (here $\overline{X}$ is the unique lift of $X$ to a horizontal vector field on $P$) defines a connection on $E$. In short

$$\widehat{\nabla_X\psi}(p) = \overline{X}_p(\widehat{\psi}). \tag{1.9}$$

We also have a local description of the situation. Given a set of trivializations $(U_\alpha, \Phi_\alpha)_{\alpha\in I}$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{GL}(n, \mathbb{K})$ there is a 1-1 correspondence between smooth sections of $E$ and collections $(\psi_\alpha)_{\alpha\in I}$ of smooth functions $\psi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ satisfying $\psi_\alpha(x) = g_{\alpha\beta}(x)\psi_\beta(x)$ for $x \in U_{\alpha\beta}$: Given a section $\psi \in \Gamma(E)$, $\psi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ is the unique function satisfying $\Phi_\alpha \circ \psi(x) = (x, \psi_\alpha(x))$, i.e.

$$\psi_\alpha(x) = \mathrm{pr}_2 \circ \Phi_\alpha(\psi(x)).$$

So the question arises: how does the induced connection on $E$ look locally? Well, given a local section $s_\alpha : U_\alpha \longrightarrow P$ of the principal bundle, the local function of $\nabla_X\psi$ relative to the corresponding trivialization of $E$ is given by (see L. Claessens "Field Theory from a Bundle Point of View", Section 6.2, in [3] Section 6.2)

$$(\nabla_X\psi)_\alpha(x) = X_x\psi_\alpha - \rho_*(A^\alpha(X_x))\psi_\alpha(x) \tag{1.10}$$

where $\rho_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is the induced Lie algebra representation of $\rho$ and $A^\alpha = s_\alpha^*\omega$ is the so-called *local gauge potential.*

In the case of the Levi-Civita connection on $E = TM$, the formula reads

$$(\nabla_X Y)_\alpha(x) = X_x Y_\alpha - \Big(\sum_{i\in I} A_i^\alpha(X_x)B_i\Big)Y_\alpha(x) \tag{1.11}$$

where $(B_i)_{i\in I}$ is some basis for the Lie algebra $\mathfrak{so}(n)$ where $m = \dim M$.

Getting back to the spin bundle case, the connection $\widetilde{\omega}$ on $P_{\mathrm{Spin}}(E)$ induces a connection $\widetilde{\nabla}$ on the spinor bundle $S(E)$. Let's try to unveil (1.10) in this particular setting. Let $t_\alpha : U_\alpha \longrightarrow P_{\mathrm{Spin}}(E)$ be a local section of the spin bundle and put $s_\alpha := \Phi \circ t_\alpha$. This is a local section of the frame bundle $P_{\mathrm{SO}}(E)$. Let

$$\widetilde{A}^\alpha := t_\alpha^*\widetilde{\omega} \qquad \text{and} \qquad A^\alpha := s_\alpha^*\omega$$

denote the local gauge potentials of the connection $\omega$, resp. the connection $\widetilde{\omega}$. Then we have

$$\begin{aligned}\widetilde{A}^\alpha &= t_\alpha^*\widetilde{\omega} = t_\alpha^*((d\Lambda)^{-1} \circ \Phi^*\omega) = (d\Lambda)^{-1} \circ t_\alpha^*\Phi^*\omega \\ &= (d\Lambda)^{-1} \circ s_\alpha^*\omega = (d\Lambda)^{-1} \circ A^\alpha,\end{aligned}$$

so the gauge potentials are related in the nicest possible way.

Putting this into (1.10) we obtain

$$\begin{aligned}(\widetilde{\nabla}_X\psi)_\alpha(x) &= X_x\psi_\alpha - (\kappa_n)_*(\widetilde{A}^\alpha(X_x))\psi_\alpha(x)\\ &= X_x\psi_\alpha - \rho_n((d\Lambda)^{-1}(A^\alpha(X_x)))\psi_\alpha(x)\end{aligned}$$

for $x \in U_\alpha$ (recall that the action of $(\kappa_n)_*$ is just $\rho_n$ itself). Now we pick the usual basis $(B_{ij})_{i<j}$ for $\mathfrak{so}(n)$ (where $B_{ij}$ is the $n \times n$-matrix whose $ij$'th entry is $-1$, the $ji$'th entry is 1 and all other entries are 0) and write the $\mathfrak{so}(n)$-valued 1-form $A^\alpha$ in terms of this basis: $A^\alpha = \sum_{i<j} A^\alpha_{ij} B_{ij}$. Then (remembering that $d\Lambda : \mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ maps $e_ie_j$ to $2B_{ij}$, where $(e_i)$ is the standard basis for $\mathbb{R}^n$) we get

$$\begin{aligned}(\widetilde{\nabla}_X\psi)_\alpha(x) &= X_x\psi_\alpha - \rho_n\Big((d\Lambda)^{-1}\Big(\sum_{i<j} A^\alpha_{ij}(X_x)B_{ij}\Big)\Big)\psi_\alpha(x)\\ &= X_x\psi_\alpha - \rho_n\Big(\frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big)\psi_\alpha(x)\\ &= X_x\psi_\alpha - \frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j \cdot \psi_\alpha(x). \qquad (1.12)\end{aligned}$$

Now, let us show compatibility of the connection $\widetilde{\nabla}$ with the fiber metric. We will use the local expression above. Locally $\psi(x) = [s_\alpha(x), \psi_\alpha(x)]$ (for some section $s_\alpha : U_\alpha \longrightarrow P_{\mathrm{Spin}}(E)$ defined around $x$) we get by definition of the fiber metric

$$\langle \psi(x), \psi'(x)\rangle = \langle \psi_\alpha(x), \psi'_\alpha(x)\rangle_{\Delta_n}$$

for $x \in U_\alpha$. Thus

$$\begin{aligned}\langle \widetilde{\nabla}_X\psi(x), \psi'(x)\rangle &= \Big\langle X_x\psi_\alpha - \frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j \cdot \psi_\alpha(x)\,,\ \psi'_\alpha(x)\Big\rangle_{\Delta_n}\\ &= \langle X_x\psi, \psi'(x)\rangle_{\Delta_n} - \frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)\langle e_ie_j \cdot \psi_\alpha(x), \psi'_\alpha(x)\rangle_{\Delta_n}.\end{aligned}$$

Note that

$$\begin{aligned}\langle e_ie_j \cdot \psi_\alpha(x), \psi'_\alpha(x)\rangle_{\Delta_n} &= -\langle e_j \cdot \psi_\alpha(x), e_i \cdot \psi'_\alpha(x)\rangle_{\Delta_n} = \langle \psi_\alpha(x), e_je_i \cdot \psi'_\alpha(x)\rangle_{\Delta_n}\\ &= -\langle \psi_\alpha(x), e_ie_j \cdot \psi'_\alpha(x)\rangle_{\Delta_n},\end{aligned}$$

and therefore

$$\langle \widetilde{\nabla}_X\psi(x), \psi'(x)\rangle + \langle \psi(x), \widetilde{\nabla}_X\psi'(x)\rangle = \langle X_x\psi, \psi'(x)\rangle_{\Delta_n} + \langle \psi(x), X_x\psi'\rangle_{\Delta_n}.$$

As mentioned $X_x\psi_\alpha$ should be interpreted componentwise, i.e. pick a complex basis $\{v_1, \ldots, v_N\}$ for $\Delta_n$ (of course $N = 2^{\lfloor \frac{n}{2} \rfloor}$) and write

$$\psi_\alpha = \sum_{i=1}^N \psi_{\alpha,i} v_i$$

where $\psi_{\alpha,i}$, the $i$'th component of $\psi_\alpha$, is a complex-valued function on $U_\alpha$, then

$$X_x\psi_\alpha = \sum_{i=1}^N (X_x\psi_{\alpha,i})v_i \in \Delta_n.$$

Since we have a metric in play, it would be wise of us to assume the basis $\{v_1,\dots,v_N\}$ to be orthonormal. Then

$$\begin{aligned}\langle X_x\psi_\alpha,\psi'_\alpha(x)\rangle_{\Delta_n}+\langle\psi_\alpha(x),X_x\psi'_\alpha\rangle_{\Delta_n}&=\sum_{i=1}^N(X_x\psi_{\alpha,i})\psi'_{\alpha,i}(x)+\sum_{i=1}^N\psi_{\alpha,i}(x)X_x\psi'_{\alpha,i}\\&=X_x\Big(\sum_{i=1}^N\psi_{\alpha,i}\psi'_{\alpha,i}\Big)=X_x\langle\psi_\alpha,\psi'_\alpha\rangle_{\Delta_n}.\end{aligned}$$

Thus we have proved that the spin connection is compatible with the metric.

Finally we need to check condition 2 in the definition of a Dirac bundle, that is

$$\widetilde{\nabla}_X(Y\cdot\psi)(x)=(\nabla_XY)(x)+Y_x\cdot(\widetilde{\nabla}_X\psi(x)) \tag{1.13}$$

for each $x\in M$. Again we use the local expressions, i.e. we consider a cover $(U_\alpha)$ which are domains of trivializations of both $TM$ and $E$. Let $\Phi_\alpha$ denote the trivializations of the tangent bundle (we may assume it to preserve the metric on $M$, i.e. $\Phi_x:T_xM\xrightarrow{\sim}\mathbb{R}^m$ is an isometry). Since $\widetilde{\nabla}_X$ is $\mathbb{C}$-linear and satisfies the Leibniz rule, it is sufficient to verify the above condition for $Y=E_k$ where $E_k(x)=\Phi_\alpha^{-1}(x,e_k)$ are local orthonormal vector fields.

But first, recall the following formula for the differential of the double covering

$$d\Lambda(X)v=Xv-vX$$

for $X\in\mathfrak{spin}(n)$ and $v\in\mathbb{R}^n\subseteq\mathrm{Cl}_{0,n}$ and replace in that $X$ by $\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\in\mathfrak{spin}(n)$ and $v$ by $e_k$ to get

$$\begin{aligned}\Big(\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\Big)e_k&=e_k\Big(\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\Big)+d\Lambda\Big(\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\Big)e_k\\&=e_k\Big(\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\Big)+2\sum_{i<j}A^\alpha_{ij}(X_x)B_{ij}e_k.\end{aligned} \tag{1.14}$$

Note that concatenation here means multiplication inside the Clifford algebra, and *not* the Clifford action. Recall also formula (1.11) for the local form of the Levi-Civita connection. It will be used in the following calculations (for explanations see below):

$$\begin{aligned}(\widetilde{\nabla}_X(E_k\cdot\psi))_\alpha(x)&=X_x(e_k\cdot\psi_\alpha)-\Big(\frac{1}{2}\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\Big)e_k\cdot\psi_\alpha(x)\\&=e_k\cdot(X_x\psi_\alpha)-e_k\Big(\frac{1}{2}\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\Big)\cdot\psi_\alpha(x)\\&\quad-\sum_{i<j}(A^\alpha_{ij}(X_x)B_{ij}e_k)\cdot\psi_\alpha(x)\\&=e_k\cdot(\widetilde{\nabla}_X\psi)_\alpha(x)+(\nabla_XE_k)_\alpha(x)\cdot\psi_\alpha(x)\end{aligned}$$

and this is precisely the local form of the right-hand side of (1.13). For the first identity we used that $(E_k\cdot\psi)_\alpha=e_k\cdot\psi_\alpha$ and in the second we used (1.14) as well as the fact, that $e_k\cdot$ is a linear map, and thus commutes with $X_x$. This verifies condition 2 and hence we have shown that the spinor bundle $S(E)$ is a Dirac bundle.

The second most important example of a Dirac bundle is the complex spinor bundle $S^c(E)$. We can proceed in almost the same way we did before and we begin by giving $S^c(E)$ a metric: On $\Delta_n$ we can, as before, find an inner product $\langle\,,\,\rangle_{\Delta_n}$ such that $\kappa^c_n(g)$ is unitary for each $g\in\mathrm{Spin}^c(n)$ and such that $\rho_n(v)$ is skew-adjoint for any $v \in \mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$. We transfer this inner product to a fiber metric on $S^c(E)$ in the usual way by defining

$$\langle [p, v], [p, w] \rangle := \langle v, w \rangle_{\Delta_n}.$$

Unitarity of $\kappa_n^c(g)$ guarantees that this is well-defined.

Thanks to the isomorphism $\mathrm{Cl}(E) \cong P^c_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}^c} \mathrm{Cl}_{0,n}$ we can also define a Clifford action $\mathrm{Cl}(E) \times S^c(E) \longrightarrow S^c(E)$ by

$$([p, \xi], [p, v]) \longmapsto [p, \rho_n(\xi)v].$$

Checking condition 1 in the definition of a Dirac bundle is done as above.

In the previous section we gave $P^c_{\mathrm{Spin}}(E)$ a connection depending on the choice of connection $\mathcal{A}$ on $L^0$. The corresponding connection 1-form is given in the same way as for the spin bundle

$$\widetilde{\omega}^{\mathcal{A}} = (d\Lambda^c)^{-1} \circ (\Xi^* \omega^{\mathcal{A}})$$

where $\Xi : P^c_{\mathrm{Spin}}(E) \longrightarrow Q$ is the double covering and $Q$ is the bundle is as defined in the previous section. This connection induces a connection $\widetilde{\nabla}^{\mathcal{A}}$ on $S^c(E)$. We are interested in calculating its local expression from (1.10). Note first that $d\Lambda^c : \mathfrak{spin}^c(n) \cong \mathfrak{spin}(n) \oplus i\mathbb{R} \longrightarrow \mathfrak{so}(n) \oplus i\mathbb{R}$ (we identify the Lie algebra of $\mathrm{U}(1)$ with $i\mathbb{R}$) is simply given by

$$(t, i\theta) \longmapsto (d\Lambda(t), 2i\theta). \tag{1.15}$$

Consider trivializations of $P^c_{\mathrm{Spin}}(E)$ and $Q$ over $U_\alpha$, let $t_\alpha : U_\alpha \longrightarrow P^c_{\mathrm{Spin}}(E)$ denote the corresponding local section and put $s_\alpha := \Xi \circ t_\alpha$. If

$$A^\alpha := s_\alpha^* \omega^{\mathcal{A}} \qquad \text{and} \qquad \widetilde{A}^\alpha := t_\alpha^* \widetilde{\omega}^{\mathcal{A}}$$

denote the local gauge potentials (local connection 1-forms), then we have $\widetilde{A}^\alpha = (d\Lambda^c)^{-1} \circ A^\alpha$. Note also that since $A^\alpha$ is an $\mathfrak{so}(n) \oplus i\mathbb{R}$-valued 1-form, we may split it: $A^\alpha = A^\alpha_\omega \oplus A^\alpha_{\mathcal{A}}$ where $A^\alpha_\omega$ is the gauge potential for the connection $\omega$ on $P_{\mathrm{SO}}(E)$ and $A^\alpha_{\mathcal{A}}$ is the gauge potential for the connection $\mathcal{A}$ on $L^0$. >From (1.15) we get

$$(d\Lambda^c)^{-1} A^\alpha = (d\Lambda^{-1}(A^\alpha_\omega), \tfrac{1}{2} A^\alpha_{\mathcal{A}})$$

and the induced Lie algebra representation of $\kappa_n^c$ is just $\rho_n$ restricted to the Lie algebra. $\kappa_n(\frac{1}{2} A^\alpha_{\mathcal{A}}(X_x))$ is just multiplication with the imaginary number $\frac{1}{2} A^\alpha_{\mathcal{A}}(X_x)$ and $(d\Lambda)^{-1} A^\alpha_\omega$ is already known to us from our discussion above. Hence we get:

$$(\widetilde{\nabla}^{\mathcal{A}}_X \psi)_\alpha(x) = X_x \psi_a - \frac{1}{2} \sum_{i<j} (A^\alpha_\omega)_{ij}(X_x) e_i e_j \cdot \psi_\alpha(x) + \frac{1}{2} A^\alpha_{\mathcal{A}}(X_x) \psi_\alpha(x). \tag{1.16}$$

With this local expression at our disposal we can show that the requirements of Definition 1.11 are satisfied, and hence that $S^c(E)$ is a Dirac bundle. The arguments are identical to the ones above for the spin bundle and so we skip them.

**Definition 1.12 (Dirac Operator).** Let $S$ be a Dirac bundle. The *Dirac operator* $\not{\partial}$ is then defined as the composition

$$\Gamma(S) \xrightarrow{\widetilde{\nabla}} \Gamma(T^*M \otimes S) \xrightarrow{\sim} \Gamma(TM \otimes S) \longrightarrow \Gamma(S) \tag{1.17}$$

where the last map is Clifford multiplication $\mathfrak{X}(M) \otimes \Gamma(S) \longrightarrow \Gamma(S)$.

Next follow three elementary facts about general Dirac operators. Proofs are not included here, they can be found in any standard treatment of the subject, for instance in [13] Part II.5.

**Lemma 1.13.** *The Dirac operator is a first order differential operator, and given a local orthonormal frame* $\{E_1, \dots, E_m\}$ *for* $TM$ *over* $U$*, the Dirac operator takes the local form*

$$\not\partial\psi|U = \sum_{j=1}^{m} E_j \cdot (\widetilde{\nabla}_{E_j}\psi). \tag{1.18}$$

As a differential operator, we can compute its symbol:

**Proposition 1.14.** *For* $\xi_x \in T_x^*M$*, let* $\xi_x^\sharp\cdot : S_x \longrightarrow S_x$ *be Clifford multiplication with the metric dual* $\xi_x^\sharp$ *of* $\xi_x$*. Then*

$$\sigma(\not\partial)(\xi_x) = i\xi_x^\sharp\,\cdot \qquad \text{and} \qquad \sigma(\not\partial^2)(\xi_x) = \|\xi_x\|^2.$$

*Thus both* $\not\partial$ *and* $\not\partial^2$ *(called the* Dirac Laplacian*) are elliptic.*

**Proposition 1.15.** *The Dirac operator is formally self-adjoint, i.e. for* $\psi_1$ *and* $\psi_2$ *in* $\Gamma_c(S)$ *(the set of sections of* $S$ *with compact support) we have*

$$(\not\partial\psi_1|\psi_2) = (\psi_1|\not\partial\psi_2). \tag{1.19}$$

We augment this list of elementary properties of the Dirac operator with a perhaps less renowned result. The proof may be found in [2] Theorem 8.2:

**Theorem 1.16 (Unique Continuation Property).** *If* $\psi \in \Gamma(S)$ *is in the kernel of the Dirac operator* $\not\partial$ *and* $\psi$ *is* 0 *on some open set, then* $\psi$ *is identically equal to* 0*.*

In the next example we present two of the most important Dirac operators.

**Example 1.17.** (**Spin-Dirac operator**). We consider the spinor bundle $S(E)$ associated to some oriented Riemannian vector bundle $E$ carrying a spin structure. We saw earlier that this vector bundle is a Dirac bundle and thus it carries a Dirac operator, $\not{D}$ called the *spin-Dirac operator* or just *the* Dirac operator (in [13] it is called the *Atiyah-Singer operator*). Thanks to Lemma 1.13 and the calculations done in the previous example, we arrive at the following local description

$$\begin{aligned}
(\not{D}\,\psi)_\alpha(x) &= \Big(\sum_{k=1}^{m} E_k \cdot \widetilde{\nabla}_{E_k}\psi\Big)_\alpha(x) = \sum_{k=1}^{m} e_k \cdot (\widetilde{\nabla}_{E_k}\psi)_\alpha(x) \\
&= \sum_{k=1}^{m} e_k \cdot \Big((E_k)_x\psi_\alpha - \frac{1}{2}\sum_{i<j}^{n} A_{ij}^{\alpha}((E_k)_x)e_ie_j \cdot \psi_\alpha(x)\Big) \\
&= \sum_{k=1}^{m} e_k \cdot (E_k)_x\psi_\alpha - \frac{1}{2}\sum_{i<j} A_{ij}^{\alpha}((E_k)_x)e_ke_ie_j \cdot \psi_\alpha(x)
\end{aligned} \tag{1.20}$$

where $\{E_1, \dots, E_m\}$ is a local tangent frame ($m = \dim M$) such that $E_k(x) = \Phi_\alpha^{-1}(x, e_k)$ [1].

---

[1] Note that often $(E_k)_x\psi_\alpha$ is written as $\frac{\partial\psi_\alpha}{\partial e_k}(x)$.

(**Geometric Dirac operator**). The Dirac operator associated to a complex spinor bundle of a principal $\mathrm{Spin}^c(n)$-bundle is called the *geometric Dirac operator*. Inserting the local expression (1.16) of the connection into (1.18) we get

$$\begin{aligned}(\not{D}_{\mathcal{A}}\psi)_\alpha(x) = \sum_{k=1}^m \Big( & e_k \cdot (E_k)_x \psi_\alpha - \frac{1}{2}\sum_{i<j}(A_\omega^\alpha)_{ij}((E_k)_x) e_k e_i e_j \cdot \psi_\alpha \\ & + \frac{1}{2} A_{\mathcal{A}}^\alpha((E_k)_x) e_k \cdot \psi_\alpha(x)\Big). \end{aligned} \tag{1.21}$$

where $\mathcal{A}$ is a choice of a connection on the determinant line bundle, and where $m$ is the dimension of the base manifold. If we replace the connection $\mathcal{A}$ on the determinant line bundle by $\mathcal{A}+\beta$ for some $\beta \in i\Omega^1(M)$ the local connection 1-forms change to from $A_{\mathcal{A}}^\alpha$ to $A_{\mathcal{A}}^\alpha + \beta$ and from the local expression for the Dirac operator we immediately deduce

$$\not{D}_{\mathcal{A}+\beta}\psi = \not{D}_{\mathcal{A}}\psi + \frac{1}{2}\beta \cdot \psi. \tag{1.22}$$

We will make extensively use of this result. □

In the Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$ we have the *volume form* given unambiguously by $\omega_{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n$ whenever $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathbb{R}^n$. In the same manner we may define a *volume section* (also denoted $\omega_{\mathbb{C}}$) of the *complexified* Clifford bundle $\mathrm{Cl}(M)\otimes\mathbb{C}$ by the local formula $\omega_{\mathbb{C}}|_U = i^{\lfloor \frac{m+1}{2} \rfloor} E_1 \cdots E_m$ when $(E_1, \ldots, E_m)$ is a local orthonormal tangent frame over $U$.

Assume now that $m = 2k$, then $\omega_{\mathbb{C}} = i^k E_1 \cdots E_{2k}$. In this dimension it is well-known that the volume form commutes with everything in $(\mathrm{Cl}_m^{\mathbb{C}})^0$ - the even part of the Clifford algebra. For a Dirac bundle $S$ over $E$, $\widetilde{\nabla}$ the connection and $\psi$ a section of $S$, we get since $X_x(\omega_{\mathbb{C}} \cdot \psi)_\alpha = X_x(\omega_{\mathbb{C}} \cdot \psi_\alpha) = \omega_{\mathbb{C}} \cdot (X_x \psi_\alpha)$ (in the first expression $\omega_{\mathbb{C}}$ denotes the volume *section* and in the two last expressions it denotes the volume *form*) that

$$\begin{aligned}[\widetilde{\nabla}_X(\omega_{\mathbb{C}} \cdot \psi)]_\alpha(x) &= X_x(\omega_{\mathbb{C}} \cdot \psi)_\alpha - \frac{1}{2}\sum_{i<j} A_{ij}^\alpha(X_x) e_i e_j \cdot (\omega_{\mathbb{C}} \cdot \psi_\alpha(x)) \\ &= \omega_{\mathbb{C}} \cdot (X_x \psi_\alpha) - \omega_{\mathbb{C}} \cdot \frac{1}{2}\sum_{i<j} A_{ij}^\alpha(X_x) e_i e_j \cdot \psi_\alpha(x) \\ &= \omega_{\mathbb{C}} \cdot (\widetilde{\nabla}_X \psi)_\alpha(x).\end{aligned}$$

If $E$ has a spin structure and $S(E)$ is the associated spinor bundle, then we have the splitting $S = S^+ \oplus S^-$ induced by the decomposition $\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-$. The subspaces $\Delta_{2k}^\pm$ are the $\pm1$-eigenspaces of the action of $\omega_{\mathbb{C}}$ on $\Delta_{2k}$. Therefore, sections of $S(E)^\pm$ are exactly the spinor fields satisfying $\omega_{\mathbb{C}} \cdot \psi = \pm\psi$. Therefore the formula above implies that $\widetilde{\nabla}_X$ maps $\Gamma(S^\pm) \longrightarrow \Gamma(S^\pm)$. To see how the Dirac operator reacts to this splitting, note the following:

$$\begin{aligned}\not{D}(\omega_{\mathbb{C}} \cdot \psi) &= \sum_{i=1}^m E_i \cdot \widetilde{\nabla}_{E_i}(\omega_{\mathbb{C}} \cdot \psi) = \sum_{i=1}^m E_i \cdot \omega_{\mathbb{C}} \cdot \widetilde{\nabla}_{E_i}(\psi) \\ &= -\omega_{\mathbb{C}} \cdot \sum_{i=1}^m E_i \cdot \widetilde{\nabla}_{E_i}(\psi) = -\omega_{\mathbb{C}} \cdot \not{D}\psi\end{aligned}$$

($m = \dim M$). From this it is apparent, that $\not{D}$ maps $\Gamma(S(E)^\pm) \longrightarrow \Gamma(S(E)^\mp)$. If $\not{D}^\pm$ denotes the restriction of $\not{D}$ to $\Gamma(S(E)^\pm)$ we may write the Dirac operator relative to the splitting as a matrix

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix}.$$

Exactly the same holds true for the geometric Dirac operator. Also the spin$^c$-representation $\kappa^c_{2k} : \mathrm{Spin}^c(2k) \longrightarrow \mathrm{Aut}(\Delta_{2k})$ decomposes into irreducible representation spaces $\Delta_{2k} = \Delta^+_{2k} \oplus \Delta^-_{2k}$ and hence also the complex spinor bundle $S^c(E)$ exhibits a splitting $S^c(E) = S^c(E)^+ \oplus S^c(E)^-$, relative to which the geometric Dirac operator $\not{D}_{\mathcal{A}}$ takes the form

$$\not{D}_{\mathcal{A}} = \begin{pmatrix} 0 & \not{D}^-_{\mathcal{A}} \\ \not{D}^+_{\mathcal{A}} & 0 \end{pmatrix}$$

where $\not{D}^\pm_{\mathcal{A}} : \Gamma(S^c(E)^\pm) \longrightarrow \Gamma(S^c(E)^\mp)$.

## 1.3 Curvature Identities

Originally, back in the 30's, when Dirac set out to find his Dirac operator, his goal was to find an operator which was a "square root" of the Laplacian. So to justify the name "Dirac operator" for the operators defined above (and, not least, to acquire some tools which will prove indispensable later on), lets see how the Dirac Laplacian $\not{\partial}^2$ relates to the so-called *connection Laplacian* $\widetilde{\nabla}^*\widetilde{\nabla}$ where $\widetilde{\nabla}$ is the connection on the Dirac bundle and $\widetilde{\nabla}^*$ is its formal adjoint. What we will eventually realize is that the difference between the Dirac Laplacian and the connection Laplacian is an expression involving the curvature of the underlying Riemannian manifold.

Let $E$ be any Riemannian vector bundle (real or complex, that doesn't matter) over a Riemannian manifold $M$ and let $\nabla$ be a metric connection on $E$. We will also let $\nabla$ denote the Levi-Civita connection on $M$, there should be little possibility for confusion. Recall that the *curvature transformation* of the connection $\nabla$ on $E$ is defined to be the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ given by

$$R_{X,Y}\varphi = \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi.$$

This is a tensor field, i.e. the value of $R_{X,Y}\varphi$ at $x \in M$ depends only on $X_x$, $Y_x$ and $\varphi(x)$, thus for each $x \in M$ we get a multilinear map $T_xM \times T_xM \times E_x \longrightarrow E_x$.

For a connection $\nabla$ on the vector bundle $E$ we define the *local connection* 1-*forms* relative to some local frame $\{s_1, \dots, s_N\}$ for $E$ over $U_\alpha$ to be the unique $\mathfrak{gl}(N, \mathbb{K})$-valued 1-forms $\omega^\alpha_\nabla$ over $U_\alpha$ satisfying

$$\nabla s_j = \sum_{i=1}^N (\omega^\alpha_\nabla)_{ij} \otimes s_i.$$

The frame $(s_1, \dots, s_N)$ corresponds to a local section $s$ of the frame bundle $P_{\mathrm{GL}}(E)$, defined over $U$. If $\omega$ is the connection on the frame bundle corresponding to $\nabla$, then $\omega^\alpha_\nabla$ is just the pullback of $\omega$ along $s$.

Similarly we define the *local curvature* 2-*forms* to be the $\mathfrak{gl}(N, \mathbb{K})$-valued 2-form $F^\alpha_\nabla$ defined over $U_\alpha$ satisfying

$$Rs_j = \sum_{i=1}^N (F^\alpha_\nabla)_{ij} \otimes s_i.$$

The curvature 2-forms are related to the connection 1-forms by the formula

$$(F^\alpha_\nabla)_{ij} = d(\omega^\alpha_\nabla)_{ij} + \sum_{k=1}^{N} (\omega^\alpha_\nabla)_{ik} \wedge (\omega^\alpha_\nabla)_{kj}. \tag{1.23}$$

On the overlap $U_\alpha \cap U_\beta$ between two trivialization neighborhoods the curvature 2-forms are related by

$$F^\beta_\nabla = g^{-1}_{\alpha\beta} \circ F^\alpha_\nabla \circ g_{\alpha\beta}$$

where $g_{\alpha\beta}$ are the transitions functions of the vector bundle $E$. From this we see, in particular, that if the vector bundle $E$ has rank 1 (in which case we will often denote the connection on it by $\mathcal{A}$) that the local curvature 2-forms piece together to a *globally defined* 2-form $F_\mathcal{A}$ on $M$. If the connection is a metric connection, one can show that the curvature 2-form is purely imaginary valued.

Assume that $L$ is a Riemannian line bundle and $\mathcal{A}$ is a metric connection on it (usually, we are only interested in metric connections). Adding to $\mathcal{A}$ an $\alpha \in i\Omega^1(M)$ gives again a metric connection on $L$ (in fact this parametrizes all possible metric connections on $L$). If $\omega^\beta_\mathcal{A}$ is the local connection 1-form of $\mathcal{A}$ over some trivialization $U_\beta$ then $\omega^\beta_\mathcal{A} + \alpha$ is the local connection 1-form of $\mathcal{A} + \alpha$. By (1.23) we see that $F_\mathcal{A}|_{U_\beta} = d\omega^\beta_\mathcal{A}$ and hence that $F_{\mathcal{A}+\alpha}|_{U_\beta} = d\omega^\beta_\mathcal{A} + d\alpha|_{U_\beta} = (F_\mathcal{A} + d\alpha)|_{U_\beta}$. Therefore we get the following important formula to be used extensively in the calculations ahead:

$$F_{\mathcal{A}+\alpha} = F_\mathcal{A} + d\alpha. \tag{1.24}$$

Now define the operator

$$\nabla^2 : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

by $\nabla^2_{X,Y}(\varphi) = \nabla_X \nabla_Y \varphi - \nabla_{\nabla_X Y}\varphi$. Since the Levi-Civita connection is torsion-free: $\nabla_X Y - \nabla_Y X = [X, Y]$, we see

$$\begin{aligned}\nabla^2_{X,Y}\varphi - \nabla^2_{Y,X}\varphi &= \nabla_X \nabla_Y \varphi - \nabla_{\nabla_X Y}\varphi - \nabla_Y \nabla_X \varphi + \nabla_{\nabla_Y X}\varphi \\ &= \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{\nabla_X Y - \nabla_Y X}\varphi \\ &= R_{X,Y}\varphi.\end{aligned}$$

It is apparent from the definition and from the fact that $\nabla_X \varphi(x)$ only depends on $X$ at $x$, that $\nabla^2_{X,Y}\varphi(x)$ depends only on $X$ at $x$. The similar fact for $Y$ follows from the above formula. Hence for a given $\varphi \in \Gamma(E)$ and $x \in M$ we have bilinear map

$$\nabla^2_{\cdot,\cdot}\, \varphi : (X_x, Y_x) \longmapsto \nabla^2_{X_x,Y_x}\varphi.$$

Taking the trace of this bilinear form and changing sign give us a map $A : \Gamma(E) \longrightarrow \Gamma(E)$:

$$A\varphi(x) = -\operatorname{Tr}(\nabla^2_{\cdot,\cdot}\, \varphi).$$

In terms of a local orthonormal frame $\{E_1, \ldots, E_m\}$ over $U$ we get

$$A\varphi|_U = -\sum_{i=1}^{m} \nabla^2_{E_i,E_i}\varphi.$$

In the following a *proper tangent frame* around some point $x \in M$ will mean a local frame $\{E_1, \ldots, E_m\}$ for $TM$ which is an orthonormal basis for $T_xM$ and such that $\nabla_{E_i} E_j(x) = 0$ where $\nabla$ is the Levi-Civita connection. The existence of such a local frame is guaranteed by the Normal Coordinate Theorem of Riemannian geometry.

**Lemma 1.18.** *The map $A$ equals the connection Laplacian, i.e.*

$$\nabla^*\nabla\varphi = -\operatorname{Tr}(\nabla^2_{\cdot,\cdot}\varphi).$$

*Locally in terms of a proper tangent frame around $x$ we get*

$$\nabla^*\nabla\varphi(x) = -\sum_{i=1}^m \nabla^2_{E_i,E_i}\varphi(x) = -\sum_{i=1}^m \nabla_{E_i}\nabla_{E_i}\varphi(x).$$

Proof. Pick $x \in M$ arbitrarily and let $\{E_1,\dots,E_m\}$ be a proper tangent frame for $TM$ around $x$, then

$$\sum_{i=1}^m \nabla^2_{E_i,E_i}\varphi(x) = \sum_{i=1}^m \nabla_{E_i}\nabla_{E_i}\varphi(x) - \nabla_{\nabla_{E_i}E_i}\varphi(x) = \sum_{i=1}^m \nabla_{E_i}\nabla_{E_i}\varphi(x).$$

Hence if $\langle\,,\,\rangle$ denotes the fiber metric on $TM$ we get

$$\begin{aligned}\langle A\varphi(x),\psi(x)\rangle &= -\Big\langle\sum_{i=1}^m \nabla_{E_i}\nabla_{E_i}\varphi(x),\psi(x)\Big\rangle\\ &= -\sum_{i=1}^m\big(E_i|_x\langle\nabla_{E_i}\varphi,\psi\rangle - \langle\nabla_{E_i}\varphi(x),\nabla_{E_i}\psi(x)\rangle\big) \qquad (1.25)\end{aligned}$$

where $\varphi,\psi \in \Gamma_c(E)$ are arbitrary (the last equation is just the condition for $\nabla$ to be a metric connection).

Now, let $V$ be the unique vector field on $M$ such that $\langle V,W\rangle = \langle\nabla_W\varphi,\psi\rangle$ for all $W \in \mathfrak{X}(M)$ and recall the local expression for the divergence:

$$\operatorname{div} V = \sum_{i=1}^m\langle\nabla_{E_i}V,E_i\rangle = \sum_{i=1}^m E_i\langle V,E_i\rangle - \sum_{i=1}^m\langle V,\nabla_{E_i}E_i\rangle$$

(the first identity follows from [14], Problem 5-6, and the second identity is just the metric condition). Since $\nabla$ is metric and since $\nabla_{E_i}E_j(x) = 0$, the divergence at $x$ just equals $\sum_{i=1}^n E_i|_x\langle\nabla_{E_i}\varphi,\psi\rangle$ (by definition of $V$). Plugging this into (1.25) gives

$$\langle A\varphi(x),\psi(x)\rangle = -\operatorname{div} V(x) + \langle\nabla\varphi(x),\nabla\psi(x)\rangle$$

and by integrating we get $(A\varphi \mid \psi) = (\nabla\varphi \mid \nabla\psi) = (\nabla^*\nabla\varphi \mid \psi)$ [2] and hence $A = \nabla^*\nabla$. ☐

With this local formula for the connection Laplacian we may investigate its relations first to the Hodge Laplacian (Laplace-Beltrami operator) and later to the Dirac Laplacian.

**Proposition 1.19** (**Kato's Inequality**). *Let $M$ be a spin$^c$-manifold and let $S$ be the spinor bundle determined by a spin$^c$-structure. Let $\psi \in \Gamma(S)$ be a spinor field and $\nabla = \nabla^{\mathcal{A}}$ be the connection on $S$ determined by the Levi-Civita connection on $M$ and a connection $\mathcal{A}$ on the determinant line bundle. Then the following inequality holds*

$$\Delta_M|\psi|^2 \le 2\langle\nabla^*\nabla\psi,\psi\rangle \qquad (1.26)$$

*where $\Delta_M = dd^* + d^*d$ is the Hodge Laplacian for $M$.*

---

[2] We will adopt the convention that $\langle\,,\rangle$ denotes the fiberwise inner product in a Riemannian vector bundle $E$, and that $(s_1 \mid s_2) := \int_M\langle s_1,s_2\rangle_x$ denotes the inner product in $\Gamma(E)$ induced from the metric.

Proof. Again, let us pick a proper local frame $\{E_1, \dots, E_k\}$ around a point $x$. By (1.25) we get (by manipulations involving only the metric condition on $\nabla$):

$$\begin{aligned}\langle \nabla^*\nabla\psi, \psi\rangle_x &= -\sum_{i=1}^{k}\Big(E_i|_x\langle\nabla_{E_i}\psi,\psi\rangle - \langle\nabla_{E_i}\psi,\nabla_{E_i}\psi\rangle_x\Big)\\ &= -\sum_{i=1}^{k}\Big(E_i|_x(E_i|_x\langle\psi,\psi\rangle) - E_i|_x\langle\psi,\nabla_{E_i}\psi\rangle - |\nabla_{E_i}\psi(x)|^2\Big)\\ &= -\sum_{i=1}^{k}\Big(E_i|_x(E_i\langle\psi,\psi\rangle) - \langle\psi,\nabla_{E_i}\nabla_{E_i}\psi\rangle - 2|\nabla_{E_i}\psi(x)|^2\Big)\\ &= -\sum_{i=1}^{k}\Big(E_i|_x(E_i\langle\psi,\psi\rangle) - 2|\nabla_{E_i}\psi(x)|^2\Big) - \langle\psi,\nabla^*\nabla\psi\rangle_x\end{aligned}$$

At $x$ we can assume (by the Normal Coordinate Theorem) that $E_i|_x = \partial_i|_x$ and therefore $\Delta_M|\psi|^2(x) = -\sum_{k=1} E_i|_x(E_i\langle\psi,\psi\rangle)$ by which we conclude

$$\Delta_M|\psi|^2(x) = 2\langle\nabla^*\nabla\psi,\psi\rangle_x - 2\sum_{i=1}^{k}|\nabla_{E_i}\psi(x)|^2 \leq 2\langle\nabla^*\nabla\psi,\psi\rangle_x. \qquad \square$$

To investigate the relation of $\nabla^*\nabla$ to the Dirac Laplacian, let $S$ be a Dirac bundle over $TM$ ($TM$ equipped with the Levi-Civita connection), $\not{\partial}$ the Dirac operator and $\widetilde{\nabla}$ the connection on $S$. We define $\mathfrak{R} : \Gamma(S) \longrightarrow \Gamma(S)$ by

$$(\mathfrak{R}\varphi)(x) = \frac{1}{2}\sum_{i,j=1} E_i(x)\cdot E_j(x)\cdot(\widetilde{R}_{E_i,E_j}\varphi)(x) \tag{1.27}$$

where $\{E_1, \dots, E_n\}$ is a local proper tangent frame around $x$, and $\widetilde{R}$ is the curvature transformation of $\widetilde{\nabla}$.

**Theorem 1.20 (The Bochner-Weitzenböck Identity).** *Let $\not{\partial}$ be the Dirac operator of the Dirac bundle $S$ with connection $\widetilde{\nabla}$. Then*

$$\not{\partial}^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \mathfrak{R}. \tag{1.28}$$

Proof. Let $x \in M$ be given and pick a proper tangent frame $\{E_1, \dots, E_n\}$ around $x$. Using the local expression for $\not{\partial}$ (Lemma 1.13) we get at $x$

$$\begin{aligned}\not{\partial}^2\psi &= \not{\partial}\Big(\sum_{j=1}^{n}E_j\cdot\widetilde{\nabla}_{E_j}\psi\Big) = \sum_{i=1}^{n}E_i\cdot\widetilde{\nabla}_{E_i}\Big(\sum_{j=1}^{n}E_j\cdot\widetilde{\nabla}_{E_j}\psi\Big)\\ &= \sum_{i,j=1}^{n}E_i\cdot\widetilde{\nabla}_{E_i}(E_j\cdot\widetilde{\nabla}_{E_j}\psi) = \sum_{i,j=1}^{n}E_i\cdot E_j\cdot\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_j}\psi\\ &= \sum_{i,j=1}^{n}E_i\cdot E_j\cdot\widetilde{\nabla}^2_{E_i,E_j}\psi\\ &= -\sum_{i=1}^{n}\widetilde{\nabla}^2_{E_i,E_i}\psi + \sum_{i<j}E_i\cdot E_j\cdot(\widetilde{\nabla}^2_{E_i,E_j} - \widetilde{\nabla}^2_{E_j,E_i})\psi\\ &= \widetilde{\nabla}^*\widetilde{\nabla}\psi + \frac{1}{2}\sum_{i,j=1}^{n}E_i\cdot E_j\cdot\widetilde{R}_{E_i,E_j}(\psi) = \widetilde{\nabla}^*\widetilde{\nabla}\psi + \mathfrak{R}\psi.\end{aligned}$$

Some of these manipulations require an explanation: Identity 3 is a consequence of (1.8), since $\widetilde{\nabla}_{E_i}E_j(x) = 0$, identity 6 follows since $E_i \cdot E_i = -1$ and finally 7 follows since $\widetilde{R}_{E_i,E_i}\psi = 0$. $\square$

Given this result we immediately ask ourselves how this result manifests itself in the case of the spin-Dirac operator and the geometric Dirac operator. Here are the answers:

**Corollary 1.21** (**The Bochner-Lichnerowicz Identity I**). *Let $M$ be a spin manifold and $S(M)$ the spinor bundle associated to a spin structure on $M$. The spin-Dirac operator $\not{D}$ and the spin connection $\widetilde{\nabla}$ are related by*

$$\not{D}^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \frac{1}{4}\kappa \tag{1.29}$$

*where $\kappa$ is the* scalar curvature *of $M$.*

Proof. Similarly to (1.12) one can prove the following local formula for the curvature corresponding to the spin connection

$$(\widetilde{R}_{X,Y}(\psi))_\alpha(x) = -\frac{1}{2}\sum_{i<j}(F_\nabla^\alpha)^i_j(X_x, Y_x)e_ie_j \cdot \psi_\alpha(x)$$

where $F_\nabla^\alpha$ is the curvature 2-form over $U_\alpha$ of the Levi-Civita connection. If $\{E_1, \ldots, E_n\}$ is the local frame for $TM$ over $U_\alpha$ corresponding to the trivialization we get, since $R$ is a tensor, that

$$\widetilde{R}_{X,Y}\psi = -\frac{1}{2}\sum_{i<j}(F_\nabla^\alpha)^i_j(X,Y)E_iE_j \cdot \psi.$$

With this formula at hand we can calculate $\mathfrak{R}$

$$\begin{aligned}\mathfrak{R}\psi &= \frac{1}{2}\sum_{i,j=1} E_iE_j \cdot \widetilde{R}_{E_i,E_j}\psi \\ &= \frac{1}{2}\sum_{i,j=1} E_iE_j \cdot \Big(-\frac{1}{4}\sum_{k,l=1}(F_\nabla^\alpha)^k_l(E_i,E_j)E_kE_l \cdot \psi\Big) \\ &= -\frac{1}{8}\sum_{i,j,k,l}(F_\nabla^\alpha)^k_l(E_i,E_j)E_iE_jE_kE_l \cdot \psi.\end{aligned}$$

Recall that $(F_\nabla^\alpha)^k_l = -(F_\nabla^\alpha)^l_k$ and that it is given by the Riemann tensor $R$ of the Levi-Civita connection by

$$R_{ijkl} := \langle R_{E_i,E_j}E_k, E_l\rangle = (F_\nabla^\alpha)^l_k(E_i,E_j) = -(F_\nabla^\alpha)^k_l(E_i,E_j),$$

thus

$$\mathfrak{R}\psi = \Big(\sum_{i,j,k,l} R_{ijkl}E_iE_jE_kE_l\Big) \cdot \psi.$$

Thanks to the many symmetries (and anti-symmetries) of the Riemann tensor $R_{ijkl}$ as well as properties of the Clifford algebra (for instance that $E_kE_k = -1$) one can show (but I will not go into the details) that the term in the parenthesis above reduces to

$$\frac{1}{4}\sum_{\substack{i,j,l\\ i\neq j\\ i\neq l}} R_{ijil}E_jE_l.$$

Flipping $j$ and $l$ would result in a change of sign, so all terms with $j \neq l$ will cancel, so we are left with the terms where $j = l$, i.e.

$$\mathfrak{R}\psi = \frac{1}{4}\sum_{i\neq j} R_{ijij}E_jE_j = -\frac{1}{4}\sum_{i,j} R_{ijij} = \frac{1}{4}\kappa$$

where the second identity follows by $R_{iiii} = 0$. □

For the geometric Dirac operator we prove a similar formula

**Corollary 1.22 (The Bochner-Lichnerowicz Identity II).** *Let $M$ be a spin$^c$-manifold and $S^c(M)$ the spinor bundle associated to a spin$^c$-structure on $M$. The geometric Dirac operator $\not{D}_{\mathcal{A}}$ and the spin connection $\widetilde{\nabla}$, given by some connection $\mathcal{A}$ on the determinant line bundle, are related as follows*

$$\not{D}_{\mathcal{A}}^2\psi = \widetilde{\nabla}^*\widetilde{\nabla}\psi + \frac{1}{4}\kappa\psi + \frac{1}{2}F_{\mathcal{A}} \cdot \psi. \tag{1.30}$$

Proof. Again we just have to calculate $\mathfrak{R}$ where $\widetilde{\mathbb{R}}$ is the curvature of the connection induced by $\mathcal{A}$. From (1.16) one can calculate

$$\widetilde{R}_{X,Y}\psi = -\frac{1}{2}\sum_{i<j}(F_{\nabla}^{\alpha})^i_j(X,Y)E_iE_j \cdot \psi + \sum_{i<j} F_{\mathcal{A}}(X,Y)\psi$$

where again $F_{\nabla}^{\alpha}$ is the local curvature 2-form of the Levi-Civita connection and $F_{\mathcal{A}}$ is the *global* curvature 2-form of the connection $\mathcal{A}$ on the determinant line bundle. We plug this into the formula defining $\mathfrak{R}$. The first term was calculated in the proof the the first Bochner-Lichnerowicz identity, that was just $\frac{1}{4}\kappa$. The second term is

$$\frac{1}{2}\sum_{i<j} E_iE_j \cdot (F_{\mathcal{A}}(E_i, E_j)\psi)$$

which we recognize as a local expression for $\frac{1}{2}F_{\mathcal{A}} \cdot \psi$. $\square$

## 1.4 Fredholm Operators

In this section we present some basic stuff on Fredholm operators. The first couple of results are well-known and thus the proofs are omitted (for proofs in the Hilbert space case the reader is referred to the excellent presentation in [4]). After that follow some less renowned results the proofs of which are presented in great detail. These results will be utilized in the study of the local structure of the moduli space.

**Definition 1.23.** A linear operator $A : V_1 \longrightarrow V_2$ between vector spaces is called a *Fredholm operator* if $\ker A$ and $\operatorname{coker} A := V/\operatorname{im} A$ are finite-dimensional. The *Fredholm index* of a Fredholm operator is then defined as

$$\operatorname{ind}(A) := \dim\ker A - \dim \operatorname{coker} A.$$

In most of what follows, it will be sufficient for us to work with Fredholm operators on Hilbert spaces, except for Section 2.6.

Here some key features of Fredholm operators and their indices are presented. First, the so-called Atkinson Theorem which gives some equivalent conditions for a *bounded* operator to be Fredholm:

**Theorem 1.24 (Atkinson).** *Let $A : X \longrightarrow Y$ be a bounded operator between Banach spaces. Then the following conditions are equivalent*:

1) *$A$ is Fredholm.*
2) $\operatorname{im} A$ *is closed and* $\ker A$ *and* $\operatorname{coker} A$ *are finite-dimensional.*
3) *There exists a bounded operator $B : Y \longrightarrow X$ (called a* parametrix*) such that $I - AB$ and $I - BA$ are compact operators.*

The set of bounded Fredholm operators $X \longrightarrow Y$ is denoted $\mathscr{F}(X,Y)$. This is an open set inside $\mathscr{B}(X,Y)$, when the latter is given the norm-topology.

The Fredholm index turns out to be an impressively robust and well-behaved quantity.

**Proposition 1.25.** *Let $A \in \mathscr{F}(X,Y)$. Then*

1) *If $B \in \mathscr{F}(X,Y)$ is a Fredholm operator, then $BA$ is a Fredholm operator and* $\operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B)$.

2) *If $K : X \longrightarrow Y$ is a compact operator, then $A+K$ is a bounded Fredholm operator and* $\operatorname{ind}(A+K) = \operatorname{ind}(A)$.

3) *The Fredholm index is a continuous map* $\operatorname{ind} : \mathscr{F}(X,Y) \longrightarrow \mathbb{Z}$*. In particular it is constant on path components of $\mathscr{F}(X,Y)$.*

4) *If $X$ and $Y$ are Hilbert spaces, the adjoint $A^*$ is again a Fredholm operator and* $\operatorname{ind} A^* = -\operatorname{ind} A$.

The next two lemmas will be used solely in Section 2.4.

**Lemma 1.26.** *Assume $A : X \longrightarrow Y$ is a surjective Fredholm operator between Banach spaces, then there exists an $\varepsilon > 0$ such that for any $S \in \mathscr{B}(X,Y)$ with $\|S\| < \varepsilon$ the operator $A+S$ is surjective.*

Proof. Since $\ker A$ is finite-dimensional, it has a topological complement [3] which we call $Z$. The restriction of $A$ to $Z$ must be an isomorphism $Z \xrightarrow{\sim} Y$. The set of isomorphisms is open in $\mathscr{B}(Z,Y)$, hence there exists an $\varepsilon$ such that if the operator norm of $S' : Z \longrightarrow Y$ is less than $\varepsilon$, then $A|_Z + S'$ is still an isomorphism. For an operator $S : X \longrightarrow Y$ it holds that $\|S|_Z\| \leq \|S\|$ and hence if $\|S\| < \varepsilon$ we must have that $(A+S)|_Z$ is an isomorphism $Z \xrightarrow{\sim} Y$. In particular $A+S$ is surjective. ☐

**Proposition 1.27.** *Let $A : X \longrightarrow Z$ be Fredholm and $T : Y \longrightarrow Z$ be bounded ($X$, $Y$ and $Z$ are Banach spaces) and denote by $A \dotplus T$ the operator $X \oplus Y \longrightarrow Z$ given by*

$$(A \dotplus T)(x,y) = Ax + Ty.$$

*Assume that $T$ is chosen such that $A \dotplus T$ is surjective, then* $\ker(A \dotplus T)$ *has a topological complement in $X \oplus Y$, and for the projection* $\pi : \ker(A \dotplus T) \longrightarrow Y$ *we have canonical identifications*

$$\ker \pi \cong \ker A \qquad \textit{and} \qquad \operatorname{coker} \pi \cong \operatorname{coker} A,$$

*thus $\pi$ is Fredholm and* $\operatorname{ind} \pi = \operatorname{ind} A$.

Proof. The strategy is to find a continuous projection onto $\ker(A \dotplus T)$, for then we know that a topological complement exists. One way of finding continuous projections is to find split-exact sequences, so this is what we seek.

Since $\ker A$ and $\operatorname{im} A$ are of finite dimension resp. finite codimension, they have topological complements, i.e. there exist $X'$ and $Z'$ such that

$$\ker A \oplus X' = X \qquad \text{and} \qquad \operatorname{im} A \oplus Z' = Z.$$

---

[3] Recall that for a Banach space $X$, a *complement* or *algebraic complement* of a subspace $V$ is a subspace $W$ such that $V \cap W = \{0\}$ and such that $V + W = X$. A complement is called a *topological complement* if the map $V \times W \longrightarrow X$ given by $(v,w) \longmapsto v + w$ is a homeomorphism. For a generic subspace $V$ the existence of a topological complement is not á priori guaranteed. However, if $V$ is of finite dimension or finite codimension, one can show that a topological complement always exists.

Pick a basis $\{e_1, \dots, e_n\}$ for $Z'$. Since $A \dotplus T$ was assumed surjective, we must necessarily have $Z' \subseteq \operatorname{im} T$, i.e. there exists a linearly independent set $\{y_1, \dots, y_n\} \in Y$ such that $e_i = Ty_i$. Define a map $R : Z \longrightarrow X \oplus Y$ in the following way: let $z \in Z$ and decompose it $z = z_1 + z_2$ according to the splitting $Z = \operatorname{im} A \oplus Z'$. Let $x$ be the unique element in $X'$ such that $z_1 = Ax$ and let $a_1, \dots, a_n$ be the unique real numbers such that $z_2 = \sum a_i T y_i$, then the following is well-defined:

$$Rz := \Big(x, \sum_{i=1}^{n} a_i y_i\Big).$$

$R$ is a right inverse for $A \dotplus T$ for if $z$ is as above then

$$(A \dotplus T)Rz = (A \dotplus T)(x, \sum a_i y_i) = Ax + \sum a_i T y_i = z.$$

Note that this implies that the following sequence is split-exact:

$$\begin{tikzcd} 0 \arrow[r] & \ker(A \dotplus T) \arrow[r, hook] & X \oplus Y \arrow[r, shift left, "A \dotplus T"] & Z \arrow[l, shift left, "R"] \arrow[r] & 0 \end{tikzcd}$$

and cf. [16], there exists a continuous projection $P$ in $X \oplus Y$ whose image equals $\ker(A \dotplus T)$. Then $\operatorname{id} - P$ is again a projection, and its image is the desired complementary subspace.

Note that $\ker(A \dotplus T) = \{(x, y) \,|\, Ax = -Ty\}$, hence $\ker \pi = \{(x, 0) \,|\, Ax = 0\} \cong \ker A$, thus $\ker \pi$ is finite-dimensional. To identify the cokernel of $\pi$, first note that $\operatorname{im} A \cap \operatorname{im} T = T(\operatorname{im} \pi)$, this is for the following reason: if $z \in \operatorname{im} A \cap \operatorname{im} T$ then $z = Ax = Ty$, i.e. $(-x, y) \in \ker(A \dotplus T)$ and $z = Ty = T(\pi(-x, y))$. And vice versa. Thus we calculate (using a Noether Isomorphism Theorem, best known from group theory)

$$\begin{aligned} \operatorname{coker} A &= \frac{Z}{\operatorname{im} A} = \frac{\operatorname{im} A + \operatorname{im} T}{\operatorname{im} A} \cong \frac{\operatorname{im} T}{\operatorname{im} A \cap \operatorname{im} T} \\ &= \frac{\operatorname{im} T}{T(\operatorname{im} \pi)} \cong \frac{Z / \ker T}{\operatorname{im} \pi / \ker T} \cong \frac{Z}{\operatorname{im} \pi} = \operatorname{coker} \pi, \end{aligned}$$

hence also $\operatorname{coker} \pi$ is finite-dimensional, and thus $\pi$ is a Fredholm operator. □

**Proposition 1.28.** *Let $A : X \longrightarrow Z$ be Fredholm and let $T : Y \longrightarrow Z$ be a bounded operator such that $A \dotplus T$ is surjective and Fredholm. Then for a $P : X \longrightarrow Z$ with sufficiently small operator norm the operator $(A + P) \dotplus T$ is surjective and* $\ker((A + P) \dotplus T)$ *admits a complement.*

Proof. If $\widetilde{P} : X \oplus Y \longrightarrow Z$ denotes the extension by 0 of $P$ then $\|\widetilde{P}\| = \|P\|$ and $(A + P) \dotplus T = (A \dotplus T) + \widetilde{P}$ and the first claim now follows from Lemma 1.26.

For sufficiently small $P$, $A + P$ is again Fredholm, and since $(A + P) \dotplus T$ is surjective $\ker((A + P) \dotplus T)$ has a complement by Proposition 1.27. □

**Proposition 1.29.** *If $A : X \longrightarrow Z$ is Fredholm and $T : \mathbb{R}^n \longrightarrow Z$ is chosen such that $A \dotplus T$ is surjective, then $A \dotplus T$ is Fredholm and*

$$\operatorname{ind}(A \dotplus T) = \operatorname{ind} A + n.$$

Proof. $A \dotplus T$ is Fredholm since addition by $T$ can only change the kernel by something finite-dimensional.

As $A \dotplus T$ is surjective we have $\operatorname{coker}(A \dotplus T) = \{0\}$. Furthermore

$$\operatorname{coker} A = Z / \operatorname{im} A = (\operatorname{im} A + \operatorname{im} T) / \operatorname{im} A = \operatorname{im} T / (\operatorname{im} A \cap \operatorname{im} T)$$

in the last equation we invoked the Noether isomorphism theorem. Thus

$$\begin{aligned}\dim\operatorname{coker} A &= \dim\operatorname{im} T - \dim(\operatorname{im} A \cap \operatorname{im} T)\\ &= n - \dim\ker T - \dim(\operatorname{im} A \cap \operatorname{im} T).\end{aligned}$$

Next we recall that $\ker(A \dotplus T) = \{(x,y) \,|\, Ax = -Ty\}$, and we define $\widehat{A} : \ker(A \dotplus T) \longrightarrow \operatorname{im} A \cap \operatorname{im} T$ by $(x,y) \longmapsto Ax$. This is surjective onto $\operatorname{im} A \cap \operatorname{im} T$ ($\widehat{A}$ clearly maps into this space), for if $v \in \operatorname{im} A \cap \operatorname{im} T$ then there exists $(x,y)$ such that $v = Ax = Ty$, meaning that $v = \widehat{A}(x,-y)$. Observe that $\ker \widehat{A} = \ker A \times \ker T$, hence we get the isomorphism

$$\ker(A \dotplus T) = (\ker A \times \ker T) \oplus (\operatorname{im} A \cap \operatorname{im} T)$$

and therefore

$$\begin{aligned}\operatorname{ind}(A \dotplus T) &= \dim\ker A + \dim\ker T + \dim(\operatorname{im} A \cap \operatorname{im} T)\\ &= \dim\ker A + n - \dim\operatorname{coker} A = \operatorname{ind} A + n.\end{aligned}$$ ☐

## 1.5 Continuous Fields of Fredholm Operators

In the next chapter when we will study the smooth local structure of the moduli space we will make extensively use of the properties of Fredholm operators. However, when we have to orient the moduli space in Section 2.5 it will not be sufficient for us to consider just one operator at a time. In fact, we will realize that the moduli space will parametrize a family of Fredholm operators and that the tangent space at each point is isomorphic to the kernel of the Fredholm operator an that point. In this section we will introduce the theory necessary to cope with this situation. It will not be presented in complete generality but will rather be adapted to our specific needs.

First we let $(M,g)$ denote a closed, oriented Riemannian manifold and let $E^i \longrightarrow M$, $i = 0,1$ two *real* finite-rank vector bundles over $M$, and let $D_0 : \Gamma(E^0) \longrightarrow \Gamma(E^1)$ be a first order elliptic differential operator. Let $X$ (the parameter space) denote a (possibly infinite-dimensional) manifold. Via the projection map $\pi_2 : X \times M \longrightarrow M$ we obtain pullback bundles: $E^i_X := \pi_2^* E^i$ over the space $X \times M$. Let $T$ be a section of the bundle $\operatorname{Hom}(E^0_X, E^1_X)$, thus for every point $(x,p) \in X \times M$ we have a linear map

$$T_{(x,p)} : E^0_p = E^0_X|_{(x,p)} \longrightarrow E^1_X|_{(x,p)} = E^1_p,$$

hence the section $T$ corresponds to a smooth family $(T_x)_{x \in X}$ of bundle maps $T_x : E^0 \longrightarrow E^1$. For each $x \in X$ we define $D_x : \Gamma(E^0) \longrightarrow \Gamma(E^1)$ by $D_x := D_0 + T_x$. Since $T_x$ is just a zeroth order perturbation, $D_x$ will still be an elliptic differential operator, and since the index depends only on highest order part we have

$$\operatorname{ind} D_x = \operatorname{ind} D_0. \tag{1.31}$$

Defining $H^i := L^2(E^i)$, we may view $D_x$ either as an unbounded Fredholm operator $D_x : H_0 \longrightarrow H_1$ with domain $L^2_1(E^0)$, or as a bounded Fredholm operator $D_x : L^2_k(E^0) \longrightarrow L^2_{k-1}(E^1)$.

**Definition 1.30** (**Field of Fredholm Operators**). By a *continuous field of Fredholm operators* $D_\bullet$ we understand a map $X \ni x \longmapsto D_x$ of the form described above [4].

---

[4] Even though it is a *smooth* field we still call it a *continuous* field, simply because continuity is all we need for the proofs ahead.

This definition is quite restrictive but it is sufficient for our purpose (for the remainder of this thesis we will not encounter differential operators of order higher than 1).

**Definition 1.31** (**Determinant**)**.** For a finite-dimensional vector space $V$ we define the *determinant* $\det V$ of $V$ to be the top exterior power of $V$

$$\det V := \Lambda^{\mathrm{top}} V := \Lambda^{\dim V} V.$$

For a Fredholm operator $F : H_0 \longrightarrow H_1$ (bounded or unbounded) between Hilbert spaces we define the *determinant* of $F$ to be the 1-dimensional vector space

$$\det F := (\det \ker F) \otimes (\det \ker F^*)^*$$

where $(\det \ker F^*)^*$ is the dual space of the kernel of $F^*$.

Given a continuous field $D_\bullet$ of Fredholm operators we would like to define a line bundle $\det D_\bullet$ over $X$ whose fiber over $x$ is isomorphic to $\det D_x$. Constructing this bundle requires some work which will keep us occupied for most of this section.

**Definition 1.32** (**Stabilizer**)**.** For every closed subspace $V \subseteq H_1$ and for every $x \in X$ we define the operator $D_{V,x} : H_0 \oplus V \longrightarrow H_1$ with domain $L^2_1(E^0) \oplus V$ by

$$D_{V,x}(\varphi, v) := D_x \varphi + v.$$

In the notation from the previous section we have $D_{V,x} = D_x \dotplus \mathrm{id}_V$.

A *stabilizer* for $D_\bullet$ is a finite-dimensional subspace $V \subseteq H_1$ such that $D_{V,x}$ is surjective for any $x \in X$. The map $x \longmapsto D_{V,x}$ defines a field of Fredholm operators, denoted $D_{V,\bullet}$. The set of stabilizers for $D_\bullet$ is denoted $\mathrm{Stab}(D_\bullet)$.

Let $V \in \mathrm{Stab}(D_\bullet)$, then for any $x \in X$ we have a short-exact sequence (of unbounded operators!):

$$\begin{tikzcd} 0 \arrow[r] & \ker D_{V,x} \arrow[r, hook] & H_0 \oplus V \arrow[r, "D_{V,x}"] & H_1 \arrow[r] & 0 \end{tikzcd}$$

and the sequence splits by the map $R_{V,x} : H_1 \longrightarrow (\ker D_{V,x})^\perp \subseteq H_0 \oplus V$ which is just the inverse of the bijection $D_{V,x}|_{(\ker D_{V,x})^\perp}$.

**Proposition 1.33.** *Let $D_\bullet$ be a field of Fredholm operators over $X$ where $X$ is compact. Then the following elementary properties hold*:

1) $\mathrm{Stab}(D_\bullet)$ *is non-empty.*

2) *If $V \in \mathrm{Stab}(D_\bullet)$ and if $V \subseteq W$, then $W \in \mathrm{Stab}(D_\bullet)$.*

3) $\ker D_{V,\bullet}$ *is a smooth trivial vector bundle for every $V \in \mathrm{Stab}(D_\bullet)$.*

Proof. First the existence of a stabilizer: Let $x_0 \in X$ be any point and pick a finite-dimensional subspace $V$ such that $D_{V,x_0}$ is surjective (we can take any subspace which contains the cokernel of $D_{x_0}$). For any bounded operator $T : L^2_1(E^0) \longrightarrow H_1$ with sufficiently small operator norm we know by Lemma 1.28 that $(D_{x_0} + T) \dotplus \mathrm{id}_V$ is still surjective and Fredholm. Since the map $x \longmapsto \|T_x\|$ is continuous, there exists an open set $U$ around $x_0$ such that $T_x - T_{x_0}$ (for $x \in U$) is small enough for $(D_{x_0} + T_x - T_{x_0}) \dotplus \mathrm{id}_V$ to be surjective. Then

$$\begin{aligned} D_{V,x} &= D_x \dotplus \mathrm{id}_V = (D_0 + T_{x_0} + (T_x - T_{x_0})) \dotplus \mathrm{id}_V \\ &= (D_{x_0} + T_x - T_{x_0}) \dotplus \mathrm{id}_V \end{aligned}$$

is surjective for any $x \in U$. Since $X$ is compact we can cover $X$ by a finite set of open sets $X = U_{x_1} \cup \cdots \cup U_{x_n}$ and there exist finite-dimensional subspaces $V_1, \ldots, V_n$ such that $D_{V_k,x}$ is surjective for each $x \in U_{x_k}$. Then $V := V_1 + \cdots + V_n$ is in $\operatorname{Stab}(D_\bullet)$.

The second claim of the proposition is obviously true. For the third claim we consider the following

$$D_{V,x} = D_x \dotplus \mathrm{id}_V = (D_0 \dotplus \mathrm{id}_V) + \widetilde{T}_x$$

where $V \in \operatorname{Stab}(D_\bullet)$ and $\widetilde{T}_x : L_1^2(E^0) \oplus V \longrightarrow L_1^2(E^1)$ is just $\widetilde{T}_x(\varphi, v) = T_x\varphi$. Clearly $D_0 \dotplus \mathrm{id}_V$ is Fredholm and $\widetilde{T}_x$ is just a compact perturbation, thus $D_{V,x}$ is a surjective Fredholm operator for each $x$, implying that $\dim\ker D_{V,x}$ is constant. To construct the global trivialization, pick $x_0$ and let $A$ be the right inverse for the surjective map $D_{V,x_0}$, i.e. $D_{V,x_0}A = \mathrm{id}$. Define $\Phi_V : \ker D_{V,\bullet} \longrightarrow X \times \ker D_{V,x_0}$ by

$$\Phi(v_x) := (x, v_x + A(\widetilde{T}_x - \widetilde{T}_{x_0})v_x)$$

where $v_x \in \ker D_{V,x}$. It is easy to check that $v_x + A(\widetilde{T}_x - \widetilde{T}_{x_0})v_x$ is in fact in $\ker D_{V,x_0}$ and that the map $v_x \longmapsto v_x + A(\widetilde{T}_x - \widetilde{T}_{x_0})v_x$ is bijective, thus $\Phi_V$ is a trivialization. □

Assume we have $V, W \in \operatorname{Stab}(D_\bullet)$ and that $V \subseteq W$, then for every $x \in X$ we obviously have $\ker D_{V,x} \subseteq \ker D_{W,x} \subseteq H_0 \oplus W$. Letting $V^\perp$ be the orthogonal complement of $V$ in $W$:

$$V^\perp = \{w \in W \mid \forall v \in V : \langle v, w\rangle = 0\}$$

we get a short-exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker D_{V,x} \arrow[r, hook] & \ker D_{W,x} \arrow[r] & V^\perp \arrow[r] & 0 \end{tikzcd} \tag{1.32}$$

where the second map is just the composition $H_0 \oplus W \longrightarrow W \longrightarrow V^\perp$ of orthogonal projections. The sequence splits by the map $s : V^\perp \longrightarrow \ker D_{W,x}$ given by

$$w \longmapsto -R_{W,x}(w) + (0, w)$$

for $w \in V^\perp \subseteq W$. This gives a vector space isomorphism

$$\ker D_{W,x} \xrightarrow{\sim} \ker D_{V,x} \oplus V^\perp$$

and letting $x$ vary gives us a bundle isomorphism

$$\ker D_{W,\bullet} \xrightarrow{\sim} \ker D_{V,\bullet} \oplus V^\perp$$

where $V^\perp$ is shorthand notation for the trivial bundle $X \times V^\perp$. Since the determinant sends direct sums to tensor products and since tensoring with the trivial line bundle $\det V^\perp$ gives nothing new, we obtain a bundle isomorphism

$$\det\ker D_{V,\bullet} \xrightarrow{\sim} \det\ker D_{W,\bullet}$$

which shows that the following is well-defined:

**Definition 1.34** (**Determinant Line Bundle**). Let $D_\bullet$ be a continuous field of Fredholm operators over a compact or connected manifold $X$. We define the *determinant line bundle* $\det D_\bullet$ of $D_\bullet$ by

$$\det D_\bullet := \det\ker D_{V,\bullet}$$

where $V \in \operatorname{Stab}(D_\bullet)$ is arbitrary.

We only need to see that this is a reasonable definition, in the sense that the fiber over $x$ is in fact the determinant of $D_x$, i.e. that we have a natural isomorphism

$$\Phi_{V,x} : \Lambda^{\mathrm{top}}(\ker D_x) \otimes \Lambda^{\mathrm{top}}(\ker D_x^*)^* \xrightarrow{\sim} \Lambda^{\mathrm{top}} \ker(D_x \dot{+} \mathrm{id}_V).$$

We will define the isomorphism in the following way: let $z \in \det D_x$ be arbitrary (but non-zero). It is of the form

$$z = (v_1 \wedge \cdots \wedge v_n) \otimes (\xi_1 \wedge \cdots \wedge \xi_m)$$

where $\{v_1, \ldots, v_n\}$ is a basis for $\ker D_x$ and $\{\xi_1, \ldots, \xi_m\}$ is a basis for the dual $(\ker D_x^*)^*$. We must necessarily have $\ker D^* \subseteq V$ and therefore, if $\{w_1, \ldots, w_m\}$ denotes the dual system of $\{\xi_1, \ldots, \xi_m\}$, i.e $\xi_i(w_j) = \delta_{ij}$, we can extend it to an oriented basis $\{w_1, \ldots, w_m, \ldots, w_N\}$ for $V$. The vectors $w_{m+1}, \ldots, w_N$ will lie in $\operatorname{im} D_x$ and therefore we can find linearly independent vectors $y_{m+1}, \ldots, y_N \in H_0$ such that $D_x y_k = -w_k$ (for $k \geq m+1$). Thus we see that $(v_1, 0), \ldots, (v_n, 0)$ as well as $(y_{m+1}, w_{m+1}), \ldots, (y_N, w_N)$ are in $\ker D_{V,x}$. For dimension reason, this is a basis for $\ker D_{V,x}$. Thus we define

$$\Phi_{V,x}(z) := (v_1, 0) \wedge \cdots \wedge (v_n, 0) \wedge (y_{m+1}, w_{m+1}) \wedge \cdots \wedge (y_N, w_N).$$

Clearly this is an element of $\Lambda^{\mathrm{top}} \ker(D_x \dot{+} \mathrm{id}_V)$. One can check that the map is independent of the choice of bases and thus that it gives a well-defined natural bundle map

$$\Lambda^{\mathrm{top}} \ker D_\bullet \otimes \Lambda^{\mathrm{top}}(\ker D_\bullet^*)^* \xrightarrow{\sim} \Lambda^{\mathrm{top}} \ker(D_{V,\bullet}).$$

Assume now that $T^0, T^1 \in \mathrm{Hom}(E_X^0, E_X^1)$ are two families of bundle maps $E^0 \longrightarrow E^1$ parametrized by $X$. We say $T^0$ and $T^1$ are *homotopic* if there exists $\widetilde{T} \in \mathrm{Hom}(E^0_{[0,1]\times X}, E^1_{[0,1]\times X})$ such that $T|_{\{i\}\times X} = T^i$.

**Proposition 1.35.** *The fields of Fredholm operators corresponding to the homotopic families $T^0$ and $T^1$ have isomorphic determinant line bundles.*

Proof. Let $\widetilde{D}_\bullet$ be the field of Fredholm operators corresponding to $\widetilde{T}$ and let $V \in \mathrm{Stab}(\widetilde{D}_\bullet)$. Then it is easy to see that $V \in \mathrm{Stab}(D^0_\bullet) \cap \mathrm{Stab}(D^1_\bullet)$, i.e. this intersection is non-empty. But then we have (by definition)

$$\det D^i_\bullet = \det \ker(D^i_{V,\bullet})$$

so in order to show $\det D^0_\bullet \cong \det D^1_\bullet$, it suffices to show that we have an isomorphism $\det \ker D^0_{V,\bullet} \xrightarrow{\sim} \det \ker D^1_{V,\bullet}$. But this follows since the two vector bundles are homotopic: we consider the bundle $\ker \widetilde{D}_{V,\bullet}$ over $[0,1] \times X$, pick a connection on this and let $P_x : \ker \widetilde{D}_{V,(0,x)} \xrightarrow{\sim} \ker \widetilde{D}_{V,(1,x)}$ denote parallel translation along the curve $[0,1] \ni t \longmapsto (t,x) \in [0,1] \times X$. The family $(P_x)_{x\in X}$ gives a smooth bundle map $\ker D^0_{V,\bullet} \xrightarrow{\sim} \ker D^1_{V,\bullet}$ which is fiberwise an isomorphism, hence a bundle isomorphism. It then induces an isomorphism between the determinant line bundles. □

We end this section by stating the *Smale-Sard Theorem*, a generalization to infinite-dimensional manifolds of the well-known *Sard Theorem* (which states that the set of critical values of a smooth map between finite-dimensional manifolds has measure 0). Even though this result is only remotely connected to what we have just discussed in this section, I find that this is a proper place to put it.

**Definition 1.36 (Fredholm Map).** Let $M$ and $N$ be two Banach manifolds. A *Fredholm map* is a smooth map $F : M \longrightarrow N$ such that its differential $dF_x : T_xM \longrightarrow T_{F(x)}N$ is a bounded Fredholm operator.

A point $y \in Y$ is called a *regular value* for the map $F$ if $dF_x$ is surjective for all $x \in F^{-1}(y)$.

Observe that if $M$ is connected, then the Fredholm indices of the operators $dF_x$ are independent of $x$, and hence it makes sense to talk about the *Fredholm index* $\operatorname{ind} F$ of $F$.

In the following theorem we will use the phrase "for most $y$", to mean that the $y$'s in question belong to some set which contains the intersection of at most countably many dense open subsets. In some textbooks such sets are called *generic*. The *Baire Category Theorem* states that generic subsets of complete metric spaces or of locally compact Hausdorff spaces are always dense.

**Theorem 1.37 (Smale-Sard).** *Let $F : M \longrightarrow N$ be a Fredholm map between smooth paracompact Banach manifolds $M$ and $N$, and assume $M$ is connected such that $F$ has a well-defined Fredholm index* $\operatorname{ind} F$*.*

1) *If* $\operatorname{ind} F < 0$*, then $F^{-1}(y) = \emptyset$ for most $y \in N$.*

2) *If* $\operatorname{ind} F \geq 0$*, then most $y \in Y$ are regular values for $F$, and for a regular value $y$ the fiber $F^{-1}(y)$ is a smooth embedded* $\operatorname{ind} F$*-dimensional submanifold of $M$.*

The proof may be found in Smale's original 1965-article [19] or in [6] Proposition 4.3.8.

If we want the set of regular values to be open as well, some further conditions are required:

**Proposition 1.38.** *Let $F : M \longrightarrow N$ be a Fredholm map between Hilbert manifolds. Then the set of regular values of $F$ whose fibers are compact, is open in $N$.*

Proof. Let $\mathcal{R}(F)$ denote the set of regular values of $F$. Taking $y_0 \in \mathcal{R}$, we need to find an open neighborhood around this point inside $\mathcal{R}$. For each $x_0 \in F^{-1}(y_0)$ we have that $dF_{x_0}$ is surjective and hence there exists an open neighborhood $U_{x_0}$ around $x_0$ such that $dF_x$ is surjective for each $x \in U_{x_0}$. In other words $F|_{U_{x_0}}$ is a submersion and submersions are always open maps, hence $F(U_{x_0})$ is open. We can cover the fiber $F^{-1}(y_0)$ by such open sets $U_x$, and since $F^{-1}(y_0)$ is compact we only need finitely many such sets, i.e. $F^{-1}(y_0) \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. Now put $V := F(U_{x_1}) \cap \cdots \cap F(U_{x_n})$. This is an open set in $N$ and since $F^{-1}(V) \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$ we must have $dF_x$ surjective for each $x \in F^{-1}(V)$, i.e. $y_0 \in V \subseteq \mathcal{R}(F)$. □

# CHAPTER 2

# The Seiberg-Witten Invariants

## 2.1 The Seiberg-Witten Equations

In this section we give a detailed description of the Seiberg-Witten equation and the different entities entering them.

We begin with the setup. Let $M$ be a connected closed oriented Riemannian 4-manifold and $\mathrm{Cl}(M)$ its Clifford bundle. As mentioned in the previous chapter we can define a global volume section $\omega_{\mathbb{C}}$ which relative to some local orthonormal frame takes the form $\omega_{\mathbb{C}} = -E_1E_2E_3E_4$. By Theorem 1.3, $M$ can be given a spin$^c$-structure $\sigma$ and from the corresponding principal $\mathrm{Spin}^c(4)$-bundle $P^c_{\mathrm{Spin}}(M)$ we can form the determinant line bundle $L$ as well as the complex spinor bundle $S := S^c(M)$ which splits $S = S^+ \oplus S^-$. Of course, all these bundles depend on the choice of spin$^c$-structure!

Let $\mathfrak{A}$ denote the set of *metric* connections on $L$. It is well-known that this is an affine space, modeled on $i\Omega^1(M)$, the imaginary-valued 1-forms, i.e.

$$\mathfrak{A} = \mathcal{A}_0 + i\Omega^1(M)$$

for some fixed metric connection $\mathcal{A}_0$ called the *reference connection* which will remain fixed for the rest of the thesis. Picking a connection $\mathcal{A}$ on $L$ determines (together with the Levi-Civita connection) a spin connection $\widetilde{\nabla}^{\mathcal{A}}$ on $S$ as well as a geometric Dirac operator $\not{D}_{\mathcal{A}} : \Gamma(S) \longrightarrow \Gamma(S)$. These are the basic ingredients in the Seiberg-Witten equations.

To describe the other we need to digress briefly to discuss some algebraic features of the exterior algebra and Clifford algebra over $\mathbb{R}^4$. First we recall that we have a canonical vector space isomorphism $Q : \Lambda^*\mathbb{R}^4 \xrightarrow{\sim} \mathrm{Cl}_{0,4}$, called the *quantization map*, given by $e_{i_1} \wedge \cdots \wedge e_{i_k} \longmapsto e_{i_1} \cdots e_{i_k}$ whenever $(e_1, e_2, e_3, e_4)$ is an orthonormal basis for $\mathbb{R}^4$. We also recall the *Hodge star operator* $* : \Lambda^k\mathbb{R}^4 \longrightarrow \Lambda^{4-k}\mathbb{R}^4$ defined by $\omega \wedge *\eta = \langle \omega, \eta \rangle e_1e_2e_3e_4$. On 2-forms the Hodge star is an involution: $*^2 = (-1)^{2(4-2)} = 1$, so $*$ has two eigenvalues, $\pm 1$ and $\Lambda^2\mathbb{R}^4$ splits into the corresponding eigenspaces

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4.$$

$\frac{1}{2}(1 \pm *)$ are the corresponding projections onto $\Lambda^2_\pm\mathbb{R}^4$. The elements of $\Lambda^2_+\mathbb{R}^4$ are called *self-dual* 2-forms whereas elements of $\Lambda^2_-\mathbb{R}^4$ are called *anti-self-dual* 2-forms. We can perform these constructions pointwise on the bundle $\Lambda^2T^*M$, obtaining the subbundles $\Lambda^2_\pm T^*M$. Sections of these bundles are denoted $\Omega^2_\pm(M)$.

Finally, on the complexified Clifford algebra $\mathrm{Cl}_{0,4}^{\mathbb{C}}$ the volume form $\omega_{\mathbb{C}}$ acts by left multiplication (*not* by Clifford action), and again since $\omega_{\mathbb{C}}^2 = 1$ it gives a splitting of the Clifford algebra

$$\mathrm{Cl}_{0,4}^{\mathbb{C}} = (\mathrm{Cl}_{0,4}^{\mathbb{C}})_+ \oplus (\mathrm{Cl}_{0,4}^{\mathbb{C}})_- .$$

The same splitting pertains to the even and odd parts of the algebra:

$$(\mathrm{Cl}_{0,4}^{\mathbb{C}})^0 = (\mathrm{Cl}_{0,4}^{\mathbb{C}})_+^0 \oplus (\mathrm{Cl}_{0,4}^{\mathbb{C}})_-^0 \qquad \text{and} \qquad (\mathrm{Cl}_{0,4}^{\mathbb{C}})^1 = (\mathrm{Cl}_{0,4}^{\mathbb{C}})_+^1 \oplus (\mathrm{Cl}_{0,4}^{\mathbb{C}})_-^1$$

The first result we record, is that $\omega_{\mathbb{C}} \circ Q|_{\Lambda^2\mathbb{R}^4} = Q \circ *|_{\Lambda^2\mathbb{R}^4}$ i.e. that $Q$ intertwines $*|_{\Lambda^2\mathbb{R}^4}$ and the action of $\omega_{\mathbb{C}}$ restricted to $Q(\Lambda^2\mathbb{R}^4)$. For instance we can calculate

$$\begin{aligned}\omega_{\mathbb{C}}(Q(e_1 \wedge e_2)) &= -(e_1e_2e_3e_4)e_1e_2 = -e_1e_2(e_1e_2e_3e_4) = e_1^2e_2^2e_3e_4 = e_3e_4 \\ &= Q(e_3 \wedge e_4) = Q(*(e_1 \wedge e_2)).\end{aligned}$$

Similarly, one can show that $\omega_{\mathbb{C}} \circ Q|_{\Lambda^0\mathbb{R}^4} = -Q \circ *|_{\Lambda^0\mathbb{R}^4}$ and $\omega_{\mathbb{C}} \circ Q|_{\Lambda^4\mathbb{R}^4} = -Q \circ *|_{\Lambda^4\mathbb{R}^4}$. With this fact established we can prove

**Lemma 2.1.** *There is an isomorphism of vector spaces*

$$(\mathrm{Cl}_4^{\mathbb{C}})_+^0 \cong Q(\Lambda_+^2\mathbb{R}^4 \otimes \mathbb{C}) \oplus \mathbb{C}\frac{1+\omega_{\mathbb{C}}}{2}. \tag{2.1}$$

Proof. The proof is just a simple calculation:

$$\begin{aligned}(\mathrm{Cl}_4^{\mathbb{C}})_+^0 &= \tfrac{1}{2}(1+\omega_{\mathbb{C}})(\mathrm{Cl}_4^{\mathbb{C}})^0 \cong \tfrac{1}{2}(1+\omega_{\mathbb{C}})Q(\Lambda^0\mathbb{R}^4 \oplus \Lambda^2\mathbb{R}^4 \oplus \Lambda^4\mathbb{R}^4) \otimes \mathbb{C} \\ &= \tfrac{1}{2}Q((1+*)(\Lambda^2\mathbb{R}^4 \otimes \mathbb{C})) + \tfrac{1}{2}Q((1-*)(\Lambda^0\mathbb{R}^4 \oplus \Lambda^4\mathbb{R}^4)) \otimes \mathbb{C} \\ &= Q(\Lambda_+^2\mathbb{R}^4 \otimes \mathbb{C}) \oplus \mathbb{C}\frac{1+\omega_{\mathbb{C}}}{2}.\end{aligned}$$

The last identity is a consequence of the fact that $\frac{1}{2}(1+\omega_{\mathbb{C}})$ is a basis for the space $\frac{1}{2}Q((1-*)(\Lambda^0\mathbb{R}^4 \oplus \Lambda^4\mathbb{R}^4)) \otimes \mathbb{C}$. □

Thus any $\xi \in (\mathrm{Cl}_4^{\mathbb{C}})_+^0$ can be written as $\xi_0 + \frac{\lambda}{2}(1+\omega_{\mathbb{C}})$. It is well-known that the spin representation $\rho_4 : \mathrm{Cl}_{0,4}^{\mathbb{C}} \longrightarrow \mathrm{End}_{\mathbb{C}}(\Delta_4)$ restricts to an isomorphism $(\mathrm{Cl}_4^{\mathbb{C}})_+^0 \cong \mathrm{End}(\Delta_4^+)$ and then (since $\rho_4(\frac{1}{2}(1+\omega_{\mathbb{C}})) = \mathrm{id}_{\Delta_4^+}$)

$$\rho_4(\xi) = \rho_4(\xi_0) + \tfrac{\lambda}{2}\rho_4(1+\omega) = \rho_4(\xi_0) + \lambda\,\mathrm{id}_{\Delta_4^+}.$$

In [18] Example 1.3.3 it is shown that $\rho_4(\xi_0)$ is traceless. Thus $\rho_4(\xi)$ is traceless if and only if $\lambda = 0$, in other words the composition $\rho_4 \circ Q$ gives an isomorphism $\Lambda_+^2\mathbb{R}^4 \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{End}_0(\Delta_4^+)$, the space being the space of traceless endomorphisms of $\Delta_4^+$.

Now we can return to the manifold situation. The connection $\mathcal{A}$ on the determinant line bundle gives a curvature 2-form $F_{\mathcal{A}}$ on $M$. Since $M$ has dimension 4, we can split $F_{\mathcal{A}}$ it into self-dual and anti-self-dual parts: $F_{\mathcal{A}} = F_{\mathcal{A}}^+ + F_{\mathcal{A}}^-$.

Next, if $\psi \in \Gamma(S^+)$ is a positive Weyl spinor field, we get an endomorphism $\psi \otimes \psi^* : \Gamma(S^+) \longrightarrow \Gamma(S^+)$ by defining

$$(\psi \otimes \psi^*)(\varphi)(x) := \langle \varphi(x), \psi(x) \rangle_x \psi(x).$$

Define $q(\psi) : \Gamma(S^+) \longrightarrow \Gamma(S^+)$ to be the 2-form corresponding to the traceless part of this map:

$$q(\psi)(x) := (\rho_4 \circ Q)^{-1}\Big[\psi \otimes \psi^*(x) - \frac{1}{2}\,\mathrm{Tr}(\psi \otimes \psi^*(x))\,\mathrm{id}\Big]$$

(we need the constant $\frac{1}{2}$ since $S_x^+$ is of dimension 2). To determine the trace of $\psi\otimes\psi^*$, let $\psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$ be the representation of $\psi(x)$ relative to some basis for $S_x^+$. It is not hard to check that the matrix representation of $\psi\otimes\psi^*(x)$ is

$$\begin{pmatrix}|\psi_1(x)|^2 & \psi_1(x)\overline{\psi_2(x)}\\ \psi_2(x)\overline{\psi_1(x)} & |\psi_2(x)|^2\end{pmatrix}.$$

>From this we see that $\mathrm{Tr}(\psi\otimes\psi^*(x))=|\psi(x)|^2$ and hence that

$$q(\psi)=(\rho_4\circ Q)^{-1}\Big[\psi\otimes\psi^*-\frac{1}{2}|\psi|^2\,\mathrm{id}\Big].$$

The matrix for $\psi\otimes\psi-\frac{1}{2}|\psi|^2\,\mathrm{id}$ is then given by

$$\begin{pmatrix}\frac{1}{2}(|\psi_1(x)|^2-|\psi_2(x)|^2) & \psi_1(x)\overline{\psi_2(x)}\\ \psi_2(x)\overline{\psi_1(x)} & \frac{1}{2}(|\psi_2(x)|^2-|\psi_1(x)|^2\end{pmatrix}. \tag{2.2}$$

Let $\{E_1,E_2,E_3,E_4\}$ be a local orthonormal frame for $TM$ over a neighborhood which trivializes both $TM$ and $S^+$ and let $\{\varepsilon^1,\varepsilon^2,\varepsilon^3,\varepsilon^4\}$ be the dual orthonormal frame for $T^*M$. From the theory of Clifford algebras it is known that the isomorphism $\rho_4\circ Q$ in the bundle setting is given by

$$\rho_4\circ Q(\varepsilon^1\wedge\varepsilon^2+\varepsilon^3\wedge\varepsilon^4)=\begin{pmatrix}-2i & 0\\ 0 & 2i\end{pmatrix},$$

$$\rho_4\circ Q(\varepsilon^1\wedge\varepsilon^3-\varepsilon^2\wedge\varepsilon^4)=\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix},$$

$$\rho_4\circ Q(\varepsilon^1\wedge\varepsilon^4+\varepsilon^2\wedge\varepsilon^3)=\begin{pmatrix}0 & -2i\\ -2i & 0\end{pmatrix}.$$

Over this trivialization neighborhood we may view $\psi$ as just a map into $\Delta_4^+=\mathbb{C}^2$, $\psi=(\psi_1,\psi_2)$. From the matrix representation (2.2) combined with the appearance of $\rho_4\circ Q$ we get the following local expression for $q(\psi)$:

$$\begin{aligned}q(\psi)=\frac{i}{2}(|\psi_1|^2-|\psi_2|^2)(\varepsilon^1\wedge\varepsilon^2+\varepsilon^3\wedge\varepsilon^4)+i\,\mathrm{Im}(\psi_1\overline{\psi_2})(\varepsilon^1\wedge\varepsilon^3-\varepsilon^2\wedge\varepsilon^4)\\ +i\,\mathrm{Re}(\psi_1\overline{\psi_2})(\varepsilon^1\wedge\varepsilon^4+\varepsilon^2\wedge\varepsilon^3).\end{aligned} \tag{2.3}$$

**Lemma 2.2.** *For the pointwise* operator norm $|\rho_4\circ Q(q(\psi)(x))|$ *it holds that*

$$|\rho_4\circ Q(q(\psi)(x))|=\frac{1}{2}|\psi(x)|^2. \tag{2.4}$$

*For the 2-form $q(\psi)$ we only have a bound*

$$|q(\psi)(x)|\le|\psi(x)|^2. \tag{2.5}$$

Proof. Since $\rho_4\circ Q(q(\psi)(x))$ is self-adjoint we just have to find the eigenvalues. Its matrix is given by

$$\begin{pmatrix}|\psi_1(x)|^2-\frac{1}{2}|\psi(x)|^2 & \psi_1(x)\overline{\psi_2(x)}\\ \psi_2(x)\overline{\psi_1(x)} & |\psi_2(x)|^2-\frac{1}{2}|\psi(x)|^2\end{pmatrix}$$

and a routine calculations yields the eigenvalues $\pm\frac{1}{2}|\psi(x)|^2$, thus the operator norm is as desired.

The proof of the inequality requires a careful analysis of the relations involving $\rho_4\circ Q$ above. Details can be found in [1] in Section 1.4 and p. 44. □

**Definition 2.3** (**The Seiberg-Witten Equations**)**.** Given the setup above, the *Seiberg-Witten Equations* read

$$\not{D}^+_{\mathcal{A}}\psi = 0 \tag{2.6a}$$

$$F^+_{\mathcal{A}} = q(\psi) - \eta \tag{2.6b}$$

where $\eta \in i\Omega^2_+(M)$ is a perturbation parameter.

Solutions to these equations (also known as *Seiberg-Witten monopoles*) are pairs $(\psi, \mathcal{A})$ where $\psi$ is a complex positive Weyl spinor field and $\mathcal{A}$ is a connection on $L$ satisfying the above equations. The set of solutions to (2.6) is denoted $\mathcal{S}(\eta)$.

The set of all *possible* solutions, i.e. $\Gamma(S^+)\times\mathfrak{A}$ is called the *configuration space*, $\mathcal{C}$, and the elements *configurations*. We will often use the shorthand notation $\mathsf{C}$ for a configuration.

>From (2.3) and the calculations there we can deduce a local version of the second Seiberg-Witten equation: If we let $F_{ij} := F^+_{\mathcal{A}}(E_i, E_j)$ be the components of $F^+_{\mathcal{A}}$ and $\eta_{ij} := \eta(E_i, E_j)$ be the components of $\eta$, plugging in $E_1, \dots, E_4$ into (2.3) would give us

$$F_{12} + F_{34} = i(|\psi_1|^2 - |\psi_2|^2) - \eta_{12} - \eta_{34} \tag{2.7a}$$

$$F_{13} - F_{24} - i(F_{14} + F_{23}) = 2\psi_1\overline{\psi_2} - \eta_{13} + \eta_{24} + i(\eta_{14} + \eta_{23}). \tag{2.7b}$$

In fact this is how the Seiberg-Witten equation originally appeared in Witten's 1994 article [23].

The basic idea is to consider the space of solution and then "mod out" some excess degrees of freedom. These extra degrees of freedom are the ones which are related to the part of the spin$^c$-structure sitting "above" the orthonormal frame bundle so to speak, i.e. the part of the principal Spin$^c$(4)-bundle which can be twisted and turned without effecting the underlying frame bundle. We formalize it in the following definition:

**Definition 2.4** (**Gauge Group**)**.** Define $\mathcal{G}$ to be the set of automorphisms $\varphi$ of the principal Spin$^c$(4)-bundle $P^c_{\mathrm{Spin}}(M)$ which cover the identity on the frame bundle $P_{\mathrm{SO}}(M)$, i.e. automorphisms which make the following diagram commutative

$$\begin{tikzcd} P^c_{\mathrm{Spin}}(M) \arrow[rr, "\varphi"] \arrow[dr, "\Phi^c"'] & & P^c_{\mathrm{Spin}}(M) \arrow[dl, "\Phi^c"] \\ & P_{\mathrm{SO}}(M) & \end{tikzcd} \tag{2.8}$$

$\mathcal{G}$ is called the *gauge group* of the spin$^c$-structure and elements in $\mathcal{G}$ are called *gauge transformations*.

We will, shortly, provide a more handy description of this gauge group but before we move on, let's recall a basic result on principal bundle maps. If $G \hookrightarrow P \xrightarrow{\pi} M$ is a principal $G$-bundle, $(U_\alpha, \Phi_\alpha)$ a set of trivializations and $s_\alpha(x) = \Phi_\alpha^{-1}(x, e)$ the corresponding local sections, a bundle map $\varphi : P \longrightarrow P$ produces a system $(\varphi_\alpha)$ of functions $\varphi_\alpha : U_\alpha \longrightarrow G$ by defining

$$\varphi_\alpha(x) := \pi_2(\Phi_\alpha \circ \varphi(s_\alpha(x))). \tag{2.9}$$

If $U_\alpha \cap U_\beta \neq \emptyset$ and $g_{\alpha\beta}$ is the transition function, it is easy to see that $\varphi_\alpha$ and $\varphi_\beta$ are related by

$$\varphi_\beta(x) = g^{-1}_{\alpha\beta}(x)\varphi_\alpha(x)g_{\alpha\beta}(x) \tag{2.10}$$

for $x \in U_{\alpha\beta}$. In particular, if $G$ is abelian, the local functions agree on their overlap, and thus they piece together to a globally defined function $M \longrightarrow G$.

Conversely, given a set $(\varphi_\alpha)$ of functions $U_\alpha \longrightarrow G$ satisfying (2.10), we can construct a bundle map $\varphi : P \longrightarrow P$ locally by

$$\varphi|_{\pi^{-1}(U_\alpha)} = \Phi_\alpha^{-1}(\mathrm{id}_{U_\alpha} \times \varphi_\alpha) \circ \Phi_\alpha.$$

One can check that this is well-defined, i.e. independent of the choice of trivialization.

Over a local trivialization, $\varphi$ just acts as left multiplication by $\varphi_\alpha(x)$:

$$\begin{aligned}\Phi_\alpha \circ \varphi \circ \Phi_\alpha^{-1}(x,g) &= \Phi_\alpha \circ \varphi(\Phi_\alpha^{-1}(x,e)) \cdot g = \Phi_\alpha \circ \varphi(s_\alpha(x) \cdot g)\\ &= [\Phi_\alpha \circ \varphi(s_\alpha(x))] \cdot g = (x, \varphi_\alpha(x)) \cdot g\\ &= (x, \varphi_\alpha(x) g).\end{aligned}$$

Assume $\varphi \in \mathcal{G}$. The requirement that $\varphi$ covers the identity of the frame bundle seriously limits how crazy the local functions $\varphi_\alpha$ can behave. Assume we have an open set $U_\alpha$ and trivializations $\Phi_\alpha$ of $P^c_{\mathrm{Spin}}(M)$ and $\Psi_\alpha$ of $P_{\mathrm{SO}}(M)$. By (2.8) we must have

$$\Psi_\alpha \circ \Phi^c \circ \Phi_\alpha^{-1}(x,g) = \Psi_\alpha \circ \Phi^c \circ \varphi \circ \Phi_\alpha^{-1}(x,g)$$

for $(x,g) \in U_\alpha \times \mathrm{Spin}^c(4)$. The left-hand side is just $(x, \Lambda^c(g))$ and by squeezing $\Phi_\alpha^{-1} \circ \Phi_\alpha$ in between $\Phi^c$ and $\varphi$ we can calculate the right-hand side to be $(x, \Lambda^c(\varphi_\alpha(x))\Lambda^c(g))$. Hence $\varphi_\alpha(x)$ must be in the kernel of $\Lambda^c$, but this kernel is (isomorphic to) $\mathrm{U}(1) \subseteq \mathrm{Spin}^c(4)$. Thus $\varphi_\alpha$ maps into $\mathrm{U}(1)$ and since this group is abelian we can piece the local functions together to obtain a globally defined function $f_\varphi : M \longrightarrow \mathrm{U}(1)$. Hence we have proved the following

**Lemma 2.5.** *There is a natural group isomorphism* $\mathcal{G} \xrightarrow{\sim} C^\infty(M, \mathrm{U}(1))$.

A gauge transformation, $\varphi$, induces in a very natural way bundle automorphisms on the determinant line bundle as well as on the complex spinor bundle. On the determinant line bundle a bundle map $\overline{\varphi} : L \longrightarrow L$ is defined by

$$[p,z] \longmapsto [\varphi(p), z].$$

It is well-defined since

$$\begin{aligned}\overline{\varphi}([p \cdot g, \lambda(g^{-1})z]) &= [\varphi(p \cdot g), \lambda(g^{-1})z] = [\varphi(p) \cdot g, \lambda(g^{-1})z]\\ &= [\varphi(p), z] = \overline{\varphi}([p,z]),\end{aligned}$$

and it is an automorphism since the map $L \longrightarrow L$, $[p,z] \longmapsto [\varphi^{-1}(p), z]$ is an inverse. It even preserves the metric on $L$:

$$\begin{aligned}\langle \overline{\varphi}([p,z]), \overline{\varphi}([p,z]) \rangle &= \langle [\varphi(p), z], [\varphi(p), z'] \rangle = z z'\\ &= \langle [p,z], [p,z'] \rangle,\end{aligned}$$

and therefore the pullback of a metric connection on $L$ is again a metric connection.

In the same way, $\varphi$ induces a bundle endomorphism $\widetilde{\varphi} : S \longrightarrow S$ by the same formula as above:

$$[p,v] \longmapsto [\varphi(p), v].$$

Obviously, it maps $S^\pm \longrightarrow S^\pm$ and hence gives a map on sections $\widetilde{\varphi} : \Gamma(S^\pm) \longrightarrow \Gamma(S^\pm)$.

**Definition 2.6** (**Action of the Gauge Group**)**.** A right action of the gauge group $\mathcal{G}$ on the configuration space $\mathcal{C}$ is defined by

$$(\psi, \mathcal{A}) \cdot \varphi := (\,\widetilde{\varphi}^{-1}(\psi)\,,\,\overline{\varphi}^{*}\mathcal{A}\,). \tag{2.11}$$

We will denote by $\mathcal{B} := \mathcal{C}/\mathcal{G}$ the quotient of $\mathcal{C}$ under the action of $\mathcal{G}$.

Let's spend some time on transforming this definition into something more manageable in terms of the isomorphism in Lemma 2.5. We let $\varphi_\alpha$ denote the local functions of $\varphi$ corresponding to a set of sections $s_\alpha : U_\alpha \longrightarrow P^c_{\text{Spin}}(M)$. The section $s_\alpha$ gives a trivialization $\Phi_\alpha$ of $L^0 = P^c_{\text{Spin}}(M) \times_\lambda \mathrm{U}(1)$ over $U_\alpha$ by $[s_\alpha(x), z] \longmapsto (x, z)$. Since $\mathrm{U}(1)$ is abelian, the local functions $\overline{\varphi}_\alpha(x) = \pi_2(\Phi_\alpha \circ \overline{\varphi} \circ \Phi_\alpha^{-1}(x, 1))$ of $\overline{\varphi}$ piece together to a globally defined function $\overline{f}_\varphi : M \longrightarrow \mathrm{U}(1)$. This is related to the function $f_\varphi$ in the following way:

$$\begin{aligned}
\overline{f}_\varphi(x) &= \pi_2(\Phi_\alpha \circ \overline{\varphi} \circ \Phi_\alpha^{-1})(x, 1) = \pi_2(\Phi_\alpha \circ \overline{\varphi}[s_\alpha(x), 1]) \\
&= \pi_2(\Phi_\alpha[\varphi(s_\alpha(x)), 1]) = \pi_2(\Phi_\alpha([s_\alpha(x) \cdot \varphi_\alpha(x), 1])) \\
&= \pi_2(\Phi_\alpha[s_\alpha(x), \lambda(\varphi_\alpha(x))]) = \pi_2(x, \lambda(f_\varphi(x))) \\
&= \lambda(f_\varphi(x)).
\end{aligned}$$

Since $\varphi_\alpha(x)$ was in $\mathrm{U}(1) \subseteq \mathrm{Spin}^c(4)$, $\overline{f}_\varphi(x)$ is really just the square of $f_\varphi(x)$.

For the bundle automorphism $\widetilde{\varphi}$ of $S^c(M)^+$ we also have local functions, $\widetilde{\varphi}_\alpha : U_\alpha \longrightarrow \mathrm{Aut}(\Delta_4^+)$ given uniquely by the equation

$$\Phi_\alpha \circ \widetilde{\varphi} \circ \Phi_\alpha^{-1}(x, v) = (x, \widetilde{\varphi}_\alpha(x)v).$$

By a calculation as above we can show that

$$\widetilde{\varphi}_\alpha(x) = (\kappa_4^c)^+(\varphi_\alpha(x)) = (\kappa_4^c)^+(f_\varphi(x)).$$

Thus we see, again, that they can be pieced together to a global function $\widetilde{f}_\varphi : M \longrightarrow \mathrm{Aut}(\Delta_4^+)$. Moreover, the $\mathrm{U}(1)$-part of $\mathrm{Spin}^c(4)$ just acts as scalar multiplication, i.e. $(\kappa_4^c)^+(f_\varphi(x)) = f_\varphi(x)$, thus $\widetilde{f}_\varphi = f_\varphi$.

The claim is now that $\widetilde{\varphi}(\psi)$ is nothing but pointwise multiplication by $f_\varphi$, i.e. $\widetilde{\varphi}(\psi)(x) = f_\varphi(x)\psi(x)$. Assume $x \in U_\alpha$, then there exists a $v \in \Delta_4^+$ such that $\psi(x) = [s_\alpha(x), v]$ and hence:

$$\begin{aligned}
\overline{\varphi}(\psi)(x) &= \widetilde{\varphi}(\psi(x)) = \widetilde{\varphi}([s_\alpha(x), v]) = [\varphi(s_\alpha(x)), v] \\
&= [s_\alpha(x) \cdot \varphi_\alpha(x), v] = [s_\alpha(x), \lambda(\varphi_\alpha(x))v] = \lambda(\varphi_\alpha(x))[s_\alpha(x), v] \\
&= f_\varphi(x)\psi(x).
\end{aligned}$$

Thus, of course, $\widetilde{\varphi}^{-1}(\psi) = f_\varphi^{-1}\psi = f_{\varphi^{-1}}\psi$.

Next we investigate the effect of the action on the connection $\mathcal{A}$, or rather on the local connection 1-forms of $\mathcal{A}$. If $A^\alpha_{\mathcal{A}}$ denotes the local connection 1-form of $\mathcal{A}$ over some trivialization neighborhood $U_{\mathcal{A}}$ and $A^\alpha_{\mathcal{A}'}$ denotes the local connection 1-forms for the pullback connection $\mathcal{A}' := \overline{\varphi}^*\mathcal{A}$, it is well-known that the two local connection 1-forms are related by

$$\begin{aligned}
A^\alpha_{\mathcal{A}'} &= A^\alpha_{\mathcal{A}} + \overline{\varphi}_\alpha^{-1} d\overline{\varphi}_a = A^\alpha_{\mathcal{A}} + \lambda(\varphi_\alpha^{-1}) d(\lambda \circ \varphi_\alpha) = A^\alpha_{\mathcal{A}} + f_\varphi^{-2} d(f_\varphi^2) \\
&= A^\alpha_{\mathcal{A}} + 2f_\varphi^{-1} df_\varphi = A^\alpha_{\mathcal{A}} - 2f_\varphi d(f_\varphi^{-1}).
\end{aligned}$$

The last identity follows since $0 = d(f_\varphi f_\varphi^{-1}) = f_\varphi d(f_\varphi^{-1}) + f_\varphi^{-1} df_\varphi$. Hence we have proved the following:

**Proposition 2.7.** *Let $\varphi \in \mathcal{G}$ correspond to the function $f_\varphi$ under the isomorphism in Lemma* 2.5*. Then*

$$(\psi, \mathcal{A}) \cdot \varphi = (f_\varphi^{-1}\psi, \mathcal{A} + 2f_\varphi df_\varphi).$$

We are then ready to prove:

**Lemma 2.8.** *The set of solutions to the Seiberg-Witten equations is invariant under the action of the gauge group.*

Proof. Observe, using (1.21), that

$$\begin{aligned}\big(\not{D}_{\overline{\varphi}^*\mathcal{A}}(f_\varphi^{-1}\psi) - f_\varphi^{-1}\not{D}_{\mathcal{A}}\psi\big)_\alpha(x) &= \sum_{k=1}^4\Big(e_k\cdot[(E_k)_x f_\varphi^{-1}\psi_\alpha(x)]\\ &\quad + \frac{1}{2}A^\alpha_{\overline{\varphi}^*\mathcal{A}}(E_k)e_k\cdot(f_\varphi^{-1}\psi_\alpha(x)) - \frac{1}{2}f_\varphi^{-1}A^\alpha_{\mathcal{A}}(E_k)e_k\cdot\psi_\alpha(x)\Big)\\ &= \sum_{k=1}^4\Big((df_\varphi^{-1})_x(E_k)e_k\cdot\psi_\alpha(x) - f_\varphi^{-1}f_\varphi(df_\varphi^{-1})_x(E_k)e_k\cdot\psi_\alpha(x)\Big)\\ &= 0.\end{aligned}$$

Thus if $(\psi,\mathcal{A})$ satisfies (2.6a), then clearly also $(f_\varphi^{-1}\psi,\overline{\varphi}^*\mathcal{A}) = (\psi,\mathcal{A})\cdot\varphi$ satisfies (2.6a).

Under the gauge transformation $\varphi$ the field strength $F_{\mathcal{A}}$ transforms into $f_\varphi^{-1}F_{\mathcal{A}}f_\varphi = F_{\mathcal{A}}$, i.e. it is unaffected by the gauge transformation. Similarly

$$q(f_\varphi^{-1}\psi) = (\rho_4\circ Q)^{-1}[(f_\varphi^{-1}\psi)\otimes(f_\varphi^{-1}\psi)^* - \frac{1}{2}|f_\varphi^{-1}\psi|^2] = |f_\varphi^{-1}|^2 q(\psi) = q(\psi).$$

Hence also (2.6b) is invariant under gauge transformations. □

**Definition 2.9 (Moduli Space).** The quotient $\mathcal{M}(\eta) := \mathcal{S}(\eta)/\mathcal{G}$ is called the *moduli space* of the Seiberg-Witten equations.

**Definition 2.10 (Stabilizer).** Given a configuration $\mathsf{C}$ we define the *stabilizer* $\mathrm{Stab}(\mathsf{C})$ to be the set of all elements in $\mathcal{G}$ which, under the action defined above, map $\mathsf{C}$ to itself, i.e.

$$\mathrm{Stab}(\mathsf{C}) = \{\varphi\in\mathcal{G}\mid \mathsf{C}\cdot\varphi = \mathsf{C}\}$$

(there should be no risk of confusing this with the set of stabilizers for a field of Fredholm operators).

A configuration $\mathsf{C}$ is called *irreducible* if $\mathrm{Stab}(\mathsf{C}) = \{1\}$. Otherwise it is called *reducible*. The set of irreducible configurations is denoted $\mathcal{C}^*$.

We note that $\mathcal{C}^*$ is gauge invariant: assume $\mathsf{C}\in\mathcal{C}^*$ and let $\varphi\in\mathcal{G}$. If $\varphi'\in\mathrm{Stab}(\mathsf{C}\cdot\varphi)$ then, since $\mathcal{G}$ is abelian

$$\mathsf{C}\cdot\varphi = (\mathsf{C}\cdot\varphi)\cdot\varphi' = (\mathsf{C}\cdot\varphi')\cdot\varphi,$$

i.e. $\mathsf{C} = \mathsf{C}\cdot\varphi'$ and thus $\varphi'\in\mathrm{Stab}(\mathsf{C})$, hence $\varphi' = 1$. Therefore it makes sense to define

$$\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}.$$

This is of course just a subset of $\mathcal{B} = \mathcal{C}/\mathcal{G}$.

In a similar fashion the irreducible solution space $\mathcal{S}^*(\eta)$ is gauge invariant and hence we can define the *irreducible moduli space*

$$\mathcal{M}^*(\eta) := \mathcal{S}^*(\eta)/\mathcal{G}.$$

**Lemma 2.11.** *A configuration $\mathsf{C} = (\psi,\mathcal{A})$ is reducible, if and only if $\psi\equiv 0$, and in this case we have under the canonical bijection $\mathcal{G}\xrightarrow{\sim}C^\infty(M,\mathrm{U}(1))$ that* $\mathrm{Stab}(\mathsf{C}) = \mathrm{U}(1)$.

Proof. If $\psi\equiv 0$ then all gauge transformations corresponding to constant functions in $C^\infty(M,\mathrm{U}(1))$ are in $\mathrm{Stab}(\mathsf{C})$, hence $\mathsf{C}$ is reducible.

Conversely, if $\mathsf{C} = (\psi,\mathcal{A})$ is reducible, and $\varphi\in\mathrm{Stab}(\mathsf{C})$ then in particular $\mathcal{A} = \mathcal{A} + 2(df_\varphi)f_\varphi^{-1}$, i.e. we must have $df_\varphi\equiv 0$ and hence $f_\varphi$ is locally constant. Since $M$ was assumed to be connected, $f_\varphi$ is globally constant. If also $f_\varphi^{-1}\psi = \psi$ should be satisfied for some (constant) $f_\varphi\neq 1$, we must have $\psi\equiv 0$. □

## 2.2 The Sobolev Setting

Up till now we have been working exclusively within the smooth category: all functions, sections, connections etc. have been smooth. It turns out, however, that this setting is not well-tailored for the work we are about to undertake: investigating the topology of the moduli space. We are forced to leave the smooth world and enter the realm of Sobolev spaces.

If $E \longrightarrow M$ is a Riemannian vector bundle, it makes sense to talk about $L^p$-sections of $E$: they are the measurable sections $\psi : M \longrightarrow E$ for which the integral

$$\int_M |\psi(x)|^p d\mu(x)$$

is finite (here $\mu$ is the unique measure on $M$ given by the metric on $M$). The space of $L^p$-sections of $E$ is denoted $L^p(M,E)$ or just $L^p(E)$. One can show that this is a Banach space, and, in the case $p = 2$, a Hilbert space. Similarly, we can define *Sobolev spaces* of sections of $E$. Assume that $\nabla$ is a metric connection on $E$. The tangent bundle $TM$ has the Levi-Civita connection, and via the bundle isomorphism $TM \xrightarrow{\sim} T^*M$ we can turn it into a connection on $T^*M$ simply by defining $\nabla_X \omega := (\nabla_X \omega^\sharp)^\flat$. The tensor product of these two connections is a connection on $T^*M \otimes E$, and we denote it by $\nabla$ also. It maps into $T^*M \otimes T^*M \otimes E$. Again, taking the tensor product with the Levi-Civita connection yields a connection $\nabla$ on $T^*M \otimes T^*M \otimes E$ and so on. Hence we have maps

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes E) \longrightarrow \cdots$$

We let $\nabla^k := \nabla \circ \cdots \circ \nabla$ denote the composition of the first $k$ of these maps.

We also have a fiber metric on $T^*M \otimes E$, namely the tensor product of the fiber metrics on $T^*M \cong TM$ and $E$. Explicitly, if $\omega_p \otimes s_x, \widetilde{\omega}_p \otimes \widetilde{s}_x \in T_x^*M \otimes E_x$, then

$$\langle \omega_x \otimes s_x, \widetilde{\omega}_x \otimes \widetilde{s}_x \rangle := \langle \omega_x, \widetilde{\omega}_x \rangle \langle s_x, \widetilde{s}_x \rangle,$$

which gives the fiberwise norm

$$|\omega_x \otimes s_x| = |\omega_x||s_x|$$

on $T_x^*M \otimes E_x$. Similarly, we obtain metrics on the higher bundles $(T^*M)^{\otimes k} \otimes E$. Define for each $1 \leq p < \infty$ and each $k \in \mathbb{N}_0$ the *Sobolev norm* $\| \cdot \|_{L_k^p}$ on $\Gamma_c(E)$ by

$$\|u\|_{L_k^p} := \left( \sum_{j=0}^k \|\nabla^j u\|_{L^p}^p \right)^{1/p} = \left( \sum_{j=0}^k \int_M \|\nabla^j u\|_E^p d\mu_g \right)^{1/p}.$$

The notation can be a bit misleading, for as a matter of fact, the Sobolev spaces depend not only on $M$ and $E$ but *also* on the metric on $M$ as well as on the metric and connection on $E$. Thus *á priori* there is no canonical choice of norm! However, one can show that if $M$ is *compact*, the above norm is indeed independent of all these choices, in the sense that any two norms defined in this way will be equivalent. We will be content with compact base manifolds and for that reason, on compact manifolds the following definition is well-posed (up to norm equivalence).

**Definition 2.12** (**Sobolev Space**). Let $1 \leq p < \infty$ and $k \in \mathbb{N}_0$ and assume that $E$ is a Riemannian vector bundle over a compact, orientable Riemannian manifold $M$. Then define the *Sobolev space* $L_k^p(M,E)$ (or just $L_k^p(E)$) as the completion of $\Gamma(E)$ in the Sobolev norm $\| \cdot \|_{L_k^p}$.

There is another approach to defining Sobolev spaces, one that relates to the "classical" Sobolev spaces and norms over $\mathbb{R}^n$. Continue to let $M$ denote a compact oriented Riemannian manifold and $E$ a smooth rank $N$ vector bundle over $M$. Pick a finite atlas $(U_i, \varphi_i)_{i=1}^J$ for $M$ for which each open set $U_i$ is a trivialization neighborhood of $E$ and let $(\rho_i)$ be a partition of unity with compact support subordinate to this cover. For any $\psi \in \Gamma(E)$ we have that $(\rho_i\psi) \circ \varphi_i^{-1}$ is a smooth map $\mathbb{R}^n \longrightarrow E$ with compact support in $\rho_i(U_i)$. Since $E$ is trivial over the support of $\rho_i\psi$, we may view $(\rho_i\psi) \circ \varphi_i^{-1}$ as a smooth compactly supported map $\mathbb{R}^n \longrightarrow \mathbb{C}^N$. Thus for *any* $k \in \mathbb{N}$ it is an element of $L_k^2(\mathbb{R}^n) \times \cdots \times L_k^2(\mathbb{R}^n)$, and we define the *Sobolev norm* by

$$\|\psi\|_k' := \sum_{i=1}^J \|(\rho_i\psi) \circ \varphi_i^{-1}\|_{L_k^2(\mathbb{R}^n)^N}$$

where of course the norm on the right-hand side is the product norm. One can show that this norm is equivalent to the previously defined norm thus showing that being in a Sobolev space is a local condition.

The next theorem, which we state without proof, comprises all we need to know about elliptic differential operators (see for instance [18] Theorem 1.2.18 and/or [13] Theorem 5.2):

**Theorem 2.13.** *Let $E \longrightarrow M$ and $F \longrightarrow M$ be smooth vector bundles over a compact manifold and let $A : \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic differential operator of order $k$. Then the following hold*:

1) *The minimal and maximal realization of the differential operator is identical, they are both unbounded operators $L^2(E) \longrightarrow L^2(F)$ with domain $L_k^2(E)$. We call it the* analytic realization, *$A_k$, of $A$.*

2) *The Hilbert space adjoint of $A_k$ equals the analytic realization of the formal adjoint.*

3) *For an integer $m \geq k$ the restriction of $A_k$ to $L_m^2(E)$ is a bounded Fredholm operator $A_m : L_m^2(E) \longrightarrow L_{m-k}^2(F)$, and there exist finite-dimensional subspaces $V \subseteq \Gamma(E)$ and $W \subseteq \Gamma(F)$ such that $\ker A_m = V$ and $(\operatorname{im} A_m)^\perp = W$ for all $m$.*

4) (*Elliptic Estimate*). *For each $m \geq k$ there exists a constant $C_m$ such that*

$$\|u\|_m \leq C_m(\|u\|_{m-k} + \|Au\|_{m-k}) \tag{2.12}$$

*for all $u \in L_m^2(E)$.*

5) (*Elliptic Regularity*). *If $Au \in L_m^2(F)$ then $u \in L_{k+m}^2(E)$, in particular, if $Au$ is smooth, then $u$ is smooth.*

5) *Let $P : L^2(E) \longrightarrow L^2(E)$ be the orthogonal projection onto $\ker A_k$. For all $1 < p < \infty$ and $m \in \mathbb{N}$ there exists a constant $C > 0$ (depending on $A$, $p$, $m$) such that*

$$\|u - Pu\|_{L_{k+m}^p} \leq C\|Au\|_{L_m^p}. \tag{2.13}$$

By Hölder's inequality we deduce that if $p$, $q$ and $r$ are real numbers in $[1, \infty[$ satisfying $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r}$ then pointwise multiplication gives a map

$$L^p(M) \times L^q(M) \longrightarrow L^r(M).$$

This is an example of *Sobolev multiplication*. We will need a more advanced version of this, involving Sobolev spaces. This can be proved quite generally by use of the Hölder inequality and the Sobolev Embedding Theorem. Here we restrict to the special cases of our interests:

**Theorem 2.14 (Sobolev multiplication).** *Let* $\dim M = 4$ *and* $k_1 \geq 3$ *and* $k_2 \geq 2$ *be integers satisfying* $k_1 \geq k_2$*, then there exists a continuous map, the Sobolev multiplication:*

$$L^2_{k_1}(M) \times L^2_{k_2}(M) \longrightarrow L^2_{k_2}(M).$$

*In particular,* $L^2_k(M)$ *is a Banach algebra for* $k \geq 3$*. Moreover, if* $k \geq 3$ *is an integer then there exists a Sobolev multiplication*

$$L^{m+2}_{k-m}(M) \times L^2_k(M) \longrightarrow L^{m+2}_{k-m}(M)$$

*for each* $m \in \{1, \ldots, k\}$*.*

The proofs of these statements may be found in [1] Corollary 1.53 and 1.54.

Assume we have two vector bundles $E_1$ and $E_2$. Does there exist a Sobolev multiplication

$$L^{p_1}_{k_1}(E_1) \times L^{p_2}_{k_2}(E_2) \longrightarrow L^p_k(E_1 \otimes E_2)$$

under proper requirements on the constants? Yes, for since we are over a compact manifold, being in a Sobolev space is a local property, and thus over some local neighborhood we can pick trivializations and then the tensor product reduces to plain pointwise multiplication. Hence in the theorem above we can replace $M$ by vector bundles and product by tensor product or wedge product.

For the Sobolev multiplication to work and for $L^2_k$ to be an algebra under Sobolev multiplication we had to require that $k \geq 3$. Often one of the factors will be the differential of something and for this reason we have to require $k \geq 4$. This is also sufficient to ensure that the elements in the Sobolev spaces are continuous. Therefore we apply the following convention: When nothing else is mentioned, $k$ is an integer *at least* 4! Later in this chapter we will show that $k$ really doesn't matter.

Given a $k \geq 4$ we want to "Sobolev-complete" the configuration space $\mathcal{C} = \Gamma(S^+) \times \mathfrak{A}$. There is no problem in completing $\Gamma(S^+)$ we simply take the Sobolev space $L^2_k(S^+)$. To complete $\mathfrak{A}$ recall that $\mathfrak{A} = \mathcal{A}_0 + i\Omega^1(M)$ where $\mathcal{A}_0$ was the fixed reference connection $\mathcal{A}_0$. $\Omega^1(M)$ can be Sobolev-completed to $L^2_k(T^*M)$ and thus we define

$$\mathfrak{A}_k := \mathcal{A}_0 + L^2_k(iT^*M) \qquad \text{and} \qquad \mathcal{C}_k := L^2_k(S^+) \times \mathfrak{A}_k.$$

The space $\mathcal{C}_k$ is called the *configuration space.*

It still makes sense to talk about $\not{D}_{\mathcal{A}}$ and $F_{\mathcal{A}}$ even for $\mathcal{A} \in \mathfrak{A}_k$. Inspired by (1.22) we define for $\mathcal{A} = \mathcal{A}_0 + \alpha \in \mathfrak{A}_k$ the *Dirac operator* $\not{D}_{\mathcal{A}} : L^2_k(S) \longrightarrow L^2_{k-1}(S)$ by

$$\not{D}_{\mathcal{A}}\psi := \not{D}_{\mathcal{A}_0}\psi + \frac{1}{2}\alpha \cdot \psi.$$

By Sobolev multiplication, $\psi \longmapsto \frac{1}{2}\alpha\cdot\psi$ is continuous as a map $L^2_k(S) \longrightarrow L^2_k(S)$, and thus as a map $L^2_k(S) \longrightarrow L^2_{k-1}(S)$ it is *compact*, i.e. the only difference between $\not{D}_{\mathcal{A}}$ and $\not{D}_{\mathcal{A}_0}$ is a compact perturbation. Thus $\not{D}_{\mathcal{A}}$ is still Fredholm. One can show (and this is done in [1]) that for this "new" Dirac operator the elliptic estimate, the elliptic regularity and the unique continuation property still hold.

Inspired by (1.24) we define for $\mathcal{A} = \mathcal{A}_0 + \alpha \in \mathfrak{A}_k$ the curvature $F_{\mathcal{A}} := F_{\mathcal{A}_0} + d\alpha$. Furthermore we can define $q(\psi)$ for $\psi \in L^2_k(S^+)$ in the following way: take a representative for $\psi$ and apply the pointwise operations on this representative (recall that $q(\psi)$ was defined pointwise) and then take its Sobolev class. By Sobolev multiplication we still have $q(\psi) \in L^2_k(i\Lambda^2T^*M)$.

Hence it makes sense to talk about the *Seiberg-Witten equations* in this setting. The space of solutions to these equations among the configurations in $\mathcal{C}_k$

(where we perturb by a 2-form $\eta \in L^2_{k-1}(\Lambda^2_+ T^*M)$) is denoted $\mathcal{S}_k(\eta)$. If we for $\eta \in L^2_{k-1}(i\Lambda^2_+ T^*M)$ define the map

$$F^\eta : \mathcal{C}_k \longrightarrow L^2_{k-1}(S^-) \times L^2_{k-1}(i\Lambda^2_+ T^*M)$$

by

$$F^\eta(\psi, \mathcal{A}) = \begin{pmatrix} \not{D}_{\mathcal{A}}\psi \\ F^+_{\mathcal{A}} - q(\psi) + \eta \end{pmatrix}, \tag{2.14}$$

it should be clear that $\mathcal{S}_k(\eta) = (F^\eta)^{-1}(0)$.

**Proposition 2.15.** *The map $F^\eta$ is a differentiable map and its differential at the point $\mathsf{C} = (\psi, \mathcal{A})$ is given by*

$$dF^\eta_{\mathsf{C}} \begin{pmatrix} \varphi \\ \alpha \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha \cdot \psi + \not{D}_{\mathcal{A}}\varphi \\ d^+\alpha - dq_\psi\varphi \end{pmatrix} \tag{2.15}$$

*where*

$$dq_\psi(\varphi) = (\rho_4 \circ Q)^{-1}\Big[\varphi \otimes \psi^* + \psi \otimes \varphi^* - \frac{1}{2}(\langle \varphi, \psi\rangle + \langle \psi, \varphi\rangle)\,\mathrm{id}\Big]. \tag{2.16}$$

*In particular $\mathcal{S}_k(\eta)$ is a closed subset of $\mathcal{C}_k$.*

Proof. To calculate the differential, first note that if $F : M \times N \longrightarrow K$ is a differentiable map between manifolds, then for $(X, Y) \in T_{(x_0,y_0)}M \times N \cong T_{x_0}M \times T_{y_0}N$

$$dF_{(x_0,y_0)}(X, Y) = dF^1_{x_0}(X) + dF^2_{y_0}(Y) \tag{2.17}$$

where $F^1(x) = F(x, y_0)$ and $F^2(y) = F(x_0, y)$.

Accordingly we may calculate the differential componentwise. First we compute the differential of the map $(\psi, \mathcal{A}) \longmapsto \not{D}_{\mathcal{A}}\psi$, and again we use (2.17): For a fixed $\mathcal{A}$, the map $\psi \longmapsto \not{D}_{\mathcal{A}}\psi$ is $\mathbb{C}$-linear and hence is its own differential, and for a fixed $\psi$ we have $\not{D}_{\mathcal{A}}\psi = \not{D}_{\mathcal{A}_0}\psi + \frac{1}{2}\alpha \cdot \psi$ but here $\not{D}_{\mathcal{A}_0}\psi$ is just a constant and in $\alpha \cdot \psi$ the dependence on $\alpha$ is again linear and is thus also its own differential. Hence the differential of the first component is just the map $(\varphi, \alpha) \longmapsto \not{D}_{\mathcal{A}}\varphi + \frac{1}{2}\alpha \cdot \psi$.

For the second component note that $F_{\mathcal{A}_0+\alpha} = F_{\mathcal{A}_0} + d\alpha$ and thus $F^+_{\mathcal{A}_0+\alpha} = F^+_{\mathcal{A}_0} + d^+\alpha$, and since $\alpha \longmapsto d^+\alpha$ depends linearly on $\alpha$ this is its own differential. For a fixed $\mathcal{A}$, the only $\psi$-dependence is on $q$ ($\eta$ is just a constant) and thus we get the term $dq_\psi(\varphi)$. The identity (2.16) can be proved in a manner analogous to calculating the derivative of a product of functions. □

The equation $dF^\eta_{\mathsf{C}} = 0$ is the linearized version of the Seiberg-Witten equation near $\mathsf{C}$. Solutions to this equation may approximate solutions to the real Seiberg-Witten equations.

In the same spirit as above we complete the gauge group:

**Definition 2.16 (Gauge Group).** Define the *gauge group* $\mathcal{G}_k := L^2_k(M, \mathrm{U}(1))$, to be the set of functions in $L^2_k(M)$ mapping into $\mathrm{U}(1)$ almost everywhere.

This is a closed subset of $L^2_k(M)$ since a sequence $(f_n) \subseteq L^2_k(M, \mathrm{U}(1))$ converging in the Sobolev norm will converge pointwise almost everywhere. As $\mathrm{U}(1)$ is a closed subset of $\mathbb{C}$, the claim follows.

**Lemma 2.17.** *The gauge group $\mathcal{G}_k$ is a Hilbert-Lie group for $k \geq 3$. The Lie algebra of $\mathcal{G}_k$ is $\mathfrak{g}_k = iL^2_k(M)$ with the zero bracket and the exponential map* $\exp : iL^2_k(M) \longrightarrow \mathcal{G}_k$ *is just the usual exponential map* $\exp(f)(x) = e^{f(x)}$.

Proof. Consider the map $L^2_k(M,\mathbb{C}) \longrightarrow L^2_k(M)$ given by $f \longmapsto |f|^2$. This is a smooth map between the Hilbert manifolds $L^2_k(M,\mathbb{C})$ and $L^2_k(M)$ and its differential $L^2_k(M,\mathbb{C}) \longrightarrow L^2_k(M)$ is given by $f_1 + if_2 \longmapsto 2(f_1+f_2)$. Thus the map is a *submersion* [1] and therefore the level set corresponding to the constant function 1, which equals $\mathcal{G}_k$, is an embedded submanifold of $L^2_k(M,\mathbb{C})$, in particular a manifold in its own right. Sobolev multiplication $L^2_k(M,\mathbb{C}) \longrightarrow L^2_k(M,\mathbb{C})$ is continuous and bilinear, and therefore smooth, and it restricts to a smooth product in $\mathcal{G}_k$. Likewise with inversion. Hence $\mathcal{G}_k$ is a Hilbert-Lie group.

The Lie algebra is just the tangent space at the identity. Let $g_t = e^{f_t}$ be a smooth curve in $\mathcal{G}_k$ with $g_0$ equal to the identity element, i.e. $f_t \in iL^2_k(M,\mathbb{R})$ and $f_0 \equiv 0$. Then we see

$$\frac{d}{dt}\bigg|_{t=0} g_t = \frac{d}{dt}\bigg|_{t=0} f_t.$$

Any element in $iL^2_k(M)$ can be obtained as the derivative $\frac{d}{dt}\big|_{t=0} f_t$ of some smooth curve $f_t$ and therefore

$$T_1\mathcal{G}_k = iL^2_k(M).$$

Since the group $\mathcal{G}_k$ is abelian, the bracket $[\,,\,]$ is trivial. ☐

**Definition 2.18** (**Gauge Group Action**). On the configuration space $\mathcal{C}_k$ we define for $k \geq 3$ an action $\rho$ of the gauge group $\mathcal{G}_{k+1}$ by

$$(\psi,\mathcal{A})\cdot\lambda = (\lambda^{-1}\psi, \mathcal{A} + 2\lambda^{-1}d\lambda).$$

By Proposition 2.7 this definition is consistent with our previous definition in the smooth setting. The presence of the differential operator $d$ lowers the degree of regularity of $\lambda$ by 1 (that's why we had to require extra regularity of the gauge transformation) and therefore $d\lambda \in L^2_k(i\Lambda^1T^*M)$. The Sobolev multiplication

$$L^2_k \times L^2_k \longrightarrow L^2_k$$

guarantees that $2\lambda^{-1}d\lambda \in L^2_k$ since $\lambda^{-1} = \overline{\lambda}$ is of course also in $\mathcal{G}_{k+1} \subseteq \mathcal{G}_k$.

**Lemma 2.19.** *The action* $\rho : \mathcal{C}_k \times \mathcal{G}_{k+1} \longrightarrow \mathcal{C}_k$ *is differentiable and its differential at the point* $(\mathsf{C},1) = ((\psi,\mathcal{A}),1)$

$$d\rho_{((\psi,\mathcal{A}),1)} : L^2_k(S^+) \times L^2_k(i\Lambda^1T^*M) \times iL^2_{k+1}(M) \longrightarrow L^2_k(S^+) \times L^2_k(i\Lambda^1T^*M)$$

*is given by*

$$d\rho_{((\psi,\mathcal{A}),1)}\begin{pmatrix}\varphi\\ \alpha\\ f\end{pmatrix} = \begin{pmatrix}\varphi - f\psi\\ \alpha + 2df\end{pmatrix}.$$

Proof. As noted above, we just have to calculate the differentials of the maps $\rho^1$, $\mathsf{C}' \longmapsto \mathsf{C}'\cdot 1 = \mathsf{C}'$ and $\rho^{\mathsf{C}}$ given by $\lambda \longmapsto \mathsf{C}\cdot\lambda$. Since $\rho^1$ is just the identity, its differential is also just the identity. To calculate the differential $d\rho^{\mathsf{C}}_1$ we use the exponential map. Let $f \in L^2_{k+1}(M, i\mathbb{R})$, then

$$\begin{aligned} d\rho^{\mathsf{C}}_1(f) &= \frac{d}{dt}\bigg|_{t=0} \rho^{\mathsf{C}}(\exp tf) \\ &= \frac{d}{dt}\bigg|_{t=0} \big(\exp(-tf)\psi, \mathcal{A} + 2\exp(-tf)d(\exp tf)\big) \\ &= \frac{d}{dt}\bigg|_{t=0} \big(\exp(-tf)\psi, \mathcal{A} + 2tdf\big) = (-f\psi, 2df). \end{aligned} \tag{2.18}$$

---

[1] When dealing with Banach manifolds, showing that a map is a *submersion* also requires verifying that the kernel of the differential has a complement, but in this case we are dealing with Hilbert manifolds and in Hilbert spaces any subspace has a complement, and thus this requirement is superfluous.

Adding the two contributions gives the desired result. □

**Definition 2.20 (Stabilizer).** For a given configuration $\mathsf{C} \in \mathcal{C}_k$ we define its *stabilizer* $\mathrm{Stab}(\mathsf{C})$ to be the subgroup of elements of $\mathcal{G}_{k+1}$ which fix $\mathsf{C}$.

A configuration $\mathsf{C}$ is called *irreducible* if $\mathrm{Stab}(\mathsf{C}) = \{1\}$. The set of irreducible configurations is denoted $\mathcal{C}_k^*$. A configuration which is not irreducible is called *reducible*.

**Lemma 2.21.** *The configuration* $\mathsf{C} = (\psi, \mathcal{A}) \in \mathcal{C}_k$ *is reducible if and only if* $\psi \equiv 0$ *and in this case* $\mathrm{Stab}(\mathsf{C}) = \mathrm{U}(1) \subseteq \mathcal{G}_{k+1}$.

Proof. If $\mathsf{C} = (\psi, \mathcal{A})$ is a reducible configuration and $\lambda \in \mathrm{Stab}(\mathsf{C})$, then we must have $d\lambda = 0$ and then by elliptic regularity, the function $\lambda$ will be smooth, and the argument is exactly identical to the one in the smooth case above (Lemma 2.11). □

In accordance with the notation in the previous section we define

$$\mathcal{B}_k := \mathcal{C}_k/\mathcal{G}_{k+1} \qquad \text{as well as} \qquad \mathcal{B}_k^* := \mathcal{C}_k^*/\mathcal{G}_{k+1}.$$

We put $\mathcal{S}_k^*(\eta) := \mathcal{S}_k(\eta) \cap \mathcal{C}_k^*$ (the irreducible solutions to the Seiberg-Witten equations) and define the *moduli spaces*

$$\mathcal{M}_k(\eta) := \mathcal{S}_k(\eta)/\mathcal{G}_{k+1} \qquad \text{and} \qquad \mathcal{M}_k^*(\eta) := \mathcal{S}_k^*(\eta)/\mathcal{G}_{k+1}.$$

But wait a minute, originally we set out to study the moduli space of smooth solutions, it seems now that we have extended this space dramatically? Yes, the space of solutions has been extended, but so has the gauge group and these two increments are, as we shall soon realize, perfectly balanced. A first step on this path to enlightenment is Proposition 2.23. At the end of Section 2.3, when we have shown that the moduli spaces are compact, we will undertake a more detailed study of the relations of the different moduli spaces among each other

A major player in the proofs ahead is the operator $d^+ + 2d^*$ where $d^+ : \Omega^1(M) \longrightarrow \Omega^2_+(M)$ denote the self-dual part of $d$, explicitly:

$$d^+\omega := (d\omega)^+ = \frac{1}{2}(1 + *)d\omega.$$

The formula of the index of this operator contains some entities which we have not yet defined. First we recall that the Hodge star operator maps the space of harmonic forms to itself, thus $* : \mathscr{H}^2(M) \longrightarrow \mathscr{H}^2(M)$ is an involution and we can split the space of harmonic 2-forms into eigenspaces

$$\mathscr{H}^2(M) = \mathscr{H}^2_+(M) \oplus \mathscr{H}^2_-(M)$$

where $\mathscr{H}^2_\pm(M) = \frac{1}{2}(1 \pm *)\mathscr{H}^2(M)$. Now define

$$b_2^\pm := \dim_{\mathbb{R}} \mathscr{H}^2_\pm(M) \qquad \text{and} \qquad \sigma(M) := b_2^+ - b_2^-.$$

The integer $\sigma(M)$ is called the *signature* of the manifold. Writing only $\sigma(M)$ and not $\sigma(M, g)$ indicates that $\sigma(M)$ is independent of the metric, but how? The Betti numbers $b_2^\pm$ and hence $\sigma(M)$ were defined in terms of the Hodge star, which is certainly metric-dependent? Luckily there is an equivalent and entirely topological definition of the signature. Define the *intersection form* of the manifold $I : H^2(M;\mathbb{R}) \times H^2(M;\mathbb{R}) \longrightarrow \mathbb{R}$ by

$$I([\alpha], [\beta]) = \int_M \alpha \wedge \beta$$

(here the singular cohomology group $H^2(M;\mathbb{R})$ is identified with the de Rham cohomology group). This is obviously bilinear and symmetric, so it is represented by a symmetric $b_2 \times b_2$ matrix. Furthermore $I$ is non-degenerate: given a non-zero 2-form $\alpha$, we let $(U, x_1, x_2, x_3, x_4)$ be a coordinate patch on which $\alpha$ is everywhere non-zero. $\alpha$ is a linear combination of $dx_1 \wedge dx_2,\ dx_1 \wedge dx_3,\ dx_1 \wedge dx_4,\ dx_2 \wedge dx_3,\ dx_2 \wedge dx_4$ and $dx_3 \wedge dx_4$ and we may assume (by shrinking $U$ if necessary) that the coefficient function $\alpha_{12}$ of $dx_1 \wedge dx_2$ is strictly positive on $U$. Then if $f$ denotes a positive bump function supported in $U$ we put $\beta := f\, dx_3 \wedge dx_4$ (this is now a globally defined 2-form) and then we see that $I(\alpha, \beta) \neq 0$, since $\alpha \wedge \beta = f\alpha_{12} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. Thus $I$ is non-degenerate and hence the diagonalization of its matrix contains only non-zero eigenvalues. Then it turns out (I will not prove that) that $b_2^+$ equals the number of positive eigenvalues (counted with multiplicity) and $b_2^-$ is the number of negative eigenvalues (counted with multiplicity). $I$ was defined without reference to the metric and therefore $b_2^\pm$ and $\sigma(M)$ depend only on the manifold $M$ and not the metric.

**Lemma 2.22.** *The operator*

$$d^+ + 2d^* : \Omega^1(M) \longrightarrow \Omega^2(M) \oplus C^\infty(M)$$

*is elliptic and* $\ker(d^+ + 2d^*) = \mathscr{H}^1(M)$ *(the space of harmonic* 1*-forms on* $M$*) and* $\operatorname{coker}(d^+ + 2d^*) \cong \mathscr{H}^0(M) \oplus \mathscr{H}^2_+(M)$*. Furthermore the index of the operator is given by*

$$\operatorname{ind}_{\mathbb{R}}(d^+ + 2d^*) = b_1 - 1 - b_2^+. \tag{2.19}$$

Proof. It is a well-known result from the theory of differential operators that the differential operator in question is elliptic if and only if the complex

$$\begin{tikzcd} 0 \arrow[r] & C^\infty(M) \arrow[r, "2d"] & \Omega^1(M) \arrow[r, "d^+"] & \Omega^2_+(M) \arrow[r] & 0 \end{tikzcd}$$

(which we will call $\Omega^*$) is elliptic, i.e. if and only if the corresponding symbol complex

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r, "2i\xi\wedge\cdot"] & \Lambda^1 T_x^* M_{\mathbb{C}} \arrow[r, "(i\xi\wedge\cdot)_+"] & \Lambda^2_+ T_x^* M_{\mathbb{C}} \arrow[r] & 0 \end{tikzcd} \tag{2.20}$$

is exact for any $0 \neq \xi \in T_x^* M_{\mathbb{C}}$ (the complexified tangent space) for all $x \in M$. The first map in (2.20) is clearly injective. Pick a complex basis $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ for $T_x M_{\mathbb{C}}$ such that $\xi_1 = i\xi$. Then one can check that $\Lambda^2_+ T_x^* M_{\mathbb{C}}$ is spanned by

$$\begin{aligned} \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 &= 2(\xi_1 \wedge \xi_2)_+, \\ \xi_1 \wedge \xi_3 + \xi_4 \wedge \xi_2 &= 2(\xi_1 \wedge \xi_3)_+, \\ \xi_1 \wedge \xi_4 + \xi_2 \wedge \xi_3 &= 2(\xi_1 \wedge \xi_4)_+. \end{aligned}$$

This shows that $(i\xi \wedge \cdot)_+$ is surjective and that the kernel of the map exactly is the span of $\xi_1 = i\xi$, i.e. that the complex (2.20) is exact.

To show the identity $\ker(d^+ + 2d^*) = \mathscr{H}^1(M)$ note first that the inclusion "$\supseteq$" is obvious since a form $\omega$ is harmonic if and only if it satisfies $d\omega = d^*\omega = 0$. To show the converse inclusion note the following for $\omega \in \Omega^1(M)$

$$\begin{aligned} 0 = \int_M d(\omega \wedge d\omega) &= \int_M d\omega \wedge d\omega = \int_M d\omega \wedge (**d\omega) \\ &= (d\omega \mid *d\omega) = (d^+\omega + d^-\omega \mid d^+\omega - d^-\omega) \\ &= \|d^+\omega\|^2 - \|d^-\omega\|^2 + (d^-\omega \mid d^+\omega) - (d^+\omega \mid d^-\omega) \\ &= \|d^+\omega\|^2 - \|d^-\omega\|^2, \end{aligned}$$

thus $d^+\omega = 0$ if and only if $d^-\omega = 0$. If $\omega \in \ker(d^+ + 2d^*)$ we must have $\omega \in \ker d^+$ and $\omega \in \ker(2d^*) = \ker(d^*)$ (since the two operators map into different spaces), and then by the calculations above, $\omega \in \ker d^+$ implies $\omega \in \ker d$, thus $\omega$ is harmonic.

By the theory of elliptic complexes the index of the operator $d^+ + 2d^*$ originating from the complex $\Omega^*$ is just the Euler characteristic of this complex, i.e. the alternating sum of the dimensions of the cohomology groups. First, the zeroth cohomology $H^0(\Omega^*)$ is just $\ker d$, and since $M$ is, by assumption, connected, we know that $\ker d$ is just the constant functions on $M$, i.e. $H^0(\Omega^*) = \ker d \cong \mathbb{R}$ and thus $\dim H^0(\Omega^*) = 1$. The cohomology group $H^1(\Omega^*) = \ker d^+ / \operatorname{im} d$ is by a generalized version of the Hodge decomposition theorem isomorphic to $\ker(d^+) \cap \ker(2d^*) = \ker(d^+ + 2d^*) = \ker(d^+ + d^*) = \mathscr{H}^1(M)$, and thus $\dim H^1(\Omega^*) = b_1$.

Finally, let's calculate $H^2(\Omega^*)$. Take $\beta \in \Omega^2_+(M)$, then by Hodge decomposition

$$\beta = h + d\alpha_1 + *d\alpha_2$$

(to be perfectly honest, Hodge says that $\beta = h + d\alpha_1 + d^*\widetilde{\alpha}_2$ but then we can just take $\alpha_2 := - * \widetilde{\alpha}_2$). We act by $*$ to get

$$\beta = *h + d\alpha_2 + *d\alpha_1.$$

By uniqueness of the Hodge decomposition we get $*h = h$ and $d\alpha_1 = d\alpha_2$ and therefore

$$\beta = h + d\alpha_1 + *d\alpha_1 = h + \tfrac{1}{2}(1 + *)d(2\alpha_1) = h + d^+(2\alpha_1).$$

This formula states that any self-dual 2-form, $\beta$ is cohomologous to a self-dual harmonic 2-form $h$ and hence that the natural map

$$\mathscr{H}^2_+(M) \longrightarrow H^2(\Omega^*) = \Omega^2_+(M) / \operatorname{im} d^+ \tag{2.21}$$

(the composition of the inclusion into $\Omega^2_+(M)$ and of the quotient map) is surjective. Injectivity is assured by the uniqueness of the Hodge decomposition. Thus $\dim H^2(\Omega^*) = b_2^+$ and the alternating sum of the Betti numbers gives the stated index. □

**Proposition 2.23.** *Consider the Seiberg-Witten equations perturbed by* $\eta \in L^2_k(i\Lambda^2_+T^*M)$ *where* $k \geq 4$. *For every solution* $\mathsf{C} \in \mathcal{S}_3(\eta)$ *there exists a gauge transformation* $\lambda \in \mathcal{G}_4$ *such that* $\mathsf{C}\cdot\lambda \in \mathcal{S}_{k+1}$. *If* $\mathsf{C}$ *happens to be in* $\mathcal{S}_m(\eta) \subseteq \mathcal{S}_3(\eta)$ *for some* $m \leq k$ *then we can choose* $\lambda \in \mathcal{G}_{m+1}$. *In particular, if* $\eta$ *is a smooth 2-form* (*e.g. if* $\eta = 0$), *any solution in* $\mathcal{S}_3(\eta)$ *is gauge equivalent to a smooth solution.*

Proof. First assume $(\varphi, \mathcal{B}) \in \mathcal{S}_3(\eta)$. $\mathfrak{A}_3$ is affine, translated by $\mathcal{A}_0$, and thus

$$ib := \mathcal{B} - \mathcal{A}_0$$

is in $L^2_3(iT^*M)$. Hodge decomposition states that

$$L^2_3(T^*M) = \mathscr{H}^1(M) \oplus d(L^2_4(M)) \oplus d^*(L^2_4(\Lambda^2T^*M))$$

and therefore we can write $b = b_0 + df + d^*\beta$ for some $b_0 \in \mathscr{H}^1(M)$, $f \in L^2_4(M)$ and $\beta \in L^2_4(\Lambda^2T^*M)$ ($f$ and $\beta$, however, need not be uniquely determined). Put

$$\lambda := \exp(-\tfrac{i}{2}f) \in \mathcal{G}_4 \qquad \text{and} \qquad \alpha := ib_0 + id^*\beta.$$

As $b_0$ is harmonic and since $d^* \circ d^* = 0$, we have $d^*\alpha = 0$. Now, put $(\psi, \mathcal{A}) := (\varphi, \mathcal{B}) \cdot \lambda$, i.e. $\psi = \lambda^{-1}\varphi$ and

$$\begin{aligned}\mathcal{A} &= \mathcal{B} + 2\exp(\tfrac{i}{2}f)d\exp(-\tfrac{i}{2}f) = \mathcal{B} + 2\exp(\tfrac{i}{2}f)\exp(-\tfrac{i}{2}f)(-\tfrac{i}{2})df \\ &= \mathcal{B} - idf = \mathcal{A}_0 + ib - idf = \mathcal{A}_0 + ib_0 + id^*\beta = \mathcal{A}_0 + \alpha.\end{aligned}$$

Because we have $\not{D}_{\mathcal{A}}\psi = \not{D}_{\mathcal{A}_0}\psi + \frac{1}{2}ia\cdot\psi$ and $F_{\mathcal{A}} = F_{\mathcal{A}_0} + d\alpha$ we get the following slightly modified version of the Seiberg-Witten equations for $(\psi, \mathcal{A})$:

$$\not{D}_{\mathcal{A}_0}\psi = -\frac{1}{2}\alpha\cdot\psi, \tag{2.22a}$$

$$d^+\alpha = q(\psi) - \eta - F^+_{\mathcal{A}_0}. \tag{2.22b}$$

Since $(\varphi, \mathcal{B}) \in \mathcal{C}_3$ and $\lambda \in \mathcal{G}_4$ we have $(\psi, \mathcal{A}) = (\varphi, \mathcal{B}) \cdot \lambda \in \mathcal{C}_3$, in particular we have $\psi \in L^2_3(S^+)$ and $\alpha \in L^2_3(iT^*M)$. By the Sobolev multiplication $L^2_3 \times L^2_3 \longrightarrow L^2_3$ we obtain that $\frac{1}{2}\alpha\cdot\psi$ is in $L^2_3(S^+)$ as well. By elliptic regularity of $\not{D}_{\mathcal{A}_0}$ we then get $\psi \in L^2_4(S^+)$.

We can to the left-hand side of the second equation add the term $2d^*\alpha = 0$ thus

$$(d^+ + 2d^*)\alpha = q(\psi) - \eta - F^+_{\mathcal{A}_0} \tag{2.23}$$

and the right-hand side is in $L^2_3(i\Lambda^2_+T^*M)$ since $q(\psi)$ is in $L^2_3(i\Lambda^2_+T^*M)$ and $F^+_{\mathcal{A}_0}$ is smooth. By (2.23) we have $(d^+ + 2d^*)\alpha \in L^2_3(i\Lambda^2_+T^*M)$ and since the operator $d^+ + 2d^*$ is elliptic (Lemma 2.22) we get $\alpha \in L^2_4(iT^*M)$. This in turn implies that $\frac{1}{2}\alpha\cdot\psi$ is in $L^2_4(S^+)$ which then implies (by (2.22a)) that $\psi \in L^2_5(S^+)$ which then implies that $q(\psi)$ (and thus also the right-hand side of (2.22b)) is in $L^2_4(i\Lambda^2_+T^*M)$ which implies that $\alpha \in L^2_5(iT^*M)$ and so on. We can continue this process until $q(\psi)$ reaches the regularity of $\eta$, which happens exactly when $\psi \in L^2_{k+1}(S^+)$. Elliptic regularity on (2.22b) then says that also $\alpha \in L^2_{k+1}(iT^*M)$ and hence that $(\psi, \mathcal{A}) \in \mathcal{C}_{k+1}$. If $\eta$ happens to be smooth we can continue this bootstrapping process indefinitely and by Sobolev embedding, $(\psi, \mathcal{A})$ must be a smooth configuration. □

## 2.3 Topology of the Moduli Space

In this section we will equip $\mathcal{C}_k$ and $\mathcal{M}_k$ with natural topologies (induced by metrics) and we will show that $\mathcal{M}_k$ in this topology is a compact Hausdorff space. In the section to follow we will investigate the local structure of the moduli space and eventually realize, that it is a smooth compact finite-dimensional(!) manifold.

It is not hard to guess which metric we want to equip $\mathcal{C}_k$ with, namely the following

$$\overline{d}_k\big((\psi, \mathcal{A}), (\psi', \mathcal{A}')\big) := (\|\psi - \psi'\|^2_{L^2_k} + \|\mathcal{A} - \mathcal{A}'\|^2_{L^2_k})^{\frac{1}{2}}.$$

In order to show that this metric descends to a metric on the quotient $\mathcal{B}_k$ we will need the following technical lemma

**Lemma 2.24.** *Assume $k \geq 3$ and $\mathcal{C}_k \ni (\psi_n, \mathcal{A}_n) \to (\psi, \mathcal{A})$ and $\mathcal{C}_k \ni (\psi'_n, \mathcal{A}'_n) \to (\psi', \mathcal{A}')$ (in the metric $\overline{d}_k$, of course) and that there exists $\lambda_n \in \mathcal{G}_{k+1}$ such that $(\psi'_n, \mathcal{A}'_n) = (\psi_n, \mathcal{A}_n) \cdot \lambda_n$. Then $\lambda_n$ has a convergent subsequence $\lambda_{n_k} \to \lambda$ and $(\psi', \mathcal{A}') = (\psi, \mathcal{A}) \cdot \lambda$.*

Proof. By definition of the action we see that

$$\frac{1}{2}\lambda_n(\mathcal{A}'_n - \mathcal{A}_n) = d\lambda_n. \tag{2.24}$$

Since $M$ is compact and $|\lambda_n(x)| = 1$ we obviously have $\lambda_n \in L^p(M)$ for all $p \geq 1$, in particular $\lambda_n \in L^{k+2}(M)$. Since $\mathcal{A}'_n - \mathcal{A}_n \in L^2_k(iT^*M)$ and since we have a Sobolev product $L^{k+2} \times L^2_k \longrightarrow L^{k+2}$ we see that $\lambda_n(\mathcal{A}'_n - \mathcal{A}_n) \in L^{k+2}(iT^*M)$. As $d\lambda_n = (d+d^*)\lambda_n$ and as $d+d^*$ is elliptic, elliptic regularity gives $\lambda_n \in L^{k+2}_1(M)$. The elliptic estimate for the sequence $(\lambda_n)$ says that

$$\begin{aligned}\|\lambda_n\|_{L^{k+2}_1} &\leq C(\|d\lambda_n\|_{L^{k+2}} + \|\lambda_n\|_{L^{k+2}}) \\ &= C(\|\tfrac{1}{2}\lambda_n(\mathcal{A}'_n - \mathcal{A}_n)\|_{L^{k+2}} + \|\lambda_n\|_{L^{k+2}}).\end{aligned}$$

$(\lambda_n)$ is $L^{k+2}$-bounded and $(\mathcal{A}'_n - \mathcal{A}_n)$ is $L^2_k$-bounded (the last one because it converges to $\mathcal{A}' - \mathcal{A}$) and by the continuous Sobolev multiplication $L^{k+2} \times L^2_k \longrightarrow L^{k+2}$ we get that $\frac{1}{2}\lambda_n(\mathcal{A}'_n - \mathcal{A}_n)$ is $L^{k+2}$-bounded. Consequently, by the estimate above, $(\lambda_n)$ is $L^{k+2}_1$-bounded.

By the Rellich lemma the embedding $L^{k+2}_1(M) \hookrightarrow L^{k+2}(M)$ is compact, hence the $L^{k+2}_1$-bounded sequence $(\lambda_n)$ has an $L^{k+2}$-convergent subsequence $\lambda_{n_k}$. This is our desired subsequence. However, $L^{k+2}$-convergence is not sufficient, we need it to converge in $\mathcal{G}_{k+1} = L^2_{k+1}(M, \mathrm{U}(1))$. By (2.24) and Sobolev multiplication we see that $d\lambda_{n_k}$ converges in $L^{k+1}$ and by the elliptic estimate we see that $\lambda_{n_k}$ converges in $L^{k+1}_1$. Thus by "elliptic bootstrapping" we obtain $L^2_{k+1}$-convergence of $(\lambda_{n_k})$. Since the action of $\mathcal{G}_{k+1}$ is differentiable and thus in particular continuous we get

$$(\psi', \mathcal{A}') = \lim_{k\to\infty}(\psi'_{n_k}, \mathcal{A}'_{n_k}) = \lim_{k\to\infty}(\psi_{n_k}, \mathcal{A}_{n_k}) \cdot \lambda_{n_k} = (\psi, \mathcal{A}) \cdot \lambda$$

and this proves the lemma. □

Now define a metric $d_k$ on $\mathcal{B}_k = \mathcal{C}_k/\mathcal{G}_{k+1}$ by

$$d_k([\psi, \mathcal{A}], [\psi', \mathcal{A}']) := \inf\{\overline{d}_k\big((\psi, \mathcal{A}) \cdot \lambda, (\psi', \mathcal{A}')\big) \mid \lambda \in \mathcal{G}_{k+1}\}. \tag{2.25}$$

The only condition which is nontrivial to check is that it is positive definit. For this we need the lemma above. Assume $d_k([\psi, \mathcal{A}], [\psi', \mathcal{A}']) = 0$, then by definition of $d_k$ we can find a sequence $\lambda_n \in \mathcal{G}_{k+1}$ such that

$$\overline{d}_k\big((\psi, \mathcal{A}) \cdot \lambda_n, (\psi', \mathcal{A}')\big) \to 0.$$

Lemma 2.24 applied to the situation where $(\psi_n, \mathcal{A}_n) := (\psi, \mathcal{A})$ and $(\psi'_n, \mathcal{A}'_n) := (\psi, \mathcal{A}) \cdot \lambda_n$ yields the existence of a converging subsequence $\lambda_{n_k} \to \lambda \in \mathcal{G}_{k+1}$ such that $(\psi', \mathcal{A}') = (\psi, \mathcal{A}) \cdot \lambda$. But this just means that $[\psi, \mathcal{A}] = [\psi', \mathcal{A}']$. Thus we have shown:

**Theorem 2.25.** *The pair $(\mathcal{B}_k, d_k)$ is a metric space, in particular its topology is Hausdorff and paracompact. Moreover, the metric topology equals the quotient topology induced from $\mathcal{C}_k$.*

Since the moduli space $\mathcal{M}_k(\eta)$ is a subset of $\mathcal{B}_k$ this is a metric space as well. But for this space we can do even better, in fact it is a *compact* metric space. One of the keys to the proof is the estimate in the next lemma, the proof of which relies on the 2nd Bochner-Lichnerowicz identity:

**Lemma 2.26.** *Let $\eta \in L^2_k(\Lambda^2_+T^*M)$ where $k$ is at least 4 and assume $\mathsf{C} = (\psi, \mathcal{A}) \in \mathcal{S}_5(\eta)$, then*

$$\|\psi\|^2_\infty \leq \max\{0, -\min_{x\in M}\kappa(x) + 4\|\eta\|_\infty\} \tag{2.26}$$

*where $\kappa$ is the scalar curvature of $M$ and $\|\cdot\|_\infty$ is the supremum norm.*

Proof. First, note that by Sobolev embedding $\psi$ and $\mathcal{A}$ are twice continuously differentiable and $\eta$ is continuous, thus it makes sense to talk about the supremum norm. Letting $\Delta_M$ denote the Laplace-Beltrami operator on $M$ we get by application of first the Kato inequality (1.26) followed by the second Bochner-Lichnerowicz identity and finally the Seiberg-Witten equations that

$$\begin{aligned}\Delta_M|\psi|^2(x) &\leq 2\langle(\nabla^{\mathcal{A}})^*\nabla^{\mathcal{A}}\psi,\psi\rangle_x\\ &= 2\langle \not{D}_{\mathcal{A}}\not{D}_{\mathcal{A}}\psi,\psi\rangle - \frac{1}{2}\kappa(x)|\psi(x)|^2 - \langle F_{\mathcal{A}}\cdot\psi,\psi\rangle_x\\ &= -\frac{1}{2}\kappa(x)|\psi(x)|^2 - \langle q(\psi)\cdot\psi,\psi\rangle_x + \langle\eta\cdot\psi,\psi\rangle_x.\end{aligned}$$

First, let's deal with the term $\langle q(\psi)\cdot\psi,\psi\rangle_x$. A 2-form acts on a spinor by the Clifford action of the Clifford elements corresponding to the 2-form under the quantization map. In other works $q(\psi)\cdot\psi = \rho_4\circ Q(q(\psi))\psi$ and $\rho_4\circ Q(q(\psi))$ is well-known to us. By plugging in the definition of $\rho_4\circ Q(q(\psi))$ we get

$$\langle q(\psi)\cdot\psi,\psi\rangle_x = \langle\rho_4\circ Q(q(\psi))\psi,\psi\rangle_x = \frac{1}{2}|\psi(x)|^4.$$

Furthermore, it is well-known that the operator norm of $\eta_x\cdot$ is $2|\eta_x|$ and hence

$$(\Delta_M|\psi|^2)(x) \leq -\frac{1}{2}\kappa(x)|\psi(x)|^2 - \frac{1}{2}|\psi(x)|^4 + 2\|\eta\|_\infty|\psi(x)|^2.$$

Put $u(x) := |\psi(x)|^2$. Since $\psi$ is $C^2$, so is $u$ and $u$ satisfies the differential *inequality*

$$\Delta_M u + \tfrac{1}{2}u(u+\kappa-4\|\eta\|_\infty) \leq 0.$$

If $x_0$ is a local maximum for $u$ then $\Delta_M u(x_0) \geq 0$ (for locally $\Delta_M$ is just $-\partial/\partial x_1 - \cdots - \partial/\partial x_4$ and if $x_0$ is a maximum, the double derivatives are negative) and hence if the inequality should be satisfied we must have

$$\tfrac{1}{2}u(x_0)(u(x_0)+\kappa(x_0)-4\|\eta\|_\infty) \leq 0.$$

This gives the desired inequality. □

The key estimate in this lemma has the following immediate corollary:

**Corollary 2.27.** *Assume that the scalar curvature $\kappa$ of $M$ is non-negative, and that $\eta \in L^2_4(i\Lambda^2_+T^*M)$ is such that $\|\eta\|_\infty \leq \frac{1}{4}\min_{x\in M}\kappa(x)$, then any Seiberg-Witten monopole $\mathsf{C}\in\mathcal{S}_5(\eta)$ is reducible.*

We are now able to prove compactness of the moduli space.

**Theorem 2.28.** *Let $k \geq 3$ be a fixed integer and $m = \max\{k-1,4\}$ as well as $\eta\in L^2_m(i\Lambda^2_+T^*M)$, then the moduli space $\mathcal{M}_k(\eta)$ is a compact metric space.*

Proof. We show it only in the case where $k = 3$. Assume $[\mathsf{C}_n]$ is a bounded sequence in $\mathcal{M}_3(\eta)$. If we can find a $\mathsf{C}\in\mathcal{S}_3$ and a sequence $\lambda_n\in\mathcal{G}_4$ such that there exists a subsequence $(\mathsf{C}_{n_k}\cdot\lambda_{n_k})$ for which $\lim_{k\to\infty}\overline{d}_3(\mathsf{C}_{n_k}\cdot\lambda_{n_k},\mathsf{C}) = 0$ for $k\to\infty$, then

$$d_3([\mathsf{C}_{n_k}],[\mathsf{C}]) = \inf_{\lambda\in\mathcal{G}_4}\overline{d}_3(\mathsf{C}_{n_k}\cdot\lambda,\mathsf{C}) \leq \overline{d}_3(\mathsf{C}_{n_k}\cdot\lambda_{n_k},\mathsf{C}) \to 0,$$

and we would have proved that $[\mathsf{C}_{n_k}]$ is a convergent subsequence of $[\mathsf{C}_n]$ and hence that the moduli space is compact.

We let $H^k(M;R)$ denote the $k$'th singular cohomology group of $M$ with coefficients in a certain commutative ring $R$. Consider the string of maps

$$H^k(M;\mathbb{Z}) \longrightarrow H^k(M;\mathbb{R}) \xrightarrow{\sim} \mathscr{H}^k(M)$$

where the first map is just tensoring with $\mathbb{R}$ and the second is the isomorphism given by the de Rham Theorem and Hodge Decomposition. The first maps kills all torsion in $H^k(M;\mathbb{Z})$ and maps a $\mathbb{Z}$-basis for the free part of $H^k(M;\mathbb{Z})$ to an $\mathbb{R}$-basis for $H^k(M;\mathbb{R})$. Thus the image of $H^k(M;\mathbb{Z})$ inside $\mathscr{H}^k(M)$ consists of integer linear combinations of a certain basis. This is called a *lattice* in $\mathscr{H}^k(M)$ and is denoted by $\mathscr{H}^k(M;\mathbb{Z})$. We will also need the lattice $4\pi\mathscr{H}^k(M;\mathbb{Z})$ (which is just the set of linear combinations with coefficients in $4\pi\mathbb{Z}$). Define $\delta_k$ to be the greatest possible $L^2$-distance that any point in $\mathscr{H}^k(M)$ can have to the lattice $\mathscr{H}^k(M;\mathbb{Z})$. Formally

$$\delta_k := \sup_u \inf_v \{\|u-v\|_{L^2} \mid u \in \mathscr{H}^k(M),\ v \in \mathscr{H}^k(M;\mathbb{Z})\}.$$

Obviously, the corresponding number for the lattice $4\pi\mathscr{H}^k(M;\mathbb{Z})$ is just $4\pi\delta_k$.

We have $\mathsf{C}_n = (\psi_n, \mathcal{A}_n) \in \mathcal{S}_3(\eta)$ and $\eta \in L^2_m(i\Lambda^2_+ T^*M)$ where $m$ is at least 4. By Proposition 2.23 we can boost regularity of $\mathsf{C}_n$ by a gauge transformation $\lambda_n \in \mathcal{G}_4$ such that $\mathsf{C}_n \cdot \lambda_n \in \mathcal{S}_5(\eta)$. Since we only care about the $\mathsf{C}_n$'s up to gauge equivalence, we might as well assume from now on, that $\mathsf{C}_n \in \mathcal{S}_5(\eta)$. In particular $\psi_n$ and $\mathcal{A}_n$ will be two times continuously differentiable.

Write $\mathcal{A}_n = \mathcal{A}_0 + ia_n$ where $a_n \in L^2_5(T^*M)$. Again we use the Hodge decomposition to get

$$a_n = h_n + df_n + d^*\beta_n$$

for $h_n \in \mathscr{H}^1(M)$, $f_n \in L^2_6(M)$ and $\beta_n \in L^2_6(\Lambda^2 T^*M)$. Since the maximal distance to the lattice $4\pi\mathscr{H}^1(M;\mathbb{Z})$ was $4\pi\delta_1$, we can find $\chi_n \in 4\pi\mathscr{H}^1(M;\mathbb{Z})$ such that $\|h_n + \chi_n\|_{L^2} \leq 4\pi\delta_1$.

We will show that there exists $\lambda_n \in \mathcal{G}$ such that $i\chi_n = 2\lambda_n^{-1}d\lambda_n$. Let $\varphi : \widetilde{M} \longrightarrow M$ denote the universal cover of $M$ (this exists since $M$ is assumed to be connected) and put $\widetilde{\chi}_n := \varphi^*\chi_n$. This is a 1-form on $\widetilde{M}$ and can thus be integrated along curves. Let $\widetilde{m}_0 \in \widetilde{M}$ be a fixed point and define a function $f_n : \widetilde{M} \longrightarrow \mathbb{R}$ by

$$\widetilde{f}_n(\widetilde{m}) := \int_c \widetilde{\chi}_n$$

where $c : [0,1] \longrightarrow \widetilde{M}$ is an arbitrary curve from $\widetilde{m}_0$ to $\widetilde{m}$. Obviously, $\widetilde{f}_n$ depends on $\widetilde{m}_0$ but *not* on the choice of curve (this is because $\widetilde{M}$ is simply connected so the integral along a closed curve is always 0). Now define $\widetilde{\lambda}_n : \widetilde{M} \longrightarrow \mathrm{U}(1)$ by

$$\widetilde{\lambda}_n(\widetilde{m}) := \exp(\tfrac{i}{2}\widetilde{f}_n(\widetilde{m})).$$

This induces a map $\lambda_n : M \longrightarrow \mathrm{U}(1)$ by $\lambda_n(m) = \widetilde{\lambda}_n(\widetilde{m})$ where $\widetilde{m}$ is some arbitrary element in the fiber $\varphi^{-1}(m)$. This is well-defined: Assume $\varphi(\widetilde{m}_1) = \varphi(\widetilde{m}_2) = m$, let $c_1$ be a curve from $\widetilde{m}_0$ to $\widetilde{m}_1$ and $c_2$ a curve from $\widetilde{m}_0$ to $\widetilde{m}_2$ and let $c$ be the composite curve, traversing $c_1$ backwards from $\widetilde{m}_1$ to $\widetilde{m}_0$ and traversing $c_2$ from $\widetilde{m}_0$ to $\widetilde{m}_2$. Then

$$\begin{aligned}\widetilde{f}_n(\widetilde{m}_2) - \widetilde{f}_n(\widetilde{m}_1) &= \int_{c_1} \widetilde{\chi}_n - \int_{c_2} \widetilde{\chi}_n = \int_c \widetilde{\chi}_n = \int_0^1 c^*\widetilde{\chi}_n \\ &= \int_0^1 (\varphi \circ c)^*\chi_n = \int_{\varphi\circ c} \chi_n.\end{aligned}$$

$\varphi \circ c$ is a closed curve, i.e. a singular 1-cycle and thus represents a homology class in $H_1(M;\mathbb{Z})$. $\chi_n$ can be identified with an element in $4\pi H^1(M;\mathbb{Z})$ and the identification is (by the theorem of de Rham) that it should act on cycles by integration:

$$\int_{\varphi\circ c} \chi_n = [\chi_n]([\varphi \circ c])$$

and this is an element of $4\pi\mathbb{Z}$. Thus $\widetilde{\lambda}_n(\widetilde{m}_1) = \widetilde{\lambda}_n(\widetilde{m}_2)$, hence $\lambda_n$ is well-defined.

Furthermore, one can calculate

$$2\widetilde{\lambda}_n^{-1} d\widetilde{\lambda}_n = id\widetilde{f}_n = i\widetilde{\chi}_n. \tag{2.27}$$

Since $\widetilde{\lambda}_n = \lambda_n \circ \varphi$ we get $d\widetilde{\lambda}_n(X) = d\lambda_n(d\varphi(X)) = \varphi^*(d\lambda_n)(X)$. Therefore (2.27) may be written as

$$\varphi^*(i\chi_n) = \varphi^*(2\lambda_n^{-1} d\lambda_n). \tag{2.28}$$

For a smooth covering map $\varphi$, the differential $d\varphi_{\widetilde{m}} : T_{\widetilde{m}}\widetilde{M} \longrightarrow T_{\varphi(\widetilde{m})}M$ is a bijection, and hence also the dual map $\varphi^* : T^*_{\varphi(\widetilde{m})}M \longrightarrow T^*_{\widetilde{m}}\widetilde{M}$ is a bijection and by (2.28) we learn that

$$i\chi_n = 2\lambda_n^{-1} d\lambda_n$$

as desired.

Let $P : L^2(T^*M) \longrightarrow L^2(T^*M)$ denote the orthogonal projection onto $\mathscr{H}^1(M)$. Define

$$\begin{aligned}
\mathsf{C}'_n := \mathsf{C}_n \cdot e^{-\frac{i}{2}f_n}\lambda_n &= (\psi'_n, \mathcal{A}_n + 2\lambda_n^{-1}d\lambda_n + 2d(e^{-\frac{i}{2}f_n})e^{\frac{i}{2}f_n}) \\
&= (\psi'_n, \mathcal{A}_n + i\chi_n - idf_n) \\
&= (\psi'_n, \mathcal{A}_0 + ih_n + i\chi_n + idf_n - idf_n + id^*\beta_n) \\
&= (\psi'_n, \mathcal{A}_0 + i(h_n + \chi_n) + id^*\beta_n).
\end{aligned}$$

Since $h_n \in \mathscr{H}^1(M)$ and $\chi_n \in \mathscr{H}^1(M,\mathbb{Z}) \subseteq \mathscr{H}^1(M)$ we have $h_n + \chi_n \in \mathscr{H}^1(M)$ and by assumption $\|h_n + \chi_n\|_{L^2} \leq 4\pi\delta$. If we put $a'_n := h_n + \chi_n + d^*\beta_n$ we see

$$2d^*a'_n = 0 \qquad \text{and} \qquad \|Pa'_n\|_{L^2} \leq 4\pi\delta.$$

Since $\mathsf{C}_n$ and $\mathsf{C}'_n$ are related by an element in $\mathcal{G}$, they represent the same class in the moduli space $\mathcal{M}_2(\eta)$, so from the beginning we could just as well have chosen $\mathsf{C}'_n$ instead of $\mathsf{C}_n$. Hence our achievements thus far can be summarized as follows: there exists a sequence $\mathsf{C}_n = (\psi_n, \mathcal{A}_n) = (\psi_n, \mathcal{A}_0 + ia_n) \in \mathcal{S}_5(\eta)$ of representatives for the sequence in the moduli space which satisfy the following

$$\not{D}_{\mathcal{A}_0}\psi_n = -\frac{1}{2}ia_n \cdot \psi_n \tag{2.29a}$$

$$i(d^+ + 2d^*)a_n = q(\psi_n) - \eta - F^+_{\mathcal{A}_0} \tag{2.29b}$$

$$\|Pa_n\|_{L^2} \leq 4\pi\delta. \tag{2.29c}$$

The sequence $\|i(d^+ + 2d^*)a_n\|_\infty = \|q(\psi_n) - \eta - F^+_{\mathcal{A}_0}\|_\infty$ is bounded, simply because $|q(\psi_n)(x)| \leq |\psi_n(x)|^2 \leq \|\psi_n\|^2_\infty$ which is bounded by Lemma 2.26.

By Theorem 2.13 5) we get

$$\|a_n - Pa_n\|_{L^p_1} \leq C\|(d^+ + 2d^*)a_n\|_{L^p}$$

for all $1 < p < \infty$. Since $(d^+ + 2d^*)a_n$ is uniformly bounded and $M$ is compact, then of course $\|(d^+ + 2d^*)a_n\|_{L^p}$ is bounded and thus $\|a_n - Pa_n\|_{L^p_1}$ is bounded.

$\mathscr{H}^1(M)$ is finite-dimensional, and thus all norms on this space are equivalent. (2.29c) says that the sequence $(Pa_n)$ is bounded in the $L^2$-norm, hence the sequence is bounded in any norm. In particular it is bounded in the $L^p_1$-norm. This combined with the $L^p_1$-bound on $a_n - Pa_n$ shows that $a_n$ is $L^p_1$-bounded.

$\|a_n \cdot \psi_n\|_\infty$ is bounded for the following reason: For $p > 4$ we have a continuous Sobolev embedding $L^p_1 \hookrightarrow C^0$ and therefore, since $(a_n)$ was $L^p_1$-bounded, it is uniformly bounded. Likewise $(\psi_n)$ is uniformly bounded by Lemma 2.26 and since Clifford multiplication is continuous, $(a_n \cdot \psi_n)$ is uniformly bounded as well. Compactness of $M$ guarantees that $\frac{1}{2}\|a_n \cdot \psi_n\|_{L^p} = \|\not{D}_{\mathcal{A}_0}\psi_n\|_{L^p}$ is bounded. By the elliptic estimate we get

$$\|\psi_n\|_{L^p_1} \leq C(\|\not{D}_{\mathcal{A}_0}\psi_n\|_{L^p} + \|\psi_n\|_{L^p}).$$

and since $\|\psi_n\|_{L^p}$ is bounded by the estimate in Lemma 2.26, $\|\psi_n\|_{L^p_1}$ is bounded. Since we have a continuous Sobolev multiplication $L^p_1 \times L^p_1 \longrightarrow L^p_1$ the sequence $(\frac{1}{2}ia_n \cdot \psi_n)$ is also $L^p_1$-bounded.

Applying the elliptic estimate to (2.29a) we obtain

$$\begin{aligned}\|\psi_n\|_{L^p_2} &\leq C(\|\not{D}_{\mathcal{A}_0}\psi_n\|_{L^p_1} + \|\psi_n\|_{L^p_1}) \\ &= C(\tfrac{1}{2}\|a_n \cdot \psi_n\|_{L^p_1} + \|\psi_n\|_{L^p_1}),\end{aligned}$$

and hence that $\|\psi_n\|_{L^p_2}$ is bounded. But now we can use (2.29b) and the bound $|q(\psi)|^2 \leq |\psi|^4$ from Lemma 2.2 to conclude that $\|(d^+ + 2d^*)a_n\|_{L^p_1}$ is bounded and it follows from Theorem 2.13 5) that $\|a_n - Pa_n\|_{L^p_2}$ is bounded. Since $\|Pa_n\|_{L^p_2}$ is bounded (recall that this sequence in $\mathscr{H}^1(M)$ was bounded in any norm) we have that $\|a_n\|_{L^p_2}$ is bounded.

We can take one more turn in the merry-go-round and show that the sequences $(a_n)$ and $(\psi_n)$ are actually $L^p_3$-bounded. Because of the Sobolev multiplication $(\frac{1}{2}ia_n \cdot \psi_n)$ is $L^p_2$-bounded and then by the elliptic estimate we get that

$$\begin{aligned}\|\psi_n\|_{L^p_3} &\leq C(\|\not{D}_{\mathcal{A}_0}\psi_n\|_{L^p_2} + \|\psi_n\|_{L^p_2}) \\ &= C(\tfrac{1}{2}\|a_n \cdot \psi_n\|_{L^p_2} + \|\psi_n\|_{L^p_2}),\end{aligned}$$

thus $(\psi_n)$ is $L^p_3$-bounded. But then it follows from (2.29b) that $i(d^+ + 2d^*)a_n$ is $L^p_2$-bounded and by Theorem 2.13 5) it once again follows that $(a_n - Pa_n)$ is $L^p_3$-bounded, and since $(Pa_n)$ was bounded in any norm, we must have that $(a_n)$ is $L^p_3$-bounded. At this point it should be obvious that we could in principle continue this process indefinitely to obtain boundedness in any Sobolev norm $L^p_m$.

We have a compact inclusion $L^p_4 \hookrightarrow L^p_3$ (the Rellich lemma) and hence the $L^p_4$-bounded sequences $(a_n)$ and $(\psi_n)$ will have $L^p_3$-convergent subsequences. In other words, there is a subsequence $\mathsf{C}_{n_\ell}$ which is convergent in $\mathcal{C}_3(\eta)$. Since the solution space $\mathcal{S}_3(\eta)$ is closed, this subsequence converges inside $\mathcal{S}_3(\eta)$ and this proves the statement. □

This proves compactness of the "Sobolev" moduli spaces $\mathcal{M}_k(\eta)$. But what about the smooth moduli space $\mathcal{M}(\eta)$? Well, first of all we have to give it a topology which is somehow compatible with the topologies on $\mathcal{M}_k(\eta)$ and to accomplish this we have to investigate how the different moduli spaces are related.

Assume $\eta$ is a smooth perturbation parameter. If we let $q_k : \mathcal{S}^*_k(\eta) \longrightarrow \mathcal{M}^*_k(\eta)$ denote the quotient map, we can define $\Phi_k : \mathcal{M}_k(\eta) \longrightarrow \mathcal{M}_{k-1}(\eta)$ by the commuting diagram

$$\begin{tikzcd}
\mathcal{S}_k(\eta) \arrow[r, hook, "\iota"] \arrow[d, "q_k"'] & \mathcal{S}_{k-1}(\eta) \arrow[d, "q_{k-1}"] \\
\mathcal{M}_k(\eta) \arrow[r, "\Phi_k"'] & \mathcal{M}_{k-1}(\eta)
\end{tikzcd}$$

i.e. $\Phi_k([\mathsf{C}]_k) = [\mathsf{C}]_{k-1}$. It is easy to see that it is well-defined: If $[\mathsf{C}]_k = [\mathsf{C}']_k$ then $\mathsf{C}' = \mathsf{C} \cdot \lambda$ for some $\lambda \in \mathcal{G}_{k+1} \subseteq \mathcal{G}_k$, so $[\mathsf{C}]_{k-1} = [\mathsf{C}']_{k-1}$. It is surjective: Given $\mathsf{C} \in \mathcal{S}_{k-1}(\eta)$ we can, according to Proposition 2.23, for any $\mathsf{C} \in \mathcal{S}_{k-1}(\eta)$ find

$\lambda \in \mathcal{G}_k$ such that $\mathsf{C}\cdot\lambda \in \mathcal{S}_k(\eta)$, i.e. $[\mathsf{C}]_{k-1} = \Phi_k([\mathsf{C}\cdot\lambda]_k)$. Injectivity: If $\Phi_k([\mathsf{C}]_k) = \Phi_k([\mathsf{C}']_k)$, i.e. if $[\mathsf{C}]_{k-1} = [\mathsf{C}']_{k-1}$, there exists $\lambda \in \mathcal{G}_k$, such that $\mathsf{C}' = \mathsf{C}\cdot\lambda$. Let $\mathsf{C} = (\psi, \mathcal{A})$ and $\mathsf{C}' = (\psi', \mathcal{A}')$ then we have $\mathcal{A}' - \mathcal{A} = 2\lambda^{-1}d\lambda$. The right-hand side can be written locally as $d(\log\lambda)$ and since $\mathcal{A}' - \mathcal{A} \in L^2_k(iT^*M)$ it follows from elliptic regularity that $\lambda$ is in $L^2_{k+1}$. Being an element of a Sobolev space over a compact manifold is a local property, and hence $\lambda \in \mathcal{G}_{k+1}$, i.e. $[\mathsf{C}]_k = [\mathsf{C}']_k$ and thus $\Phi_k$ is injective. By the properties of the quotient topology, $\Phi_k$ is also continuous, and since $\mathcal{M}_k(\eta)$ is compact, it is automatically a homeomorphism. Thus we have shown

**Corollary 2.29.** *The moduli spaces $\mathcal{M}_k(\eta)$ are mutually homeomorphic and a homeomorphism $\Phi_k : \mathcal{M}_k(\eta) \longrightarrow \mathcal{M}_{k-1}(\eta)$ is given by the commuting diagram above.*

In the same way we can define a map $\Phi : \mathcal{M}(\eta) \longrightarrow \mathcal{M}_3(\eta)$ (or indeed into $\mathcal{M}_k(\eta)$ for any $k \geq 3$) and use Proposition 2.23 to show that this is a bijection. We can then give $\mathcal{M}(\eta)$ the unique topology which turns $\Phi$ into a homeomorphism. Thus we have

**Corollary 2.30.** *The moduli space $\mathcal{M}(\eta)$ has a natural topology in which the space is compact and homeomorphic to $\mathcal{M}_k(\eta)$ for any $k$.*

## 2.4 Local Structure of the Moduli Space

We begin by showing that $\mathcal{B}^*_k$ is a smooth manifold. This space was the quotient space of $\mathcal{C}^*_k$ under the action of the Hilbert-Lie group $\mathcal{G}_{k+1}$, and it is a well-known fact that the quotient of a manifold by the action of a Lie group is a manifold if the action is smooth, free and proper [2]. We have already shown smoothness (Lemma 2.19) and by definition of $\mathcal{B}^*_k$ the action is free. So it only remains to show that the action is proper. This is a consequence of the following lemma:

**Lemma 2.31** (**Local Slice Lemma**)**.** *Let $\mathsf{C} = (\psi, \mathcal{A}) \in \mathcal{C}^*_k$ and consider the map $\rho^{\mathsf{C}} : \mathcal{G}_{k+1} \longrightarrow \mathcal{C}^*_k$ given by $\lambda \longmapsto \mathsf{C}\cdot\lambda$ as well as its differential at the identity of $\mathcal{G}_{k+1}$:*

$$d\rho^{\mathsf{C}}_1 : T_1\mathcal{G}_{k+1} = iL^2_{k+1}(M) \longrightarrow L^2_k(S^+ \oplus iT^*M).$$

*There exists a neighborhood $U \subseteq \mathcal{C}^*_k$ around $\mathsf{C}$ which is diffeomorphic to $V \times \mathcal{G}_{k+1}$ where $V \subseteq \ker(d\rho^{\mathsf{C}}_1)^* \subseteq L^2_k(S^+ \oplus iT^*M)$ is some open neighborhood of $0$ in $\ker(d\rho^{\mathsf{C}}_1)^*$. The diffeomorphism $H : V \times \mathcal{G}_{k+1} \longrightarrow U$ is given by*

$$H((\varphi, \alpha), \lambda) = (\psi + \varphi, \mathcal{A} + \alpha)\cdot\lambda.$$

*This map is $\mathcal{G}_{k+1}$-equivariant [3] w.r.t. the action on $U$ and the natural action on $V \times \mathcal{G}_{k+1}$.*

Proof. >From Lemma 2.19, we already know the differential of $\rho^{\mathsf{C}}$ at 1, namely

$$d\rho^{\mathsf{C}}_1(f) = (-f\psi, 2df).$$

This differential operator has closed range: First note, that its principal symbol $\sigma(d\rho^{\mathsf{C}}_1)(\xi_x) : \mathbb{R} \longrightarrow S^+_x \oplus iT^*_xM$ is given by $t \longmapsto (0, it\xi_x)$ and is thus injective.

---

[2] Recall that an action of a Lie group $G$ on a manifold $M$ is called *proper* if the map $M \times G \longrightarrow M \times M$ given by $(x, g) \longmapsto (x \cdot g, x)$ is a proper map, i.e. the preimage of compact sets is compact.

[3] If $M$ and $N$ are two manifolds carrying actions of a Lie group then a smooth map $F : M \longrightarrow N$ is called *equivariant* w.r.t. these actions if $F(x \cdot g) = F(x) \cdot g$ for all $x \in M$ and $g \in G$.

Thus the operator is *injectively elliptic*, i.e. there exists a left parametrix $Q$ such that $Q \circ d\rho_1^{\mathbb{C}} = \mathrm{id} + \mathcal{R}$ where $\mathcal{R}$ is compact. By Theorem 2.3 of [4] it has closed image.

Let $(d\rho_1^{\mathbb{C}})^*$ denote the formal adjoint of $d\rho_1^{\mathbb{C}}$. This is a first order differential operator $\Gamma(S^+ \oplus iT^*M) \to C^\infty(M)$, and hence we can extend it to $L_k^2(S^+ \oplus iT^*M)$. This extended operator is also denoted $(d\rho_1^{\mathbb{C}})^*$.

Now put $K := \ker(d\rho_1^{\mathbb{C}})^* \subseteq L_k^2(S^+ \oplus iT^*M)$. The claim is that if just $V \subseteq K$ is small enough, then the map $H : V \times \mathcal{G}_{k+1} \longrightarrow \mathcal{C}_k^*$ given by

$$H((\varphi, \alpha), \lambda) := (\psi + \varphi, \mathcal{A} + \alpha) \cdot \lambda = (\lambda^{-1}(\psi + \varphi), \mathcal{A} + \alpha + 2\lambda^{-1} d\lambda)$$

is a diffeomorphism onto its image, and we may therefore pick $U$ to be this image. In the same fashion as in the proof of Lemma 2.19 we can calculate the differential of $H$ at $((0,0),1)$

$$dH_{((0,0),1)} : K \oplus iL_{k+1}^2(M) \longrightarrow L_k^2(S^+ \oplus iT^*M)$$

and it is given by

$$dH_{((0,0),1)}((\varphi, \alpha), f) = (\varphi, \alpha) + (-f\psi, 2df).$$

Observe that this is exactly equal to $d\rho_{((\psi,\mathcal{A}),1)}$ (cf. Lemma 2.19, although the domain has changed). We want to show that this map is bijective. Since $d\rho_1^{\mathbb{C}}$ has closed range we get

$$L_k^2(S^+ \oplus iT^*M) = K \oplus \mathrm{im}(d\rho_1^{\mathbb{C}})$$

and we see that $dH_{((0,0),1)} = \mathrm{id}_K \oplus d\rho_1^{\mathbb{C}}$, and thus $dH_{((0,0),1)}$ is surjective. Why is it injective? Well we just have to show that $d\rho_1^{\mathbb{C}}$ is injective. But if

$$(0,0) = d\rho_1^{\mathbb{C}}(f) = (2df, -f\psi),$$

then $df = 0$, i.e. $f$ is constant. Since $f\psi \equiv 0$ and $\psi \not\equiv 0$ (by assumption the configuration $(\psi, \mathcal{A})$ was irreducible) we must have $f \equiv 0$.

So $H$ is a smooth map, whose differential at the point $((0,0),1)$ is bijective. By the Inverse Function Theorem, there exists a neighborhood $V \times W \subseteq K \times \mathcal{G}_{k+1}$ such that $H|_{V \times W}$ is a diffeomorphism onto its image. We want to show that we (perhaps by shrinking $V$) may choose $W$ to be the entire group $\mathcal{G}_{k+1}$. To show this, assume for contradiction that the map

$$H : \widetilde{V} \times \mathcal{G}_{k+1} \longrightarrow H(\widetilde{V} \times \mathcal{G}_{k+1})$$

is *not* bijective (i.e. not injective) no matter how small we choose an open $\widetilde{V} \subseteq V$. Let $(\widetilde{V}_n)$ be a descending sequence of open sets in $V$ such that $\bigcap \widetilde{V}_n = \{0\}$. By assumption, $H$ restricted to $\widetilde{V}_n \times \mathcal{G}_{k+1}$ is *not* injective, meaning that we can find two non-equal elements $(v_n, u_n), (v_n', u_n') \in \widetilde{V}_n \times \mathcal{G}_{k+1}$ such that

$$((\psi, \mathcal{A}) + v_n) \cdot u_n = H(v_n, u_n) = H(v_n', u_n') = ((\psi, \mathcal{A}) + v_n') \cdot u_n'.$$

By multiplying this equation by $(u_n')^{-1}$ we may just as well assume $u_n' = 1$. We now use Lemma 2.24 with $(\psi_n, \mathcal{A}_n) := (\psi, \mathcal{A}) + v_n$ and $(\psi_n', \mathcal{A}_n') := (\psi, \mathcal{A}) + v_n'$ and $\lambda_n = u_n$ to obtain a converging subsequence $u_{n_m} \to u$ such that $(\psi, \mathcal{A}) \cdot u = (\psi, \mathcal{A})$ (note that both $v_n$ and $v_n'$ must converge to 0). But as $(\psi, \mathcal{A})$ is irreducible, we must have $u = 1$, i.e. $(v_{n_m}, u_{n_m}) \to (0,1)$ and $(v_n', 1) \to (0,1)$. But then from a certain step $(v_{n_m}, u_{n_m})$ and $(v_{n_m}, 1)$ must be inside $V \times W$, but here $H$ was injective, and so we have achieved our contradiction, showing that there exists an open set $V$ such that $H|_{V \times \mathcal{G}_{k+1}}$ is bijective onto its image.

It is easily checked that the map $H$ is indeed $\mathcal{G}_{k+1}$-equivariant. □

From the Slice Lemma we easily deduce the following

**Proposition 2.32.** *The space $\mathcal{B}_k^*$ is a smooth infinite-dimensional Hilbert manifold. In fact, given $[\mathsf{C}] = [\psi, \mathcal{A}] \in \mathcal{B}_k^*$ there exists a neighborhood $\widehat{U} \subseteq \mathcal{B}_k^*$ around $[\mathsf{C}]$ and an open neighborhood $V \subseteq \ker(d\rho_1^{\mathsf{C}})^* \subseteq L_k^2(S^+ \oplus iT^*M)$ of $0$ such that the map $\widehat{H} : V \longrightarrow \widehat{U}$ given by*

$$\widehat{H}(\varphi, \alpha) = [\psi + \varphi, \mathcal{A} + \alpha] \tag{2.30}$$

*is a diffeomorphism.*

Proof. The natural $\mathcal{G}_{k+1}$-action on $V \times \mathcal{G}_{k+1}$ is proper and by equivariance of the diffeomorphism $H$, the $\mathcal{G}_{k+1}$-action on $U$ must also be proper. Being a proper action is a local property, and hence the gauge group action on $\mathcal{C}_k^*$ is proper and therefore the quotient space $\mathcal{B}_k^*$ is a manifold.

Let $[\cdot] : \mathcal{C}_k^* \longrightarrow \mathcal{B}_k^*$ denote the quotient map and put $\widehat{U} := [U]$. Since the quotient map of a smooth Lie group action is always open in the quotient topology, $\widehat{U} \subseteq \mathcal{B}_k^*$ is open. Since $H$ is equivariant and since $V$ is just the quotient of $V \times \mathcal{G}_{k+1}$ under the natural group action, we get the following commutative diagram

$$\begin{tikzcd}
V \times \mathcal{G}_{k+1} \arrow[r, "H", "\sim"'] \arrow[d, "\pi_1"'] & U \arrow[d, "{[\cdot]}"] \\
V \arrow[r, "\sim", "\widehat{H}"'] & \widehat{U}
\end{tikzcd}$$

the induced diffeomorphism $\widehat{H}$ is clearly given by (2.30). ☐

In the next lemma we will introduce an extremely important operator, or rather field of operators. It will be the cornerstone of practically everything treated in in coming sections, simply because the kernel at each point of this field of operators is the tangent space of the moduli space. In other words it is a key player in proving smoothness of the moduli space.

Before rushing to the lemma, we recall the signature $\sigma(M)$ of the manifold which we defined in Section 2.2. We also recall the *Euler characteristic* of $M$ which is just the alternating sum of the Betti numbers

$$\chi(M) = \sum_{k=0}^{4} (-1)^k b_k$$

where $b_k := \dim_{\mathbb{R}} H^k(M; \mathbb{R}) = \dim_{\mathbb{R}} \mathscr{H}^k(M)$.

**Lemma 2.33.** *Let $\mathsf{C} = (\psi, \mathcal{A}) \in \mathcal{C}_k^*$ (i.e. $\psi \neq 0$) and consider the map*

$$\mathcal{F}_{\mathsf{C}} := dF_{\mathsf{C}}^{\eta} + (d\rho_1^{\mathsf{C}})^* : L_k^2(S^+ \oplus iT^*M) \longrightarrow L_{k-1}^2(i\Lambda_+^2 T^*M \oplus S^- \oplus i\mathbb{R})$$

*given by*

$$\begin{pmatrix} \varphi \\ \alpha \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{F}_1(\varphi, \alpha) \\ \mathcal{F}_2(\varphi, \alpha) \\ \mathcal{F}_3(\varphi, \alpha) \end{pmatrix} := \begin{pmatrix} d^+\alpha - dq_\psi(\varphi) \\ \frac{1}{2}\alpha \cdot \psi + \not{D}_{\mathcal{A}}\varphi \\ 2d^*\alpha - i\,\mathrm{Im}\langle \psi, \varphi \rangle \end{pmatrix} \tag{2.31}$$

*where $\langle \psi, \varphi \rangle$ is just the fiber-wise inner product on $S^+$. The map is Fredholm and its real index is*

$$\mathrm{ind}_{\mathbb{R}}(\mathcal{F}_{\mathsf{C}}) = \frac{1}{4} c_1(L)^2([M]) - \frac{1}{2}\chi(M) - \frac{3}{4}\sigma(M).$$

*Furthermore the map $(\varphi, \alpha) \longmapsto \begin{pmatrix} \mathcal{F}_2(\varphi, \alpha) \\ \mathcal{F}_3(\varphi, \alpha) \end{pmatrix}$ is surjective.*

Proof. Note that the map

$$(\varphi,\alpha) \longmapsto \left(-dq_\psi(\varphi), \tfrac{1}{2}\alpha\cdot\psi, -i\,\mathrm{Im}\langle\psi,\varphi\rangle\right)$$

is linear and bounded as a map $L^2_k \longrightarrow L^2_k$, hence compact as a map into $L^2_{k-1}$ (because of the compact inclusion $L^2_k \hookrightarrow L^2_{k-1}$ by Rellich's Lemma). Thus we see that $\mathcal{F}_{\mathsf{C}}$ is nothing but $\not{D}_{\mathcal{A}} \oplus (d^+ + 2d^*)$ plus some compact perturbation. Hence $\mathcal{F}_{\mathsf{C}}$ is again Fredholm and its Fredholm index is the sum of the Fredholm indices of $\not{D}_{\mathcal{A}}$ and $d^+ + 2d^*$. By the Atiyah-Singer Index Theorem we get the *complex* index of the Dirac operator

$$\mathrm{ind}_{\mathbb{C}}(\not{D}_{\mathcal{A}}) = \frac{1}{8}\big(c_1(L)^2([M]) - \sigma(M)\big).$$

The index of $d^+ + 2d^*$ is known from Lemma 2.22 and adding the two (multiplying the complex index of $D_{\mathcal{A}}$ to get its real index) gives

$$\begin{aligned}\mathrm{ind}_{\mathbb{R}}(\mathcal{F}_{\mathsf{C}}) &= 2\,\mathrm{ind}_{\mathbb{C}}\,\not{D}_{\mathcal{A}} + \mathrm{ind}_{\mathbb{R}}(d^+ + 2d^*)\\ &= \tfrac{1}{4}c_1(L)^2([M]) + b_1 - 1 + \tfrac{1}{4}b_2^- - \tfrac{5}{4}b_2^+.\end{aligned}$$

We need to show that this equals $\frac{1}{4}c_1(L)^2([M]) - \frac{1}{2}\chi(M) - \frac{3}{4}\sigma(M)$. Invoking Poincaré Duality, which states that $b_{4-k} = b_k$, as well as the simple topological fact that $b_0 = b_4 = 1$ [4] we get

$$\begin{aligned}\tfrac{1}{2}\chi(M) + \tfrac{3}{4}\sigma(M) &= \tfrac{3}{4}(b_2^+ - b_2^-) + \frac{1}{2}(b_0 - b_1 + b_2 - b_3 + b_4)\\ &= \tfrac{5}{4}b_2^+ - \tfrac{1}{4}b_2^- + 1 - b_1\end{aligned}$$

and the desired identity is proved.

Verifying surjectivity of the map $(\varphi,\alpha) \longmapsto \begin{pmatrix}\mathcal{F}_2(\varphi,\alpha)\\ \mathcal{F}_3(\varphi,\alpha)\end{pmatrix}$ is the hardest part. First, we want to show the inclusion $(\ker\mathcal{F}_3)^\perp \subseteq \ker\mathcal{F}_1$ and in order to do so we first have to calculate the formal $L^2$-adjoint of $\mathcal{F}_3$: let $\varphi \in \Gamma(S+)$, $\alpha \in i\Omega^1(M)$ and $f \in iC^\infty(M)$ be arbitrary and let $(\cdot\,|\,\cdot)$ denote the $L^2$-inner product [5], then

$$\begin{aligned}\big((-f\psi, 2df), (\varphi,\alpha)\big) &= \mathrm{Re}(-f\psi \mid \varphi) + (2df \mid \alpha)\\ &= (f \mid 2d^*\alpha) + \int_M \mathrm{Re}\langle -f(x)\psi(x), \varphi(x)\rangle dV_g.\end{aligned}$$

$f$ is imaginary-valued, so $if$ is real-valued and hence

$$\mathrm{Re}\langle -f\psi,\varphi\rangle = \mathrm{Re}\langle iif\psi,\varphi\rangle = if\,\mathrm{Re}(i\langle\psi,\varphi\rangle) = -if\,\mathrm{Im}\langle\psi,\varphi\rangle.$$

In conclusion we get

$$\begin{aligned}\big((-f\psi, 2df), (\varphi,\alpha)\big) &= (f \mid 2d^*\alpha) + \int_M -if\,\mathrm{Im}\langle\psi,\varphi\rangle dV_g\\ &= (f \mid 2d^*\alpha - i\,\mathrm{Im}\langle\psi,\varphi\rangle),\end{aligned}$$

i.e. that the formal $L^2$-adjoint of $\mathcal{F}_3$ is

$$\mathcal{F}_3^* f = (-f\psi, 2df).$$

---

[4] $b_0 = 1$ since $M$ is connected, and $b_4 = 1$ since $M$ is compact and orientable.

[5] Note that the Sobolev spaces $L^2_k(i\Lambda^*_2 T^*M)$, $L^2_k(iT^*M)$ and $iL^2_k(M)$ are *real* Hilbert spaces, whereas $L^2_k(S^+)$ is conceived as a *complex* Hilbert space. In order for direct sums like $L^2_k(iT^*M) \oplus L^2_k(S^+)$ to make sense, we have to turn $L^2_k(S^+)$ into a real Hilbert space, and we simply do that by first forgetting the complex vector space structure, and second by taking the real part of the inner product.

This formal adjoint is again a first order differential operator, and thus it can be extended to the various Sobolev spaces.

We then get

$$\mathcal{F}_1\mathcal{F}_3^*f = d^+(2df) - dq_\psi(-f\psi).$$

The first term is obviously 0 and so is the second term (this is seen simply by plugging in $-f\psi$ in the formula for $dq_\psi$). This shows that $\operatorname{im}\mathcal{F}_3^* \subseteq \ker\mathcal{F}_1$. Since $\ker\mathcal{F}_1$ is closed we also have $\overline{\operatorname{im}\mathcal{F}_3^*} \subseteq \ker\mathcal{F}_1$. Recalling finally that $\overline{\operatorname{im}\mathcal{F}_3^*} = (\ker\mathcal{F}_3)^\perp$ (which holds for any map $A : H_1 \longrightarrow H_2$ between Hilbert spaces) the inclusion is proved.

Now assume that $\mathsf{C} = (\psi, \mathcal{A}) \in \mathcal{C}_k^*$ solves the Seiberg-Witten equations perturbed by $\eta$. Then

$$\begin{aligned}\mathcal{F}_2\mathcal{F}_3^*(f) &= \mathcal{F}_2(-f\psi, 2df) = df\cdot\psi + \not{D}_\mathcal{A}(-f\psi)\\ &= df\cdot\psi - df\cdot\psi - f\not{D}_\mathcal{A}\psi = 0.\end{aligned}$$

This shows that $\overline{\operatorname{im}\mathcal{F}_3^*} \subseteq \ker\mathcal{F}_2$ and by $\overline{\operatorname{im}\mathcal{F}_3^*} = (\ker\mathcal{F}_3)^\perp$ we get the inclusion $(\ker\mathcal{F}_3)^\perp \subseteq \ker\mathcal{F}_2$.

We use these inclusions to check that the map $(\varphi,\alpha) \longmapsto \begin{pmatrix}\mathcal{F}_2(\varphi,\alpha)\\ \mathcal{F}_3(\varphi,\alpha)\end{pmatrix}$ is surjective. First we note that even though it need no longer be an elliptic differential operator, it is still a *surjectively elliptic* differential operator, i.e. its principal symbol is surjective off the zero section of $T^*M$. Examining the proof of the result that an elliptic differential operator is Fredholm (e.g. [10] Theorem 8.11) we see that a surjectively elliptic differential operator has closed range and that there exists a finite-dimensional subspace $W \subseteq \Gamma(S^- \oplus i\mathbb{R})$ such that for *any* realization

$$(\mathcal{F}_2, \mathcal{F}_3) : L_k^2(S^+ \oplus iT^*M) \longrightarrow L_{k-1}^2(S^- \oplus i\mathbb{R})$$

we have that $\big(\operatorname{im}(\mathcal{F}_2,\mathcal{F}_3)\big)^\perp = W$, where the orthogonal complement is *with respect to the inner product in* $L_{k-1}^2$! In particular this means that if just one of the realizations of $(\mathcal{F}_2,\mathcal{F}_3)$ is surjective, then *any* realization is surjective. In this case, we show that the maximal realization, i.e. $(\mathcal{F}_2,\mathcal{F}_3)$ with domain $L_1^2(S^+ \oplus iT^*M)$, is surjective, i.e. that $\operatorname{im}(\mathcal{F}_2,\mathcal{F}_3)^\perp = W$ where $\perp$ now refers to the $L^2$-inner product (and we know how this behaves, cf. our discussion in the beginning of this proof). So for the rest of this proof $\perp$ refers to the orthogonal complement in the $L^2$-inner product.

Assume that $(\sigma, f) \in \operatorname{im}(\mathcal{F}_2,\mathcal{F}_3)^\perp$, i.e.

$$0 = \big((\sigma,f)\mid(\mathcal{F}_2(\varphi,\alpha),\mathcal{F}_3(\varphi,\alpha))\big)_{L^2} = \big(\sigma,\mathcal{F}_2(\varphi,\alpha)\big)_{L^2} + \big(f,\mathcal{F}_3(\varphi,\alpha)\big)_{L^2} \quad (2.32)$$

for all $\alpha$ and $\varphi$. By the inclusion $(\ker\mathcal{F}_3)^\perp \subseteq \ker\mathcal{F}_2$ we see that

$$\operatorname{im}\mathcal{F}_2 = \{\mathcal{F}_2(\varphi,\alpha) \mid (\varphi,\alpha) \in \ker\mathcal{F}_3\}$$

and using this with (2.32) we see that $\sigma \in (\operatorname{im}\mathcal{F}_2)^\perp$. Thus to show $\sigma = 0$ it suffices to prove that $(\operatorname{im}\mathcal{F}_2)^\perp = \{0\}$. We may view $\mathcal{F}_2$ and $\not{D}_\mathcal{A}$ as unbounded operators in $L^2$ with domain $L_1^2$. As such we note that $\operatorname{im}\not{D}_\mathcal{A} \subseteq \operatorname{im}\mathcal{F}_2$ since $\not{D}_\mathcal{A}\varphi = \mathcal{F}_2(0,\varphi)$, and therefore $(\operatorname{im}\mathcal{F}_2)^\perp \subseteq (\operatorname{im}\not{D}_\mathcal{A})^\perp$. We know that even for an unbounded operator $T : H_1 \longrightarrow H_2$ between Hilbert spaces we have $H_2 = \overline{(\operatorname{im}T)} \oplus \ker T^*$, i.e. $(\operatorname{im}T)^\perp = \ker T^*$. Thus, in our case, we have $(\operatorname{im}\not{D}_\mathcal{A})^\perp = \ker\not{D}_\mathcal{A}^*$ where $\not{D}_\mathcal{A}^*$ is the *Hilbert space adjoint* of $\not{D}_\mathcal{A}$ as an unbounded operator $L^2(S^+) \longrightarrow L^2(S^-)$. However, cf. Theorem 2.13 2) this adjoint is simply the $L_1^2$-realization of the formal adjoint of $\not{D}_\mathcal{A} : \Gamma(S^+) \longrightarrow \Gamma(S^-)$, which is just the Dirac operator itself mapping $\Gamma(S^-) \longrightarrow \Gamma(S^+)$. In conclusion we get $(\operatorname{im}\mathcal{F}_2)^\perp = \ker\not{D}_\mathcal{A}^* = \ker(\not{D}_\mathcal{A}|_{S^-})$, and thus $\sigma \in \ker\not{D}_\mathcal{A} \subseteq \Gamma(S^-)$.

We also have $\psi \in \ker \not{D}_{\mathcal{A}} \subseteq \Gamma(S^+)$. The Unique Continuation Property of $\not{D}_{\mathcal{A}}$ (Theorem 1.16) then says that there exists a neighborhood $U$ in which both $\sigma$ and $\psi$ are everywhere nonzero, and this is for the following reason: Since they are in the kernel of the elliptic operator $\not{D}_{\mathcal{A}}$ they are automatically smooth, thus we must be able to find an open set $\widehat{U}$ on which $\psi$ is everywhere nonzero since $\psi \neq 0$ by irreducibility of $(\psi, \mathcal{A})$. By the Unique Continuation Property, we cannot have $\sigma|_{\widehat{U}} \equiv 0$ and therefore we can find an open set $U \subseteq \widehat{U}$ such that $\sigma|_U$ is everywhere nonzero.

Pick $x_0 \in U$ and pick a neighborhood $V$ around $x_0$ above which the spinor bundle $S \longrightarrow M$ is trivial. Over $V$ we may then view $\sigma$ and $\psi$ as smooth functions $V \longrightarrow \Delta_4^-$. Now recall the isomorphism

$$\mathrm{Cl}^{\mathbb{C}}(T_{x_0}M)^1_+ \cong (\mathrm{Cl}^{\mathbb{C}}_4)^1_+ \cong \mathrm{Hom}(\Delta_4^+, \Delta_4^-)$$

where the latter isomorphism is the Clifford action. We have $\sigma(x_0) \in \Delta_4^-$ and $\psi(x_0) \in \Delta_4^+$ and we can find a map in $\mathrm{Hom}(\Delta_4^+, \Delta_4^-)$ which maps $\psi(x_0)$ to $\sigma(x_0)$. This map corresponds to an element $\xi_{\mathbb{C}} \in \mathrm{Cl}^{\mathbb{C}}(T_{x_0}M)^1_-$, hence

$$\langle \xi_{\mathbb{C}} \cdot \psi(x_0), \sigma(x_0) \rangle = |\sigma(x_0)|^2 > 0$$

where $\langle\, ,\, \rangle$ is the inner product in $\Delta_4$. Let $\xi$ be the real part of $\xi_{\mathbb{C}}$ (i.e. the image of $\xi_{\mathbb{C}}$ under the obvious projection map $\mathrm{Cl}^{\mathbb{C}}(T_{x_0}M) = \mathrm{Cl}(T_{x_0}M) \otimes \mathbb{C} \longrightarrow \mathrm{Cl}(T_{x_0}M)$). Then, upon writing $\xi_{\mathbb{C}} = \xi + i\xi'$ we get

$$\mathrm{Re}\langle \xi \cdot \psi(x_0), \sigma(x_0) \rangle > 0.$$

We can extend $\xi$ to a smooth section of the Clifford bundle $\overline{\xi}$ over $V$ and then by continuity we can find a neighborhood $x_0 \in V' \subseteq V$ such that

$$\mathrm{Re}\langle \overline{\xi}(x) \cdot \psi(x), \sigma(x) \rangle > 0$$

for all $x \in V'$. Pick a smooth function $h \in C^\infty(M)$ which satisfies $\operatorname{supp} h \subseteq V'$ and $h(x_0) = 1$ and put $\alpha := h\overline{\xi}$. Then

$$\begin{aligned}\mathrm{Re}(\mathcal{F}_2(0,\alpha) \mid \sigma) &= \frac{1}{2}\mathrm{Re}(\alpha \cdot \psi \mid \sigma) = \frac{1}{2}\mathrm{Re}\int_M \langle \alpha(x) \cdot \psi(x), \sigma(x) \rangle dV_g \\ &= \frac{1}{2}\int_{V'} \mathrm{Re}\langle h(x)\overline{\xi}(x) \cdot \psi(x), \sigma(x) \rangle dV_g > 0,\end{aligned}$$

contrasting the assumption that $\sigma \in (\operatorname{im} \mathcal{F}_2)^\perp$.

To finish the proof we only need to show that $f = 0$. But since $\sigma = 0$, the condition (2.32) reduces to $(f \mid \mathcal{F}_3(\varphi, \alpha)) = 0$, i.e. that $f$ is orthogonal to the image of $\mathcal{F}_3$, hence we just need to show that the orthogonal complement to this image is 0. $\mathcal{F}_3$ maps into $L^2_k(M, i\mathbb{R})$ and since $d^*$ maps $L^2_{k+1}(iT^*M) \longrightarrow L^2_k(M, i\mathbb{R})$ we get

$$L^2_k(M, i\mathbb{R}) = d^*(L^2_{k+1}(iT^*M)) \oplus \ker d$$

where the orthogonal direct sum is with respect to the $L^2$-inner product. But since $M$ is connected the kernel of $d$ just consists of the constant functions, in particular $\ker d$ is a 1-dimensional space. From the explicit form of the map $\mathcal{F}_3$ we see that at least the space $d^*(L^2_{k+1}(iT^*M))$ gets hit. To see that $\mathcal{F}_3$ is surjective we thus only need to find a single function in the image of $\mathcal{F}_3$ which is *not* orthogonal to the constant function $i$, i.e. a function $h$ satisfying $\int_M ih dV_g \neq 0$. But why not take the function $\mathcal{F}_3(i\psi, 0) = -i\,\mathrm{Im}\langle i\psi, \psi \rangle = -i\,\mathrm{Re}\langle \psi, \psi \rangle = -i|\psi|^2$. The integral over $M$ of $i$ times this is just $\|\psi\|^2_{L^2}$ which is certainly nonzero, since the configuration is assumed irreducible. This shows that $\mathcal{F}_3$ is surjective and hence that $f$ must be 0. $\square$

We can put the considerations of the preceding lemma into a more streamlined form. Namely, consider the following string of maps

$$0 \longrightarrow T_1\mathcal{G}_{k+1} \xrightarrow{d\rho_1^{\mathsf{C}}} T_{\mathsf{C}}\mathcal{C}_k \xrightarrow{dF_{\mathsf{C}}^{\eta}} L_k^2(S^- \oplus i\Lambda_+^2 T^*M) \longrightarrow 0 \tag{2.33}$$

This is the so-called *deformation complex*. It is an elliptic complex, and if $\mathsf{C}$ is a Seiberg-Witten monopole, it is in fact a complex. The elliptic operator originating from this complex is indeed the map $\mathcal{F}_{\mathsf{C}}$.

As it appears from Lemma 2.33, the index of $\mathcal{F}_{\mathsf{C}}$ depends not on the choice of $\mathsf{C}$, neither on $k$ but only on $M$ and on $L$, i.e. on the spin$^c$-structure.

**Definition 2.34** (**Virtual Dimension**)**.** For a choice of spin$^c$-structure $\sigma$ on $M$ the integer $d(\sigma) = \operatorname{ind}_{\mathbb{R}}(\mathcal{F}_{\mathsf{C}})$ is called the *virtual dimension* of the moduli space $\mathcal{M}_k^*(\eta)$. A spin$^c$-structure $\sigma$ is called *feasible* if $d(\sigma) \geq 0$.

The reason for this terminology is that in case of a feasible spin$^c$-structure the virtual dimension is in fact equal to the manifold-dimension of the moduli space. We will prove this shortly. The strategy employed is to show that the moduli space is the level set of a smooth Fredholm map defined on some smooth manifold and whose differential has the same index as $\mathcal{F}_{\mathsf{C}}$. Then the Implicit Function Theorem tells us that the moduli space is a smooth submanifold whose dimension equals this index.

The smooth manifold on which this map is to be defined is the parametrized moduli space defined below:

**Definition 2.35** (**Parametrized Moduli Space**)**.** Define the irreducible *parametrized configuration space* by

$$\mathcal{P}\mathcal{C}_k^* := \mathcal{C}_k^* \times L_{k-1}^2(i\Lambda_+^2 T^*M)$$

and define an action of $\mathcal{G}_{k+1}$ on this space by

$$\mathcal{P}\mathcal{C}_k^* \times \mathcal{G}_{k+1} \ni ((\mathsf{C}, \eta), \gamma) \longmapsto (\mathsf{C} \cdot \gamma, \eta).$$

We define the space $\mathcal{P}\mathcal{B}_k^*$ to be the quotient space under this action

$$\mathcal{P}\mathcal{B}_k^* := \mathcal{P}\mathcal{C}_k^*/\mathcal{G}_{k+1} = \mathcal{B}_k^* \times L_{k-1}^2(i\Lambda_+^2 T^*M).$$

Finally, define the *parametrized moduli space* $\mathcal{P}\mathcal{M}_k^*$ to be the set of equivalence classes $[\psi, \mathcal{A}, \eta] \in \mathcal{P}\mathcal{B}_k^*$ for which $F^\eta(\psi, \mathcal{A}) = 0$, i.e. where $(\psi, \mathcal{A})$ solves the Seiberg-Witten equations perturbed by $\eta$.

Note that the parametrized moduli space is well-defined since the solution space to the Seiberg-Witten equations is gauge-invariant.

As shown above, the space $\mathcal{B}_k^*$ is a smooth manifold, hence also $\mathcal{P}\mathcal{B}_k^*$ is a smooth manifold. As we will now show the parametrized moduli space is a submanifold of $\mathcal{P}\mathcal{B}_k^*$:

**Lemma 2.36.** *The parametrized moduli space* $\mathcal{P}\mathcal{M}_k^*$ *is a smooth separable Hilbert manifold and the tangent space at a point* $[\psi, \mathcal{A}, \eta]$ *can be identified with the kernel of the map*

$$\mathcal{F}_{(\psi,\mathcal{A})} \dotplus T : L_k^2(iT^*M \oplus S^+ \oplus i\Lambda_+^2 T^*M) \longrightarrow L_{k-1}^2(i\Lambda_+^2 T^*M \oplus S^- \oplus i\mathbb{R})$$

*where* $\mathcal{F}_{(\psi,\mathcal{A})}$ *is the map defined in* (2.31) *and where* $T$ *is the map given by* $\zeta \longmapsto (-\zeta, 0, 0)$.

Proof. Let $[\psi, \mathcal{A}, \zeta] \in \mathcal{PM}_k^*$ and let $\mathsf{C} := (\psi, \mathcal{A}) \in \mathcal{C}_k^*$ and $\zeta$ be fixed representatives for this class. Let $V$ be the open neighborhood around $0 \in \ker d\rho_1^{\mathsf{C}}$ whose existence is guaranteed by the Local Slice Lemma. Since $V$ is open in $\ker d\rho_1^{\mathsf{C}} \subseteq L_{k-1}^2(iT^*M \oplus S^+)$ we can find an open set $\widetilde{V} \subseteq L_{k-1}^2(iT^*M \oplus S^+)$ such that $V = \widetilde{V} \cap \ker d\rho_1^{\mathsf{C}}$. Then define

$$\widetilde{F} : \widetilde{V} \times L_{k-1}^2(i\Lambda_+^2 T^*M) \longrightarrow L_{k-1}^2(i\Lambda_+^2 T^*M \oplus S^- \oplus i\mathbb{R})$$

by

$$\widetilde{F}(\varphi, \alpha, \zeta) = \begin{pmatrix} F_{\mathcal{A}+\alpha}^+ - q(\psi + \varphi) - \zeta - \eta \\ \not{D}_{\mathcal{A}+\alpha}(\psi + \varphi) \\ 2d^*\alpha - i \operatorname{Im}\langle \psi, \varphi \rangle \end{pmatrix}.$$

By Proposition 2.32 we see that the map

$$(\varphi, \alpha, \zeta) \longmapsto [\psi + \varphi, \mathcal{A} + \alpha, \eta + \zeta]$$

is a diffeomorphism from $V \times L_{k-1}^2(i\Lambda_+^2 T^*M)$ to a neighborhood of $[\psi, \mathcal{A}, \eta]$ in $\mathcal{PB}_k^*$ and that this maps $\widetilde{F}^{-1}(0)$ to a neighborhood of $[\psi, \mathcal{A}, \eta]$ in $\mathcal{PM}_k^*$. If we can show that $\widetilde{F}^{-1}(0)$ has a smooth structure, then by the diffeomorphism above, this structure will be transferred to a smooth structure on $\mathcal{PM}_k^*$ around the point $[\psi, \mathcal{A}, \eta]$ and thus the parametrized moduli space will be a manifold.

So we consider the space $\widetilde{F}^{-1}(0)$. We want to show that this is a manifold and for this we use the Implicit Function Theorem, for then we only need to show that $d\widetilde{F}_{(\varphi,\alpha,\zeta)}$ is surjective (again, since we are working with Hilbert manifolds the requirement that the kernel of the differential should have a complement is automatically satisfied). Since we can continuously deform $d\widetilde{F}_{(\varphi,\alpha,\zeta)}$ to $d\widetilde{F}_{(0,0,0)}$ within the space of Fredholm operators we need only (cf. Lemma 1.26) check surjectivity of $d\widetilde{F}_{(0,0,0)}$, and as before one can calculate that

$$d\widetilde{F}_{(0,0,0)}(\varphi, \alpha, \zeta) = (\mathcal{F}_{(\psi,\mathcal{A})} \dot{+} T)(\varphi, \alpha, \zeta) = \begin{pmatrix} d^+\alpha - dq_\psi(\varphi) - \zeta \\ \frac{1}{2}\alpha \cdot \psi + \not{D}_{\mathcal{A}}\varphi \\ 2d^*\alpha - i \operatorname{Im}\langle \psi, \varphi \rangle \end{pmatrix}.$$

$\mathcal{F}_{(\psi,\mathcal{A})} \dot{+} T$ is surjective on the two last components (cf. Lemma 2.33), so given $(a, b, c)$ we can find $(\varphi, \alpha)$ such that $\mathcal{F}_2(\varphi, \alpha) = b$ and $\mathcal{F}_3(\varphi, \alpha) = c$. Then pick $\zeta = \mathcal{F}_1(\varphi, \alpha) - a$ and we get $(\mathcal{F}_{(\psi,\mathcal{A})} \dot{+} T)(\varphi, \alpha, \zeta) = (a, b, c)$. □

Now we are almost at the end. From the parametrized moduli space we can deduce properties of the *actual* moduli space

**Theorem 2.37.** *There exists a dense subset* $\Xi_{k-1}^{\mathrm{reg}} \subseteq L_{k-1}^2(i\Lambda_+^2 T^*M)$ *which consist of the* $\eta$*'s for which* $\mathcal{F}_{\mathsf{C}}$ *is surjective for all* $\mathsf{C} \in \mathcal{S}_k^*(\eta)$. *For such an* $\eta$ *it holds that the moduli space* $\mathcal{M}_k^*(\eta)$ *is a smooth manifold of dimension*

$$\dim(\mathcal{M}_k^*(\eta)) = \frac{1}{4}c_1(L)^2([M]) - \frac{1}{2}\chi(M) - \frac{3}{4}\sigma(M). \tag{2.34}$$

*Furthermore the tangent space* $T_{[\mathsf{C}]}\mathcal{M}_k^*(\eta)$ *can be identified with* $\ker \mathcal{F}_{\mathsf{C}}$.

Note that the dimension of the moduli space is independent of $\eta$ and of $k$!

Proof. Let $\Xi_{k-1}^{\mathrm{reg}}$ be the space mentioned above and assume we have an element $\eta_0$ in here (we don't know yet, that this space is non-empty, but we will remedy this shortly). We consider the projection map

$$\widetilde{\pi} : \mathcal{PC}_k^* = \mathcal{B}_k^* \times L_{k-1}^2(i\Lambda_+^2 T^*M) \longrightarrow L_{k-1}^2(i\Lambda_+^2 T^*M)$$

which is, of course, smooth. Let $\pi$ denote the restriction of $\widetilde{\pi}$ to $\mathcal{PM}_k^*$, then $\pi$ is also smooth, and we have $\mathcal{M}_k^*(\eta_0) = \pi^{-1}(\eta_0)$. The differential of $\pi$

$$d\pi_{[\psi,\mathcal{A},\eta_0]} : T_{[\psi,\mathcal{A},\eta_0]}\mathcal{PM}_k^* = \ker(\mathcal{F}_{(\psi,\mathcal{A})} \dotplus T) \longrightarrow L^2_{k-1}(i\Lambda^2_+T^*M)$$

at a point $[\psi,\mathcal{A},\eta_0] \in \pi^{-1}(\eta_0)$ is given by

$$(\varphi,\alpha,\zeta) \longmapsto \zeta$$

(the differential of a projection is a projection). Cf. Proposition 1.27 this map is Fredholm and its index equals the index of $\mathcal{F}_{(\psi,\mathcal{A})}$ which is exactly (2.34). Furthermore the proposition also says that $\operatorname{coker} d\pi_{[\psi,\mathcal{A},\eta_0]} \cong \operatorname{coker}(\mathcal{F}_{(\psi,\mathcal{A})} \dotplus T)$, in particular (since both maps have closed range, being Fredholm operators) $d\pi_{[\psi,\mathcal{A},\eta_0]}$ is surjective if and only if $\mathcal{F}_{(\psi,\mathcal{A})}$ is surjective. But the latter is surjective by the assumption on $\eta_0$ since $(\psi,\mathcal{A})$ is a solution. Thus $\mathcal{M}^*(\eta_0)$ is the level set of a regular value, hence by the Implicit Function Theorem it is an embedded submanifold of the parametrized moduli space, in particular it is itself a manifold. The dimension equals the index of $d\pi_{[\psi,\mathcal{A},\eta_0]} = \operatorname{ind}_{\mathbb{R}} \mathcal{F}_{(\psi,\mathcal{A})}$ which equals $\frac{1}{4}c_1(L)^2([M]) - \frac{1}{2}\chi(M) - \frac{3}{4}\sigma(M)$ as desired. One further consequence of the Implicit Function Theorem is that the tangent space $T_{[\psi,\mathcal{A}]}\mathcal{M}^*(\eta_0)$ equals the kernel of $d\pi_{[\psi,\mathcal{A},\eta_0]}$ which, by Proposition 1.27 is identified with $\ker \mathcal{F}_{(\psi,\mathcal{A})}$.

>From the discussion above we see that the space $\Xi^{\mathrm{reg}}_{k-1}$ is the set of regular values for $\pi$, and by the Smale-Sard Theorem (Theorem 1.37) this set is generic in $L^2_{k-1}(i\Lambda^2_+T^*M)$. Hence it is dense, in particular non-empty. □

It seems we can say an awful lot about the irreducible moduli spaces $\mathcal{M}_k^*(\eta)$, but originally we set out to study the space $\mathcal{M}_k(\eta)$. Thus we look for perturbation parameters $\eta$ for which these two spaces are identical, or rather, where $\mathcal{S}_k^*(\eta) = \mathcal{S}_k(\eta)$, i.e. for which there are no reducible solutions.

To this end define the space $\Xi^{\mathrm{red}}_{k-1} \subseteq L^2_{k-1}(i\Lambda^2_+T^*M)$ (not to be confused with $\Xi^{\mathrm{reg}}_{k-1}$!) to be the set of parameters $\eta$ for which *reducible* solutions exist, i.e. for which $\mathcal{S}_k^*(\eta) \neq \mathcal{S}_k(\eta)$. Similarly we define $\Xi^{\mathrm{red}}_{\infty} \subseteq i\Omega^2_+(M)$ to be the set of smooth perturbation parameters for which $\mathcal{S}^*(\eta) \neq \mathcal{S}(\eta)$. Obviously we have the following string of inclusions

$$\Xi^{\mathrm{red}}_1 \supseteq \Xi^{\mathrm{red}}_2 \supseteq \cdots \supseteq \Xi^{\mathrm{red}}_\infty.$$

The point of introducing this space is that it is, fortunately, rather small and that its size depends only on the topology of $M$:

**Proposition 2.38.** *For all $k \in \mathbb{N}_0 \cup \{\infty\}$ it holds that $\Xi^{\mathrm{red}}_k$ is an affine subspace of $L^2_k(i\Lambda^2_+T^*M)$ [6] of codimension $b_2^+$ modeled over $d^+(L^2_{k+1}(iT^*M))$.*

Proof. We do the smooth case first. Obviously the space $\Xi^{\mathrm{red}}_\infty$ is non-empty, for if we pick an arbitrary smooth connection $\mathcal{A}$ on the determinant line bundle $L$ and put $\eta := F^+_{\mathcal{A}}$, then $(0,\mathcal{A})$ is obviously a reducible solutions to the Seiberg-Witten equations perturbed by $\eta$, i.e. $\eta \in \Xi^{\mathrm{red}}_\infty$.

Let $\eta_1, \eta_2 \in \Xi^{\mathrm{red}}_\infty$ and let $(0,\mathcal{A}_i) \in \mathcal{S}(\eta_i)$ be corresponding reducible solutions. We know that $F_{\mathcal{A}_1}$ and $F_{\mathcal{A}_2}$ represent the same cohomology class, namely $-2\pi i$ times the first Chern class of the line bundle $L$. Thus there exists $\alpha \in i\Omega^1(M)$ such that

$$F_{\mathcal{A}_2} - F_{\mathcal{A}_1} = d\alpha.$$

Since by the Seiberg-Witten equations we have $\eta_i = F^+_{\mathcal{A}_i}$ we see that

$$\eta_2 - \eta_1 = (F_{\mathcal{A}_2} - F_{\mathcal{A}_1})^+ = d^+\alpha$$

---

[6] In the case $k = \infty$ this, of course, means just $i\Omega^2_+(M)$.

hence $\Xi_\infty^{\text{red}}$ is indeed an affine space modeled over $d^+(i\Omega^1(M))$. The codimension of $\Xi_\infty^{\text{red}}$ is, by definition, the codimension of the model space $d^+(i\Omega^1(M))$. Cf. (2.21) the cokernel of $d^+ : i\Omega^1(M) \longrightarrow i\Omega_+^2(M)$ equals $\mathscr{H}_+^2(M)$ and its dimension is exactly equal to $b_2^+$. Upon completing we prove the general case. □

Thus we see that $b_2^+$ which is a topological invariant of $M$ determines how many of the perturbation parameters which will allow reducible solutions. The greater the number $b_2^+$ the easier it is to find perturbation parameters which do *not* allow reducible solutions. Combining this proposition with Theorem 2.37 as well as Proposition 1.38 we obtain

**Theorem 2.39.** *If $b_2^+ \geq 1$, then for any $k \in \mathbb{N}_0$ the space $\Xi_k^{\text{reg}} \setminus \Xi_k^{\text{red}}$ is open and dense in $L_k^2(i\Lambda_+^2 T^*M)$ and for every $\eta \in \Xi_k^{\text{reg}} \setminus \Xi_k^{\text{red}}$ the moduli space $\mathcal{M}_{k+1}(\eta) = \mathcal{M}_{k+1}^*(\eta)$ is a compact smooth manifold of dimension*

$$\dim(\mathcal{M}_{k+1}(\eta)) = \frac{1}{4}c_1(L)^2([M]) - \frac{1}{2}\chi(M) - \frac{3}{4}\sigma(M). \tag{2.35}$$

*Furthermore for any point in the moduli space the tangent space $T_{[\mathsf{C}]}\mathcal{M}_k(\eta)$ can be identified with* $\ker \mathcal{F}_{\mathsf{C}}$.

At the end of the previous section we saw that for different $k$ the moduli spaces $\mathcal{M}_k(\eta)$ were mutually homeomorphic. A natural question to ask is whether they are also mutually *diffeomorphic*. Here it is important to be very precise about which spaces we are talking about. We have only showed compactness for the full moduli spaces $\mathcal{M}_k(\eta)$, whereas only the irreducible moduli spaces $\mathcal{M}_k^*(\eta)$ have been given smooth structures. Thus, to properly ask the question, we have to define yet another space: Let

$$i\Omega_+^2(M)_k^{\text{reg}} := (\Xi_k^{\text{reg}} \setminus \Xi_k^{\text{red}}) \cap i\Omega_+^2(M).$$

Since $i\Omega_+^2(M)$ is dense in $L_k^2(i\Lambda_+^2 T^*M)$ and $\Xi_k^{\text{reg}} \setminus \Xi_k^{\text{red}}$ is open in $L_k^2(i\Lambda_+^2 T^*M)$, $i\Omega^1(M)_k^{\text{reg}}$ is non-empty, and since the inclusion $i\Omega_+^2(M) \hookrightarrow L_k^2(i\Lambda_+^2 T^*M)$ is continuous, $i\Omega_+^2(M)_k^{\text{reg}}$ is open in $i\Omega_+^2(M)$. One can show moreover that $i\Omega_+^2(M)_k^{\text{reg}}$ is also dense in $i\Omega_+^2(M)$, (see [1] Corollary 2.37) and since $i\Omega_+^2(M)$ is a Fréchet space, in particular a complete metric space, the Baire Category Theorem states that

$$i\Omega_+^2(M)_\infty^{\text{reg}} := \bigcap_{k=1}^{\infty} i\Omega_+^2(M)_k^{\text{reg}}$$

is dense in $i\Omega_+^2(M)$, in particular (and that's the only thing we need) it is non-empty.

Now let $\eta \in i\Omega_+^2(M)_\infty^{\text{reg}}$. Then for all $k$ we have $\mathcal{M}_k(\eta) = \mathcal{M}_k^*(\eta)$ and $\mathcal{M}_k^*(\eta)$ is a manifold. Now it makes sense to ask whether the moduli spaces are diffeomorphic, and the answer is affirmative:

**Corollary 2.40.** *For $\eta \in i\Omega_+^2(M)_\infty^{\text{reg}}$ the map $\Phi_k : \mathcal{M}_k(\eta) \longrightarrow \mathcal{M}_{k-1}(\eta)$ described in Section* 2.3 *is a diffeomorphism.*

Proof. The map $\Phi_k$ can be described as the restriction to $\mathcal{M}_k(\eta)$ of the smooth inclusion $\iota_k : \mathcal{PM}_k^* \hookrightarrow \mathcal{PM}_{k-1}^*$. Being an inclusion $\iota_k$ must have injective differential, and since the differential of $\Phi_k$ is then just the restriction of $d\iota_k$, this differential must be injective as well. But $\Phi_k$ maps between manifolds of equal (finite) dimension and hence its differential must be an isomorphism. By the Inverse Function Theorem, $\Phi_k$ is a diffeomorphism. □

Considering again the map $\Phi : \mathcal{M}(\eta) \longrightarrow \mathcal{M}_3(\eta)$ (recall that this was, by definition of the topology on $\mathcal{M}(\eta)$, a homeomorphism). We use this map to give $\mathcal{M}(\eta)$ a smooth structure: we simply declare $\Phi$ to be a diffeomorphism. Then we may conclude:

**Corollary 2.41.** *For* $\eta \in i\Omega^2_+(M)^{\mathrm{reg}}_\infty$ *the moduli space* $\mathcal{M}(\eta)$ *can be given a structure of a compact smooth manifold of dimension*

$$\dim(\mathcal{M}(\eta)) = \frac{1}{4}c_1(L)^2([M]) - \frac{1}{2}\chi(M) - \frac{3}{4}\sigma(M). \tag{2.36}$$

*With this smooth structure* $\mathcal{M}(\eta)$ *is diffeomorphic to* $\mathcal{M}_k(\eta) = \mathcal{M}^*_k(\eta)$ *for any* $k \geq 3$.

## 2.5 Orienting the Moduli Space

Recall that our purpose is to define the Seiberg-Witten invariant as an integral *over the moduli space* of a certain differential form. To be able to integrate over the moduli space we have to show that it is orientable, and that is the goal of this section.

That is we have to show that we can "continuously" orient each of the tangent spaces $T_{[\mathsf{C}]}\mathcal{M}^*_k(\eta) \cong \ker \mathcal{F}_{\mathsf{C}}$. Choosing an orientation of $\ker \mathcal{F}_{\mathsf{C}}$ is the same as choosing a nonzero element of $\Lambda^{\mathrm{top}}(\ker \mathcal{F}_{\mathsf{C}})$. Choose $\eta$ in $\Xi^{\mathrm{reg}}_{k-1} \setminus \Xi^{\mathrm{red}}_{k-1}$, then $\mathcal{M}_k(\eta) = \mathcal{M}^*_k(\eta)$ and according to Theorem 2.37 $\mathcal{F}_{\mathsf{C}}$ is surjective for any solution $\mathsf{C} \in \mathcal{S}^*_k(\eta)$, meaning that $\operatorname{coker} \mathcal{F}_{\mathsf{C}} = \ker \mathcal{F}^*_{\mathsf{C}} = 0$. Then we have (since, by definition, $\Lambda^{\mathrm{top}}(0) = \mathbb{R}$) that:

$$\Lambda^{\mathrm{top}}(\ker \mathcal{F}_{\mathsf{C}}) \cong \Lambda^{\mathrm{top}}(\ker \mathcal{F}_{\mathsf{C}}) \otimes \Lambda^{\mathrm{top}}(\ker \mathcal{F}^*_{\mathsf{C}})^* = \det \mathcal{F}_{\mathsf{C}}.$$

This is where the determinant line bundle (as discussed in Section 1.5) enters the stage: picking an orientation for $T_{[\mathsf{C}]}\mathcal{M}^*_k(\eta) \cong \ker \mathcal{F}_{\mathsf{C}}$ is the same as picking a nonzero element of $\det \mathcal{F}_{\mathsf{C}}$. However, there are two problems here: first the fact that the determinant line bundles in Section 1.5 were only defined over compact or connected manifolds and secondly how to pick orientations on $\ker \mathcal{F}_{\mathsf{C}}$ such that the orientations agree under gauge equivalence. We will answer these two questions simultaneously by reducing the Fredholm field $\mathcal{F}_{\mathsf{C}}$ over $\mathcal{C}_k$ to a field over the moduli space (which is compact but not necessarily connected).

Recall that the tangent bundle to $\mathcal{C}_k$ is just $\mathcal{C}_k \times L^2_k(S^+ \oplus iT^*M)$. Let $\overline{V}^k := (T\mathcal{C}_k)|_{\mathcal{S}_k(\eta)} = \mathcal{S}_k(\eta) \times L^2_k(S^+ \oplus iT^*M)$ denote the restriction of this tangent bundle to the solution space. Furthermore, define the Hilbert bundle

$$\overline{W}_k = \mathcal{S}_k(\eta) \times L^2_{k-1}(i\Lambda^2_+T^*M \oplus S^- \oplus i\mathbb{R}).$$

Note how these vector bundles are tailored to match the domain and target spaces of $\mathcal{F}_\bullet$. Indeed this map is a bundle map between the two Hilbert bundles.

We give the total space $\overline{V}^k$ an action of the gauge group, $\mathcal{G}_{k+1}$, in the following way

$$(\mathsf{C}, \varphi, \alpha) \cdot \lambda = (\mathsf{C} \cdot \lambda, \lambda^{-1}\varphi, \alpha).$$

Then the map $\overline{V}^k/\mathcal{G}_{k+1} \longrightarrow \mathcal{M}_k(\eta)$ given by

$$[\mathsf{C}, \varphi, \alpha] \longmapsto [\mathsf{C}]$$

is well-defined and defines a vector bundle over $\mathcal{M}_k(\eta)$ with vector space operations

$$a[\mathsf{C}, \varphi, \alpha] + b[\mathsf{C}, \varphi', \alpha'] = [\mathsf{C}, a\varphi + b\varphi', a\alpha + b\alpha']$$

This vector bundle over $\mathcal{M}_k(\eta)$ we call $V^k$.

Similarly with the vector bundle $\overline{W}^k$. The total space of this bundle is equipped with the $\mathcal{G}_{k+1}$-action

$$(\mathsf{C}, \zeta, \varphi, f) \cdot \lambda = (\mathsf{C} \cdot \lambda, \zeta, \lambda^{-1}\varphi, f)$$

and in the same way as before we have a well-defined map $\overline{W}^k/\mathcal{G}_{k+1} \longrightarrow \mathcal{M}_k(\eta)$ by $[\mathsf{C}, \zeta, \varphi, f] \longmapsto [\mathsf{C}]$ and this is again a vector bundle, denoted $W^k$.

So the big question is now: does the bundle map $\mathcal{F}_\bullet : \overline{V}^k \longrightarrow \overline{W}^k$ induce a bundle map between $V^k$ and $W^k$? The answer is "yes". Showing this requires some calculations, first

$$\begin{aligned}\not{D}_{(\lambda^{-1}\psi, \mathcal{A}+2\lambda^{-1}d\lambda)}(\lambda^{-1}\varphi) &= \not{D}_{\mathcal{A}}(\lambda^{-1}\varphi) + \lambda^{-1}d\lambda \cdot (\lambda^{-1}\varphi) \\ &= d(\lambda^{-1}) \cdot \varphi + \lambda^{-1}\not{D}_{\mathcal{A}}\varphi - \lambda d(\lambda^{-1}) \cdot (\lambda^{-1}\varphi) \\ &= \lambda^{-1}\not{D}_{\mathcal{A}}\varphi\end{aligned}$$

from which it is easy to deduce that

$$\mathcal{F}_{\mathsf{C}\cdot\lambda}\big((\varphi, \alpha) \cdot \lambda\big) = \big(\mathcal{F}_{\mathsf{C}}(\varphi, \alpha)\big) \cdot \lambda$$

where $(\varphi, \alpha) \cdot \lambda = (\lambda^{-1}\varphi, \alpha)$ is just the restriction of the action on $\overline{V}^k$ to the fiber over $\mathsf{C}$. This shows that we have a well-defined map $\mathcal{F}_{[\mathsf{C}]} : V^k_{[\mathsf{C}]} \longrightarrow W^k_{[\mathsf{C}]}$ given by

$$\mathcal{F}_{[\mathsf{C}]}[\varphi, \alpha] = [\mathcal{F}_{\mathsf{C}}(\varphi, \alpha)].$$

Moreover, since

$$\mathcal{F}_{\mathsf{C}\cdot\lambda}(\varphi, \alpha) = \mathcal{F}_{\mathsf{C}\cdot\lambda}((\varphi, \alpha) \cdot \lambda^{-1}\lambda) = \mathcal{F}_{\mathsf{C}}((\varphi, \alpha) \cdot \lambda^{-1}),$$

we see that $(\varphi, \alpha) \cdot \lambda^{-1} \in \ker \mathcal{F}_{\mathsf{C}}$ if and only if $(\varphi, \alpha) \in (\ker \mathcal{F}_{\mathsf{C}}) \cdot \lambda$ hence

$$\ker(\mathcal{F}_{\mathsf{C}\cdot\lambda}) = (\ker \mathcal{F}_{\mathsf{C}}) \cdot \lambda \tag{2.37}$$

and similarly for the cokernel:

$$\operatorname{coker}(\mathcal{F}_{\mathsf{C}\cdot\lambda}) = (\operatorname{coker} \mathcal{F}_{\mathsf{C}}) \cdot \lambda. \tag{2.38}$$

This shows that the map $\mathcal{F}_{[\mathsf{C}]}$ is a Fredholm operator which has the same index as $\mathcal{F}_{\mathsf{C}}$.

The upshot of all this is that we have a continuous field of Fredholm operators parametrized by the compact manifold $\mathcal{M}_k(\eta)$ and so we may invoke the theory of Section 1.5, i.e. we can form the determinant line bundle $\det \mathcal{F}_{[\mathsf{C}]}$. By the remarks above, orienting this determinant line bundle is the same as orienting the manifold $\mathcal{M}_k(\eta)$, so showing orientability of $\det \mathcal{F}_{[\mathsf{C}]}$ is the last step.

Define maps (when $\mathsf{C} = (\psi, \mathcal{A})$):

$$\mathcal{F}^0_{\mathsf{C}}(\varphi, \alpha) := \begin{pmatrix} d^*\alpha \\ \not{D}_{\mathcal{A}}\varphi \\ 2d^*\alpha \end{pmatrix}$$

and $\mathscr{P}_{\mathsf{C}} := \mathcal{F}_{\mathsf{C}} - \mathcal{F}^0_{\mathsf{C}}$. They are bundle maps

$$\mathscr{P}_\bullet, \mathcal{F}^0_\bullet : \overline{V}^k \longrightarrow \overline{W}^k$$

and one can check, by the calculations above, that they are $\mathcal{G}_{k+1}$-equivariant and thus also descend to maps $\mathcal{F}^0_{[\mathsf{C}]}, \mathscr{P}_{[\mathsf{C}]} : V^k_{[\mathsf{C}]} \longrightarrow W^k_{[\mathsf{C}]}$ between the fibers of the vector bundles $V^k$ and $W^k$. Note how $\mathcal{F}^0_{\mathsf{C}}$ is just a direct sum $\not{D}_{\mathcal{A}} \oplus (d^+ + 2d^*)$ and hence

$$\det \mathcal{F}^0_{\mathsf{C}} = \det \not{D}_{\mathcal{A}} \otimes \det(d^+ + 2d^*),$$

(that is just a small calculation involving commutativity of the tensor product). The Dirac operator is a complex linear operator and therefore $\ker \not{D}_{\mathcal{A}}$ and $\operatorname{coker} \not{D}_{\mathcal{A}}$ are *realifications* of (finite-dimensional) complex vector spaces, thus they have *canonical orientations*. Another way to formulate it, is that the determinant line bundle $\det \not{D}_A$ is *canonically* orientable. The other part of the operator $\mathcal{F}^0_{\mathsf{C}}$, namely $d^+ + 2d^*$ does not depend on $\mathsf{C}$ at all, so to orient $\det \mathcal{F}^0_{\mathsf{C}}$ we just have to pick some orientation on the vector *space* $\det(d^+ + 2d^*)$. Recall from Lemma 2.22 that $\ker(d^+ + 2d^*) = \mathscr{H}^1(M)$ and $\operatorname{coker}(d^+ + 2d^*) = \mathscr{H}^0(M) \oplus \mathscr{H}^2_+(M)$. Since $\mathscr{H}^0(M)$ is canonically isomorphic to $\mathbb{R}$, this space has a canonical orientation. Thus once we have chosen orientations on the vector spaces $\mathscr{H}^1(M)$ and $\mathscr{H}^2_+(M)$ we get an orientation on $\det \mathcal{F}^0_{\mathsf{C}}$. By its canonical nature, it descends to a well-defined orientation on $\det \mathcal{F}^0_{\mathsf{C}}$ and hence to the line bundle $\det \mathcal{F}^0_{[\bullet]}$.

To transfer this to the determinant line bundle of $\mathcal{F}_\bullet$ we just need a homotopy between the two. But this is provided by

$$\mathcal{F}^t_{[\mathsf{C}]} := \mathcal{F}^0_{[\mathsf{C}]} + t\mathscr{P}_{[\mathsf{C}]}.$$

It now follows by Proposition 1.35 that the determinant line bundles $\det \mathcal{F}^0_\bullet$ and $\det \mathcal{F}_\bullet$ are isomorphic, and since the first one is orientable so is the latter. This proves

**Theorem 2.42 (Orientability of Moduli Space).** *For $\eta \in \Xi^{\mathrm{reg}}_{k-1} \setminus \Xi^{\mathrm{red}}_{k-1}$, the moduli space $\mathcal{M}_k(\eta) = \mathcal{M}^*_k(\eta)$ is an orientable manifold. A choice of orientations on $\mathscr{H}^1(M)$ and $\mathscr{H}^2_+(M)$ determines a specific orientation on $\mathcal{M}_k(\eta)$.*

## 2.6 The Seiberg-Witten Invariant

Finally we have reached a stage where we are able to define the renowned Seiberg-Witten Invariant. It is the integral over the moduli space of a certain cohomology class. The definition of this cohomology class is what will concern us at first (although for the calculations later the precise definition is really not important).

The definition utilizes the so-called *slant product* which we now define. First, let us briefly recall some definitions from homological algebra: In general we let $C_*$ and $C'_*$ be to chain complexes over a commutative ring $R$:

$$\begin{tikzcd}
0 & C_0 \arrow[l] & C_1 \arrow[l, "\partial_1"'] & C_2 \arrow[l, "\partial_2"'] & \cdots \arrow[l, "\partial_3"'] ,\\
0 & C'_0 \arrow[l] & C'_1 \arrow[l, "\partial'_1"'] & C'_2 \arrow[l, "\partial'_2"'] & \cdots \arrow[l, "\partial'_3"'] .
\end{tikzcd}$$

We can form the *tensor product* of these two complexes obtaining a new chain complex $C_* \otimes_R C'_*$ where

$$(C_* \otimes_R C'_*)_n = \bigoplus_{k=0}^{n} C_k \otimes_R C'_{n-k}$$

and with a boundary map $\overline{\partial} : (C \otimes_R C')_n \longrightarrow (C_* \otimes_R C'_*)_{n-1}$ given by

$$\overline{\partial}(c \otimes c') = \partial c \otimes c' + (-1)^k c \otimes \partial' c'$$

whenever $c \in C_k$ and $c' \in C'_{n-k}$ (one can check that it satisfies $\overline{\partial}\,\overline{\partial} = 0$). We can form the homology $H_*(C \otimes_R C')$ of this complex. If $C_* = C_*(X)$ and $C'_* = C_*(Y)$ are the singular chain complexes of two topological spaces $X$ and $Y$, then the tensor product complex corresponds to the singular chain complex of the product space $X \times Y$.

Now let $G$ be an abelian group. We can dualize any complex $C_*$ w.r.t. the group $G$ by defining $C^k := \operatorname{Hom}_{\mathbb{Z}}(C_k, G)$ and define $\delta_k : C^k \longrightarrow C^{k+1}$ by

$\delta_k(\varphi)(c) := \varphi(\partial_{k+1}(c))$ (whenever $\varphi \in C^k$ and $c \in C_{k+1}$), thus obtaining the co-complex

$$\begin{tikzcd} 0 \arrow[r] & C^0 \arrow[r, "\delta_0"] & C^1 \arrow[r, "\delta_1"] & C^2 \arrow[r, "\delta_2"] & \cdots . \end{tikzcd}$$

The (co)homology of this complex, is denoted $H^*(C_*; G)$. We can also form the complex $C_* \otimes_{\mathbb{Z}} G$

$$\begin{tikzcd} 0 & C_0 \otimes_{\mathbb{Z}} G \arrow[l] & C_1 \otimes_{\mathbb{Z}} G \arrow[l, "\partial_1 \otimes \mathrm{id}_G"'] & C_2 \otimes_{\mathbb{Z}} G \arrow[l, "\partial_2 \otimes \mathrm{id}_G"'] & \cdots \arrow[l, "\partial_3 \otimes \mathrm{id}_G"'] \end{tikzcd}$$

and the homology of this complex is denoted $H_*(C_*; G)$.

For $\varphi \in \mathrm{Hom}_{\mathbb{Z}}((C \otimes_R C')_n, G)$ and $c' = \sum_i c'_i \otimes g'_i \in C'_k \otimes_{\mathbb{Z}} G'$ we can define the *slant product* $\varphi / c' \in \mathrm{Hom}_{\mathbb{Z}}(C'_{n-k}, G \otimes_{\mathbb{Z}} G')$ by

$$\langle \varphi / c', c \rangle := \sum_i \langle \varphi, c \otimes c'_i \rangle \otimes g'_i \tag{2.39}$$

when $c \in C_{n-k}$ (here $\langle \, , \, \rangle$ denotes the duality pairing).

Observe the following identity

$$\delta(\varphi / c') = \delta\varphi / c' - (-1)^{n-k} \varphi / \partial c'$$

which is verifiable by a direct calculation. This identity is the key for the slant product to descend to the homology level, for if $\varphi$ is a cocycle and $c'$ is a cycle, then obviously $\delta(\varphi / c') = 0$, thus $\varphi / c'$ represents a cohomology class. One can check by direct verification that the cohomology class $[\varphi / c']$ is zero when either $\varphi$ or $c'$ is a boundary. Thus we wrap up our construction in the following definition:

**Definition 2.43** (**Slant Product**)**.** Given two chain complexes $C_*$ and $C'_*$ as well as two abelian groups $G$ and $G'$ we define the *slant product*

$$/ : \mathrm{Hom}((C_* \otimes_R C'_*)_n, G) \times (C'_k \otimes_{\mathbb{Z}} G') \longrightarrow \mathrm{Hom}(C'_{n-k}, G \otimes_{\mathbb{Z}} G')$$

by the formula (2.39) above. It descends to the homology level to a well-defined *slant product*

$$/ : H^n(C_* \otimes_R C'_*; G) \times H_k(C'_*; G') \longrightarrow H^{n-k}(C_*; G \otimes_{\mathbb{Z}} G')$$

defined by the formula

$$([\varphi], [c]) \longmapsto [\varphi]/[c] := [\varphi / c].$$

For much more information about slant products, one should consult Chapter 6 of [20].

Next consider the trivial complex line bundle

$$M \times \mathcal{C}^* \times \mathbb{C} \longrightarrow M \times \mathcal{C}^*$$

and equip it with the a $\mathcal{G}$-action by

$$(x, \mathsf{C}, z) \cdot \lambda = (x, \mathsf{C} \cdot \lambda, \lambda(x)^{-1} z).$$

This gives a line bundle

$$(M \times \mathcal{C}^* \times \mathbb{C}) / \mathcal{G} \longrightarrow M \times \mathcal{B}^*$$

by defining $[x, \mathsf{C}, z] \longmapsto (x, [\mathsf{C}])$. Obviously this is well-defined, and one can check that this is, indeed, a complex line bundle over $M \times \mathcal{B}^*$, the vector space operations given by

$$a[x, \mathsf{C}, z] + b[x, \mathsf{C}, z'] = [x, \mathsf{C}, az + bz'].$$

We call this line bundle $\mathbb{U}_\sigma$ (where $\sigma$ emphasizes that it depends on the spin$^c$-structure) the *Seiberg-Witten bundle*. This has a first Chern class $c_1(\mathbb{U}_\sigma) \in H^2(M \times B^*; \mathbb{Z})$. Define the $\mu$-map

$$\mu_\sigma : H_k(M \times \mathcal{B}^*; \mathbb{Z}) \longrightarrow H^{2-k}(\mathcal{B}^*; \mathbb{Z})$$

by $\mu_\sigma(a) = c_1(\mathbb{U}_\sigma)/a$ - the slant product of $c_1(\mathbb{U}_\sigma)$ with $a$. Since $H_0(M \times \mathcal{B}^*; \mathbb{Z}) \cong \mathbb{Z}$ (because $M \times \mathcal{B}^*$ is connected) it makes sense to define the *Seiberg-Witten class* [7] $\mu_\sigma(1) \in H^2(\mathcal{B}^*; \mathbb{Z})$.

We will now define the Seiberg-Witten invariant of the spin$^c$-manifold $M$. Above all we assume that $b_2^+ > 1$. Recall that the space $\Xi_\infty^{\mathrm{red}}$ of perturbation parameters $\eta$ for which $\mathcal{S}(\eta)$ contains reducible solutions, is an affine subspace of $i\Omega_+^2(M)$ of codimension $b_2^+$. Requiring $b_2^+ > 1$ implies that $i\Omega_+^2(M) \setminus \Xi_\infty^{\mathrm{red}}$ is connected. This will be important later when showing that the Seiberg-Witten invariant is independent of the perturbation parameter $\eta$.

With this requirement fixed we define the Seiberg-Witten invariant in different cases, depending on the sign of the virtual dimension $d(\sigma)$.

**Definition 2.44** (**Seiberg-Witten Invariant I**)**.** If $d(\sigma)$ happens to be negative, then we define the Seiberg-Witten invariant to be 0:

$$\mathbf{SW}(M, \sigma) := 0. \tag{2.40}$$

The second case is where $d(\sigma) \geq 0$. We pick $\eta \in i\Omega_+^2(M)_\infty^{\mathrm{reg}}$, then $\mathcal{S}(\eta) = \mathcal{S}^*(\eta)$ and $\mathcal{M}(\eta) = \mathcal{M}^*(\eta)$ is a compact, orientable manifold of dimension $d(\sigma)$. Now it's $\mu_\sigma(1)$'s time to shine: Let $[\mathcal{M}(\eta)] \in H_{d(\sigma)}(\mathcal{M}(\eta); \mathbb{Z})$ denote the *orientation class* of $\mathcal{M}(\eta)$. Define

$$(1 - \mu_\sigma(1))^{-1} := 1 + \mu_\sigma(1) + \mu_\sigma(1)^2 + \cdots \in H^{2*}(\mathcal{M}(\eta); \mathbb{Z}).$$

Since $\mathcal{M}(\eta)$ has finite dimension, this series terminates after finitely many terms.

**Definition 2.45** (**Seiberg-Witten Invariant II**)**.** In the case where $d(\sigma) \geq 0$ we define the *Seiberg-Witten invariant* of $M$ by

$$\mathbf{SW}(M, \sigma) := (1 - \mu_\sigma(1))^{-1}[\mathcal{M}(\eta)] = \int_{\mathcal{M}(\eta)} (1 - \mu_\sigma(1))^{-1}. \tag{2.41}$$

Since all the terms in $(1 - \mu_\sigma(1))^{-1}$ are of even degree (since $\mu_\sigma(1)$ is represented by a 2-form), we see that $\mathbf{SW}(M, \sigma) = 0$ if $d(\sigma)$ is odd and that

$$\mathbf{SW}(M, \sigma) = \mu_\sigma(1)^k[\mathcal{M}(\eta)] = \int_{\mathcal{M}(\eta)} \mu_\sigma(1)^k \tag{2.42}$$

if $d(\sigma) = 2k$. Furthermore, if $d(\sigma) = 0$, then the moduli space is just a finite collection of discrete points. The orientation is just an assignment of + or − to each point, and $\mu_\sigma(1)^0$ is just the constant function 1. If $\varepsilon : \mathcal{M}(\eta) \longrightarrow \{\pm 1\}$ is the function which satisfies that $\varepsilon([\mathsf{C}]) = 1$ if the orientation at $[\mathsf{C}]$ is + and $\varepsilon([\mathsf{C}]) = -1$ if the orientation at $[\mathsf{C}]$ is −, we see (by definition of integration over a discrete manifold) that

$$\mathbf{SW}(M, \sigma) = \sum_{[\mathsf{C}] \in \mathcal{M}(\eta)} \varepsilon([\mathsf{C}]).$$

Sometimes this is the definition of the Seiberg-Witten invariant.

Note how the requirement that $b_2^+$ prevents us from defining the Seiberg-Witten invariant in the simplest case of a compact four-manifold, namely the

---

[7] My own terminology.

four-sphere $S^4$. We all know that $H^2(S^4;Z) = 0$ and hence $b_2^+ = b_2^- = 0$. However, at the end of this chapter we indicate how we can remedy this.

Inherent in the our construction of the invariant (or indeed in the construction of the moduli space) there were quite a few choices: we had to choose a metric, $g$, then a spin$^c$-structure, $\sigma$, and finally a perturbation parameter, $\eta$. Our final goal in this chapter is to show that our new invariant is *independent* of the metric and of the perturbation parameter, and thus only depends on the manifold $M$ and on the spin$^c$-structure. But wait a minute, one may object, doesn't the spin$^c$-structure depend on the metric? After all isn't the spin$^c$-structure defined as some bundle which sits "on top of" the oriented orthonormal frame bundle, which depends on the metric? True, but the matter of fact is that the frame bundles $P_{\mathrm{SO}}(M, g_0)$ and $P_{\mathrm{SO}}(M, g_1)$ corresponding to two different metrics $g_0$ and $g_1$ are canonically isomorphic (this follows from [21] Theorem 12.9 on p. 57). Thus given a spin$^c$-structure $\sigma_0$ on $P_{\mathrm{SO}}(M, g_0)$, i.e. a map $P^c_{\mathrm{Spin}}(M) \longrightarrow P_{\mathrm{SO}}(M, g_0)$ we can compose it with the isomorphism $P_{\mathrm{SO}}(M, g_0) \xrightarrow{\sim} P_{\mathrm{SO}}(M, g_1)$ to obtain a spin$^c$-structure $P^c_{\mathrm{Spin}}(M) \longrightarrow P_{\mathrm{SO}}(M, g_1)$. One can check that the conditions for a spin$^c$-structure are fulfilled, thus, formally, we have inferred:

**Lemma 2.46.** *Let* $\mathrm{Spin}^c(M, g_0)$ *and* $\mathrm{Spin}^c(M, g_1)$ *be the set of isomorphism classes of spin$^c$-structures on $M$ relative to $g_0$ resp. $g_1$. Then there is a canonical bijection* $\mathrm{Spin}^c(M, g_0) \xrightarrow{\sim} \mathrm{Spin}^c(M, g_1)$ *and this bijection preserves the principal* $\mathrm{Spin}^c(4)$*-bundle and hence also the spinor spaces $S$ and $S^\pm$ as well as the determinant line bundle.*

Thus the spin$^c$-structure depends only vaguely on the metric. Something which, however, *does* depend on the metric is the Hodge star operator $*$ and thus also the spaces $\Omega^2_+(M)$ and $L^2_k(i\Lambda^2_+T^*M)$. Describing how these spaces are related is the first task to undertake. Assume we have two metrics $g_0$ and $g_1$ and a curve $g_t$ of metrics joining them (for instance a straight line - after all the space of metrics is affine). Let us by $*_t$ denote the Hodge star operator corresponding to the metric $g_t$ and let $\Omega^2_{+,t}(M)$ denote the corresponding space of $*_t$-self-dual 2-forms. Similarly we define $P^t_+ := \frac{1}{2}(1 + *_t)$, the projection onto $\Omega^2_{+,t}(M)$ as well as $d_t^+ := \frac{1}{2}(1 + *_t) \circ d$ and $\Xi^{\mathrm{red}}_{k,t}$ and $\Xi^{\mathrm{reg}}_{k,t}$, and so on.

**Lemma 2.47.** *The space*

$$i\Omega^2_{+,\bullet}(M) := \coprod_{t\in[0,1]} i\Omega^2_{+,t}(M)$$

*is a trivial* (*infinite rank*) *vector bundle over* $[0, 1]$, *i.e. we have a family of isomorphisms*

$$\Phi_t : i\Omega^2_{+,t}(M) \xrightarrow{\sim} i\Omega^2_{+,0}(M)$$

*and under the isomorphism $\Phi_t$ the affine subspace $\Xi^{\mathrm{red}}_{\infty,t} \subseteq i\Omega^2_{+,t}(M)$ is mapped isomorphically to $\Xi^{\mathrm{red}}_{\infty,0}$.*

*Similarly*

$$L^2_k(\Lambda^2_{+,\bullet}T^*M) := \coprod_{t\in[0,1]} L^2_k(\Lambda^2_{+,t}T^*M)$$

*is a trivial Hilbert bundle* [8].

Proof. First, let $x \in M$ be arbitrary. Since $M$ was assumed to be oriented, we have an *orientation preserving* isomorphism $\Lambda^4T^*_xM \cong \mathbb{R}$, the explicit form of

---

[8] Admittedly, this notation might be a bit misleading: This is the bundle of $L^2_k$ spaces, *not* the space of $L^2_k$-section of a bundle.

which would depend on a choice of metric, (having chosen such a metric we could define the map by sending the oriented volume form to 1) however the sign will remain unchanged under a change of metric. Thus the wedge product becomes a real bilinear form on $\Lambda^2 T_x^* M$ and composing with the orientation preserving isomorphism it makes sense to talk about the sign of $\omega \wedge \eta$ (independently of the metric). For a 2-form $\omega$ which is self-dual w.r.t. some arbitrary metric $g$, the wedge $\omega \wedge \omega$ is always positive, simply because

$$\omega \wedge \omega = \omega \wedge *\omega = \langle \omega, \omega \rangle_g > 0,$$

and when it is positive relative to one metric, it is positive relative to all metrics.

Now let $(g_t)$ be the curve of metrics, let $\omega \in \Lambda^2 T_x^* M$ and assume that it is $g_t$ self-dual for some $t \neq 0$, i.e. $*_t \omega = \omega$. Hence $\omega \wedge \omega$ is positive. $\omega$ need no longer be self-dual relative to the metric $g_0$, but then we split it into self-dual and anti-self-dual parts

$$\omega = \omega_+ + \omega_-$$

relative to $*_0$. We see that

$$\begin{aligned} 0 \leq \omega \wedge \omega &= (\omega_+ + \omega_-) \wedge (\omega_+ + \omega_-) \\ &= \omega_+ \wedge \omega_+ + \omega_- \wedge \omega_- + \omega_+ \wedge \omega_- + \omega_- \wedge \omega_+ \\ &= \omega_+ \wedge *_0 \omega_+ - \omega_- \wedge *_0 \omega_- - \omega_+ \wedge *_0 \omega_- + \omega_- \wedge *_0 \omega_+ \\ &= |\omega_+|_0^2 - |\omega_-|_0^2 + \langle \omega_-, \omega_+ \rangle_0 - \langle \omega_+, \omega_- \rangle_0 \\ &= |\omega_+|_0^2 - |\omega_-|_0^2, \end{aligned}$$

i.e. $|\omega_+|_0^2 \geq |\omega_-|_0^2$. We can define an embedding

$$\iota_t : \Lambda^2_{+,t} T_x^* M \hookrightarrow \Lambda^2_{+,0} T_x^* M \times \Lambda^2_{-,0} T_x^* M$$

by $\omega \longmapsto (\omega_+, \omega_-)$. The image of this map is the graph of a linear operator $\Lambda^2_{+,0} T_x^* M \longrightarrow \Lambda^2_{-,0} T_x^* M$ because it is a linear subspace and if $(\omega_+, \omega_-) \in \operatorname{im} \iota_t$ satisfies $\omega_+ = 0$ then by the inequality above we have $\omega_- = 0$. For a graph, the projection down to the domain is an isomorphism, and since the projection in this case is the map $\frac{1}{2}(1 + *_0)$ we define

$$\Phi_t := \frac{1}{2}(1 + *_0) : \Lambda^2_{+,t} T_x^* M \xrightarrow{\sim} \Lambda^2_{+,0} T_x^* M,$$

in other words $\Phi_t = \frac{1}{2}(1 + *_0)|_{\Lambda^2_{+,t}(M)}$. Obviously the inverse map

$$\Phi_t^{-1} : \Lambda^2_{+,0} T_x^* M \longrightarrow \Lambda^2_{+,t} T_x^* M$$

is just $\frac{1}{2}(1 + *_t)$. Putting all these maps together for all $x$ gives us a smooth bundle isomorphism $\Lambda^2_{+,t} T^* M \xrightarrow{\sim} \Lambda^2_{+,0} T^* M$ which induces a bijection on the spaces of sections

$$\Phi_t : \Omega^2_{+,t}(M) \xrightarrow{\sim} \Omega^2_{+,0}(M).$$

We just have to show that it maps $\Xi^{\text{red}}_{\infty,t}$ to $\Xi^{\text{red}}_{\infty,0}$. But remember that $\Xi^{\text{red}}_{\infty,t}$ was an affine space modeled on $d_t^+(i\Omega^1(M))$, i.e.

$$\Xi^{\text{red}}_{\infty,t} = \eta_t + d_t^+(i\Omega^1(M))$$

where $\eta_t$ is a fixed element. Note that $P_+^t P_+^0 = P_+^t$. If $\eta_0 \in \Xi^{\text{red}}_{\infty,t}$ there exists a reducible solution $(0, \mathcal{A})$ to the $g_0$-Seiberg-Witten equations perturbed by $\eta_0$, in particular $P_+^0 F_{\mathcal{A}} = \eta_0$. Act by $P_+^t$ to get

$$F_{\mathcal{A}}^{+_t} = P_+^t F_{\mathcal{A}} = P_+^t P_+^0 F_{\mathcal{A}} = P_+^t \eta_0 = \Phi_t^{-1}(\eta_0),$$

i.e. $(0, \mathcal{A})$ is a reducible solution to the $g_t$-Seiberg-Witten equations, so that $\Phi_t^{-1}(\eta_0) \in \Xi^{\mathrm{red}}_{\infty,t}$. Therefore we may pick $\eta_t$ such that $\Phi_t(\eta_t) = \eta_0$ and then:

$$\begin{aligned}\Phi_t(\Xi^{\mathrm{red}}_{\infty,t}) &= \Phi_t(\eta_t + d_t^+(i\Omega^1(M))) \\ &= \eta_0 + \Phi_t(d_t^+(i\Omega^1(M))) = \eta_0 P^0_+ P^t_+(d(i\Omega^1(M))) \\ &= \eta_0 + P^0_+(d(i\Omega^1(M))) = \eta_0 + d_0^+(i\Omega^1(M)) = \Xi^{\mathrm{red}}_{\infty,0}.\end{aligned}$$

By completion the claims hold also on the level of Sobolev spaces. □

Assume $\eta_j \in \Xi^{\mathrm{reg}}_{k-1,j} \setminus \Xi^{\mathrm{red}}_{k-1,j}$ for $j = 0, 1$. Let $\mathcal{K}_k(\eta_0, \eta_1)$ denote the set of $C^1$-sections of the bundle $L^2_{k-1}(i\Lambda^2_{+,\bullet}T^*M)$ such that $\gamma(0) = \eta_0$ and $\gamma(1) = \Phi_1(\eta_1)$. Since this Hilbert bundle is trivial, the space of sections is an affine space modeled over the Banach space $\mathcal{K}^0_k$ of $C^1$-curves $I \longrightarrow L^2_{k-1}(i\Lambda^2_{+,0}T^*M)$ with $\gamma(0) = \gamma(1) = 0$. Then define

$$\mathcal{PPC}^*_k := \mathcal{C}^*_k \times \mathcal{K}_k(\eta_0, \eta_1) \times [0, 1].$$

This is a Banach manifold with boundary (the boundary being $\mathcal{C}^*_k \times \mathcal{K}(\eta_0, \eta_1) \times \{0, 1\}$) whose tangent space at an arbitrary point is (isomorphic to) $L^2_k(iT^*M \times S^+) \times \mathcal{K}^0_k \times \mathbb{R}$. Define an action of the gauge group $\mathcal{G}_{k+1}$ on $\mathcal{PPC}^*_k$ by letting $\lambda \in \mathcal{G}_{k+1}$ act on the first component only:

$$(\mathsf{C}, (\eta_s), t) \cdot \lambda = (\mathsf{C} \cdot \lambda, (\eta_s), t)$$

and then define

$$\mathcal{PPB}^*_k := \mathcal{PPC}^*_k / \mathcal{G}_{k+1} = \mathcal{B}^*_k \times \mathcal{K}_k(\eta_0, \eta_1) \times [0, 1].$$

Let furthermore $\overline{\mathcal{V}} \longrightarrow [0, 1]$ be the Hilbert bundle whose fiber over $t \in [0, 1]$ is $\overline{\mathcal{V}}_t := L^2_{k-1}(i\Lambda^2_{+,t}T^*M \oplus S^-)$ (use the trivialization $\Phi_t$ above to see that $\overline{\mathcal{V}}$ is indeed a vector bundle). If $\pi_3 : \mathcal{PPC}^*_k \longrightarrow [0, 1]$ denotes the projection onto the third component, we can pull back $\overline{\mathcal{V}}$ along $\pi_3$ to obtain a Hilbert bundle $\mathcal{V} := \pi_3^*\overline{\mathcal{V}}$ over $\mathcal{PPC}^*_k$. Above the point $(\mathsf{C}, (\eta_s), t) \in \mathcal{PPC}^*_k$ we have the fiber $L^2_{k-1}(i\Lambda^2_{+,t}T^*M \oplus S^-)$. Define a section $G$ of this bundle by

$$G(\psi, \mathcal{A}, (\eta_s), t) := \begin{pmatrix} F^{+,t}_{\mathcal{A}} - q^t(\psi) + \eta_t \\ \not{D}^t_{\mathcal{A}}\psi \end{pmatrix}.$$

The section, of course, just represents the Seiberg-Witten equations for the various metrics $g_t$ and various $\eta_t$'s, more precisely, if $G(\psi, \mathcal{A}, (\eta_s), t) = 0$ then $(\psi, \mathcal{A})$ solves the Seiberg-Witten equations with metric $g_t$ and perturbation parameter $\eta_t$.

Then we define

$$\mathcal{PPM}^*_k := \{[\psi, \mathcal{A}, (\eta_s), t] \mid G(\psi, \mathcal{A}, (\eta_s), t) = 0\} \subseteq \mathcal{PPB}^*_k.$$

This is well-defined since the solution space to the Seiberg-Witten equations is gauge invariant.

**Proposition 2.48.** *The space $\mathcal{PPM}^*_k$ is a smooth Banach manifold with boundary and by choosing a representative $(\psi, \mathcal{A}, (\eta_s), t)$ for the point $[\psi, \mathcal{A}, (\eta_s), t] \in \mathcal{PPM}^*_k$ we may identify the tangent space of $\mathcal{PPM}^*_k$ at that point with the kernel of the map*

$$\mathcal{F}^t_{(\psi,\mathcal{A})} \dotplus T^t \dotplus A_w : L^2_k(iT^*M \oplus S^+) \oplus \mathcal{K}^0_k \oplus \mathbb{R} \longrightarrow L^2_{k-1}(i\Lambda^2_+T^*M \oplus S^- \oplus i\mathbb{R})$$

*where* $\mathcal{F}^t_{(\psi,\mathcal{A})}$ *is just the map defined in Lemma* 2.33 *in the* $g_t$*-setting, where* $T^t$ *is the map*: $T^t(\eta_s) := (-\eta_t, 0, 0)$, *and where* $A_w$ *for*

$$w := \frac{d}{ds}\bigg|_{s=t} \begin{pmatrix} F^{+_s}_{\mathcal{A}} + \eta_s \\ D\!\!\!/^{\,s}_{\mathcal{A}}\psi \\ 0 \end{pmatrix} \in L^2_{k-1}(i\Lambda^2_+ T^*M \oplus S^- \oplus i\mathbb{R})$$

*is the map* $\mathbb{R} \ni \tau \longmapsto \tau w$.

Proof. We will proceed in much the same way we did, when we proved that the parametrized moduli space was a manifold, namely given a point in $\mathcal{PPM}^*_k$ we can identify a neighborhood of that point with a level set and show that this level set is a manifold.

Let's consider the point $(\psi, \mathcal{A}, (\eta_s), t)$ mentioned in the statement of the proposition and put $\mathsf{C} := (\psi, \mathcal{A})$. Around the point $[\mathsf{C}] \in \mathcal{M}^*_k(\eta_t)$ we have the neighborhood $\widehat{U} \subseteq \mathcal{B}^*_k$ as constructed in Proposition 2.32. Since the question of smoothness is a local one, it is enough to give

$$\mathcal{PPM}^*_k \cap (\widehat{U} \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1])$$

(which is an open neighborhood of $[\mathsf{C}, (\eta_s), t]$ in $\mathcal{PPM}^*_k$) a smooth structure.

Define $\widetilde{G}$ on $L^2_k(iT^*M \oplus S^+) \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1]$ by

$$(\varphi, \alpha, (\eta_s), t) \longmapsto \begin{pmatrix} F^{+,t}_{\mathcal{A}+\alpha} - q^t(\psi+\varphi) + \eta_t \\ \frac{1}{2}\alpha\cdot\psi + D\!\!\!/^{\,t}_{\mathcal{A}+\alpha}(\psi+\varphi) \\ 2d^{*_t}\alpha - i\,\mathrm{Im}\langle\psi,\varphi\rangle_t \end{pmatrix}$$

(this is just a section of the Hilbert bundle obtained by pulling back $\mathcal{V}$ to $L^2_k(iT^*M \oplus S^+) \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1]$ along the map $(\dot\psi, \dot\alpha, (\zeta_s), \tau) \longmapsto (\dot\psi, \mathcal{A}_0 + \dot\alpha, (\zeta_s), \tau)$ and taking the direct sum with the trivial vector bundle $iL^2_{k-1}(\mathbb{R})$). Again by Proposition 2.32 we have a diffeomorphism $V \times \mathcal{K}_k(\eta_0,\eta_1) \times [0,1] \longrightarrow \widehat{U} \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1]$ given by

$$((\varphi, \alpha), (\eta_s), t) \longmapsto ([\psi+\varphi, \mathcal{A}+\alpha], (\eta_s), t)$$

(where $V \subseteq \ker(d\rho^{\mathsf{C}})^* \subseteq L^2_k(S^+ \oplus iT^*M)$ is an open neighborhood of 0) and since $(\varphi, \alpha) \in V \subseteq \ker(d\rho^{\mathsf{C}}_1)^* = \ker \mathcal{F}_3$ it is not hard to see that $\mathcal{PPM}^*_k \cap (\widehat{U} \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1])$ under the inverse of the diffeomorphism above is mapped to $\widetilde{G}^{-1}(0) \cap (V \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1])$ so (just as in the proof of smoothness of $\mathcal{PM}^*_k$) we will show that this intersection has a smooth structure. But since $\widetilde{G}^{-1}(0) \cap (V \times \mathcal{K}_k(\eta_0,\eta_1) \times [0,1]$ is open in $\widetilde{G}^{-1}(0)$, it is enough to consider just $\widetilde{G}^{-1}(0)$.

Since the pullback of $\mathcal{V}$ to $L^2_k(iT^*M \oplus S^+) \times \mathcal{K}_k(\eta_0,\eta_1) \times [0,1]$ is (still) a trivial Hilbert bundle, we can identify the section $\widetilde{G}$ with a map from $L^2_k(iT^*M \oplus S^+) \times \mathcal{K}_k(\eta_0, \eta_1) \times [0,1]$ into a fixed fiber. If we view it as such, we just need to show that 0 is a regular value in order to prove smoothness of $\widetilde{G}^{-1}(0)$. If $\pi$ denotes the projection in the Hilbert bundle $\mathcal{V}$, this is just the same as showing that $d\pi \circ d\widetilde{G}_{(\varphi,\alpha,(\eta_s),t)}$ is surjective. It is enough to show this for points at the form $(0, 0, (\eta_s), t)$ (since we can continuously deform $d\pi \circ d\widetilde{G}_{(\varphi,\alpha,(\eta_s),t)}$ to $d\pi \circ d\widetilde{G}_{(0,0,(\eta_s),t)}$ within the space of Fredholm operators) and over this point $d\pi \circ d\widetilde{G}$ takes the form

$$(\dot\varphi, \dot\alpha, (\delta_s), \tau) \longmapsto \begin{pmatrix} d^+_t\dot\alpha - dq^t_\psi(\dot\varphi) \\ \frac{1}{2}\dot\alpha\cdot\psi + D\!\!\!/^{\,t}_{\mathcal{A}}\dot\varphi \\ 2d^{*_t}\dot\alpha - i\,\mathrm{Im}\langle\psi,\dot\varphi\rangle \end{pmatrix} + \tau\,\frac{d}{ds}\bigg|_{s=t} \begin{pmatrix} F^{+_s}_{\mathcal{A}} + \eta_s \\ D\!\!\!/^{\,s}_{\mathcal{A}}\psi \\ 0 \end{pmatrix} + \begin{pmatrix} -\delta_t \\ 0 \\ 0 \end{pmatrix},$$

i.e.

$$d\pi \circ d\widetilde{G}_{(0,0,(\eta_s),t)} = \mathcal{F}^t_{\mathsf{C}} \dotplus A_w \dotplus T^t.$$

The map $T^t$ is bounded (since it is just the evaluation map defined on a space with the supremum norm) and $\mathcal{F}^t_{\mathsf{C}} \dotplus A_w$ is Fredholm of index $d(\sigma)+1$ (for if $w$ in the image of $\mathcal{F}^t_{\mathsf{C}}$ then $A_w$ contributes with an extra dimension to the kernel, and if $w$ is not in the image, $A_w$ reduces the codimension of the image by 1, in both cases adding 1 to the index). If we can show that $\mathcal{F}^t_{\mathsf{C}} \dotplus A_w \dotplus T^t$ is surjective, then it follows from Proposition 1.27 that $\ker(d\pi \circ d\widetilde{G}_{(0,0,(\eta_s),t)})$ has a complement. The cases $t=0$ and $t=1$ follows obviously from the fact that $\eta_j \in \Xi^{\mathrm{reg}}_{k-1,j}$ (so that $\mathcal{F}^j_{\mathsf{C}}$ is surjective). For an arbitrary $0<t<1$ it need not be true that $\mathcal{F}^t_{\mathsf{C}}$ is surjective (we have not necessarily chosen the curve $\eta_t$ such that $\eta_t \in \Xi^{\mathrm{reg}}_{k-1,t}$). But we always know that $\mathcal{F}^t_{\mathsf{C}}$ is surjective on the last two components of the target space. But then the map $T^t$ yields surjectivity on the first component. This shows that $\widetilde{G}^{-1}(0)$ has a smooth structure. □

So now the space $\mathcal{PPM}^*_k$ has been given a smooth structure, hence we can talk about smooth maps to and from $\mathcal{PPM}^*_k$. Denote by $\pi_3 : \mathcal{PPM}^*_k \longrightarrow \mathcal{K}_k(\eta_0,\eta_1)$ the natural projection. This is a smooth map. For certain curves, the preimage is a submanifold (with boundary) of $\mathcal{PPM}^*_k$ and if we are lucky (and we are!) the boundary will be equal to the disjoint union of the moduli spaces $\mathcal{M}_k(g_0,\eta_0)$ and $\mathcal{M}_k(g_1,\eta_1)$.

**Theorem 2.49** (**Existence of a Cobordism**). *There exists a dense subset* $\mathcal{K}^{\mathrm{reg}}_k \subseteq \mathcal{K}_k(\eta_0,\eta_1)$ *of sections* $(\eta_s)$ *for which the preimage* $\mathcal{M}^*_k(\{\eta_s\}) := \pi_3^{-1}[(\eta_s)]$ *is a smooth manifold with boundary of dimension* $d(\sigma)+1$ *whose tangent space at a point* $[\mathsf{C},t] = [\mathsf{C},(\eta_s),t] \in \mathcal{M}^*_k(\{\eta_s\}) \subseteq \mathcal{PPM}^*_k$ *can be identified with*

$$T_{[\mathsf{C},t]}\mathcal{M}^*_k(\{\eta_s\}) \cong \ker(\mathcal{F}^t_{\mathsf{C}} \dotplus A_w)$$

*and such that the boundary is diffeomorphic to the disjoint union of* $\mathcal{M}_k(g_0,\eta_0) = \mathcal{M}^*_k(g_0,\eta_0)$ *and* $\mathcal{M}_k(g_1,\eta_1) = \mathcal{M}^*_k(g_1,\eta_1)$. *For each such curve* $(\eta_s)$ *there is a canonical orientation on* $\pi_3^{-1}[(\eta_s)]$ *such that the boundary orientation is identical to the orientation on* $\mathcal{M}_k(g_1,\eta_1)$ *and opposite to the orientation on* $\mathcal{M}_k(g_0,\eta_0)$.

*If the section* $(\eta_s)$ *has been chosen in such a way that* $\eta_s \notin \Xi^{\mathrm{red}}_{k-1,s}$ *for all* $s \in [0,1]$, *then* $\mathcal{M}^*_k(g_s,\eta_s) = \mathcal{M}_k(g_s,\eta_s)$ *is compact and consequently* $\mathcal{M}^*_k(\{\eta_s\})$ *is compact.*

Proof. Since the differential of a projection is a projection, then at the point $[\mathsf{C},(\eta_s),t] \in \mathcal{PPM}^*_k$ the differential of $\pi_3$ is

$$(d\pi_3)_{[\mathsf{C},(\eta_s),t]} : T_{[\mathsf{C},(\eta_s),t]}\mathcal{PPM}^*_k = \ker(\mathcal{F}^t_{\mathsf{C}} \dotplus T^t \dotplus A_w) \longrightarrow \mathcal{K}_k(\eta_0,\eta_1)$$

is given by $(\dot\varphi,\dot\alpha,(\zeta_s),\tau) \longmapsto (\zeta_s)$. Since $\mathcal{F}^t_{\mathsf{C}} \dotplus A_w$ has index $d(\sigma)+1$ as shown in the preceding proof, it follows from Proposition 1.27 that $(d\pi_3)_{[\mathsf{C},(\eta_s),t]}$ is a Fredholm operator with index $d(\sigma)+1$, i.e. $\pi_3$ is a Fredholm map. Then the Smale-Sard Theorem tells us that the set of regular values for $\pi_3$ is dense in $\mathcal{K}_k(\eta_0,\eta_1)$ and for such a regular value, the preimage $\mathcal{M}^*_k(\{\eta_s\}) = \pi_3^{-1}[(\eta_s)]$ is a smooth manifold with boundary of dimension $d(\sigma)+1$ and that we can identify the tangent space $T_{[\mathsf{C},t]}\mathcal{M}^*_k(\{\eta_s\})$ with $\ker(\mathcal{F}^t_{\mathsf{C}} \dotplus A_w)$.

Define $\pi_4 : \mathcal{PPM}^*_k \longrightarrow [0,1]$ and let $\overline{\pi}_4 : \mathcal{M}^*_k(\{\eta_s\}) \longrightarrow [0,1]$ be its restriction to $\mathcal{M}^*_k(\{\eta_s\})$. It's easy to see that $\overline{\pi}_4$ is a submersion and hence that $\overline{\pi}_4^{-1}(0)$ and $\overline{\pi}_4^{-1}(1)$ are submanifolds of $\mathcal{M}^*_k(\{\eta_s\})$. The union of these two fibers equals the boundary of $\mathcal{M}^*_k(\{\eta_s\})$. Furthermore

$$\overline{\pi}_4^{-1}(0) = \pi_4^{-1}(0) \cap \pi_3^{-1}(\{\eta_s\})$$

in other words this submanifold consists of the points $([\mathsf{C}], (\eta_s), 0)$ in $\mathcal{PPM}^*_k$ which satisfy $F^{\eta_0}(\mathsf{C}) = 0$, i.e. $\overline{\pi}_4^{-1}(0) = \mathcal{M}_k(\eta_0)$. Similarly we have $\overline{\pi}_4^{-1}(1) = \mathcal{M}_k(\eta_1)$. Thus we have seen that the boundary of $\mathcal{M}^*_k(\{\eta_s\})$ is equal to the disjoint union of $\mathcal{M}_k(\eta_0)$ and $\mathcal{M}_k(\eta_1)$.

So what about orientations? Well, first of all $\mathcal{M}^*_k(\{\eta_s\})$ is orientable for the following reason: The tangent space at a point $[\mathsf{C}, t]$ can be identified with $\ker(\mathcal{F}^t_{\mathsf{C}} \dotplus A_w)$ but then the determinant of this space is, in a canonical way, isomorphic to $\det \mathcal{F}^t_{\mathsf{C}}$ (since $\mathcal{F}^t_{\mathsf{C}} \dotplus A_w$ is surjective, as is shown in Section 1.5 right after Definition 1.34). But $\det \mathcal{F}^t_{\mathsf{C}}$ has been given a gauge-invariant orientation, so this determines a gauge-invariant orientation on $\det\ker(\mathcal{F}^t_{\mathsf{C}} \dotplus A_w)$ which then gives an orientation on the tangent space $T_{[\mathsf{C},t]}\mathcal{M}^*_k(\{\eta_s\})$. Hence $\mathcal{M}^*_k(\{\eta_s\})$ can be given an orientation and the orientation induces a canonical orientation on the boundary: Any manifold with boundary has an outward pointing vector field $N$ (defined on the boundary $\partial\mathcal{M}^*_k(\{\eta_s\})$) and given a point $x$ on the boundary, we declare a basis $\{v_1, \dots, v_{d(\sigma)}\}$ for $T_x\partial\mathcal{M}^*_k(\{\eta_s\})$ to be oriented, if $\{v_1, \dots, v_{d(\sigma)}, N_x\}$ is an oriented basis for $T_x\mathcal{M}^*_k(\{\eta_s\})$. We just have to determine the induced boundary orientation. Again we will make use of the isomorphism

$$\det \mathcal{F}^t_{\mathsf{C}} \longrightarrow \Lambda^{\mathrm{top}} \ker(\mathcal{F}^t_{\mathsf{C}} \dotplus A_w)$$

from Section 1.5. By the choices of $\eta_0$ and $\eta_1$, $\mathcal{F}^t_{\mathsf{C}}$ is surjective when $t = 0, 1$. By Lemma 1.26 $\mathcal{F}^t_{\mathsf{C}}$ must also be surjective, whenever $[\mathsf{C}, t]$ is chosen close enough to the boundary. Therefore

$$\det \mathcal{F}^t_{\mathsf{C}} = \Lambda^{\mathrm{top}} \ker \mathcal{F}^t_{\mathsf{C}} \cong \Lambda^{\mathrm{top}} \ker(\mathcal{F}^t_{\mathsf{C}} \dotplus A_w)$$

and the latter isomorphism is by the construction in Section 1.5 given by the following: If $\{y_1, \dots, y_{d(\sigma)}\}$ is an oriented basis for $\ker \mathcal{F}^t_{\mathsf{C}}$ and $v$ in the domain of $\mathcal{F}^t_{\mathsf{C}}$ is chosen such that $\mathcal{F}^t_{\mathsf{C}}(v) = A_w(1)$ then the isomorphism is given by

$$y_1 \wedge \cdots \wedge y_{d(\sigma)} \longmapsto (y_1, 0) \wedge \cdots \wedge (y_{d(\sigma)}, 0) \wedge (v, 1).$$

When $t = 0$, the vector $(v, 1)$ points inward, and when $t = 1$ the vector $(v, 1)$ points outward. Thus for $t = 0$ we see that the basis $\{y_1, \dots, y_{d(\sigma)}\}$ is *not* oriented in the induced boundary orientation, whereas it *is* oriented when $t = 0$.

The last claim about compactness of $\mathcal{M}^*_k(\{\eta_s\})$ is a standard topology argument which is left to the reader. □

This cobordism result is the cornerstone in showing independence of metric and perturbation parameter. However, one last question remains: can we find sections $(\eta_s) \in \mathcal{K}^{\mathrm{reg}}_k$ such that $\eta_s \notin \Xi^{\mathrm{red}}_{k-1,s}$? Yes, we can! Finding a section starting at $\eta_0$ and ending at $\eta_1$ corresponds, under the trivialization $\Phi_t$, to finding a curve in $L^2_{k-1}(i\Lambda^2_{+,0}T^*M) \setminus \Xi^{\mathrm{red}}_{k-1,0}$ from $\eta_0$ to $\Phi_1(\eta_1)$. Since, by assumption, $b_1^+ = \operatorname{codim} \Xi^{\mathrm{red}}_{k-1,0} > 1$, such curves exist, no matter what $\eta_0$ and $\eta_1$ are (if the codimension of $\Xi^{\mathrm{red}}_{k-1,0}$ was equal to 1, there would be no such curves if $\eta_0$ and $\eta_1$ were at either side of the "barrier" $\Xi^{\mathrm{red}}_{k-1,0}$), and the space of such curves is open in the space of curves in $L^2_{k-1}(i\Lambda^2_{+,0}T^*M)$. Since $\mathcal{K}^{\mathrm{reg}}_k$ is dense, we *can* find sections of $\mathcal{K}^{\mathrm{reg}}_k$ which avoids $\Xi^{\mathrm{red}}_{k-1,s}$.

**Theorem 2.50.** *The Seiberg-Witten invariant is a well-defined invariant of $(M, \sigma)$ independent of the metric on $g$ and on the perturbation parameter $\eta$.*

Proof. Assume we have a fixed spin$^c$-structure $\sigma$ on $M$. Pick two metrics $g_0$ and $g_1$ on $M$ and pick $\eta_0 \in i\Omega^2_+(M)^{\mathrm{reg}}_{\infty,0}$ and $\eta_1 \in i\Omega^2_+(M)^{\mathrm{reg}}_{\infty,1}$, then we know that $\mathcal{M}^*_k(\eta_0) = \mathcal{M}_k(\eta_0)$ and $\mathcal{M}^*_k(\eta_1) = \mathcal{M}_k(\eta_1)$ for all $k$ are smooth compact manifolds of dimension $d(\sigma)$ and that they are diffeomorphic to $\mathcal{M}(g_0, \eta_0)$ and

$\mathcal{M}(g_1, \eta_1)$ respectively (we will just write $\mathcal{M}(\eta_0)$ and $\mathcal{M}(\eta_1)$ for now). If this dimension is odd we have nothing to show (in this case $\mathbf{SW}(M, \sigma) = 0$ no matter how the moduli spaces look). So assume $d(\sigma) = 2k$ for some $k \geq 1$. We have to show that

$$\int_{\mathcal{M}(\eta_0)} \mu_\sigma(1)^n = \int_{\mathcal{M}(\eta_1)} \mu_\sigma(1)^n.$$

Choose a curve $g_s$ between the metrics (if could be the line between them for instance) and a section $(\eta_s) \in \mathcal{K}_k^{\mathrm{reg}}$ such that $\eta_s \notin \Xi_{k-1,s}^{\mathrm{red}}$. Then $\mathcal{M}_k(\{\eta_s\})$ is a smooth *compact* manifold with boundary of dimension $d(\sigma) + 1$. Pick a representative $\Omega_\sigma$ for the cohomology class $\mu_\sigma(1)$. $\Omega_\sigma$ is a form over $\mathcal{B}_k^*$ and we can extend $\Omega_\sigma$ to $\mathcal{PPM}_k^*$ (by pullback along the projection) and restrict to $\mathcal{M}_k(\{\eta_s\})$. We still call it $\Omega_\sigma$. This is an extension of the original $\Omega_\sigma$, it is closed and hence by Stokes theorem (for which we need $\mathcal{M}_k^*(\{\eta_s\})$ to be compact) we get

$$0 = \int_{\mathcal{M}_k^*(\{\eta_s\})} d(\Omega_\sigma^n) = \int_{\partial\mathcal{M}_k^*(\{\eta_s\})} \Omega_\sigma^n = \int_{\mathcal{M}(\eta_1)} \Omega_\sigma^n - \int_{\mathcal{M}(\eta_0)} \Omega_\sigma^n$$

(the last minus follows because of the reverse orientation on the boundary compared to that of $\mathcal{M}(\eta_0)$). The integral of the representation is (by definition) the integral of the cohomology class, hence the two integrals are equal. □

So after a lot of work we have obtained the result that the Seiberg-Witten invariant **SW** depends only on the manifold $M$ and on the choice of spin$^c$-structure, $\sigma$. Thus we get a map

$$\mathbf{SW}_M : \mathrm{Spin}^c(M) \longrightarrow \mathbb{Z}$$

(it maps into $\mathbb{Z}$ since the Seiberg-Witten class is an integer cohomology class). Denote by $\mathbb{B}_M = \{\sigma \in \mathrm{Spin}^c(M) \mid \mathbf{SW}_M(\sigma) \neq 0\}$ the *support* of $\mathbf{SW}_M$ then:

**Theorem 2.51.** *$\mathbb{B}_M$ is a finite set, i.e. for a given manifold at most finitely many spin$^c$-structures give a non-zero Seiberg-Witten invariant.*

PROOF. First a preliminary calculation: if $\mathcal{A}$ is a connection on the determinant line bundle and $F_\mathcal{A}$ its curvature, then we may split it $F_\mathcal{A} = F_\mathcal{A}^+ + F_\mathcal{A}^-$ and thus

$$\begin{aligned} F_\mathcal{A} \wedge F_\mathcal{A} &= F_\mathcal{A}^+ \wedge F_\mathcal{A}^+ + F_\mathcal{A}^- \wedge F_\mathcal{A}^- \\ &= F_\mathcal{A}^+ \wedge *F_\mathcal{A}^+ - F_\mathcal{A}^- \wedge *F_\mathcal{A}^- = (|F_\mathcal{A}^+|^2 - |F_\mathcal{A}^-|^2) dV_g. \end{aligned}$$

Fix for the rest of the proof a perturbation parameter $\eta$. Assume that $\sigma$ is a spin$^c$-structure on $M$ such that $\mathbf{SW}(M, \sigma) \neq 0$. That implies $d(\sigma) \geq 0$, i.e.

$$0 \leq c_1(L)^2[M] - 2\chi(M) - 3\sigma(M).$$

We want to show that the first Chern class of such a spin$^c$-structure must belong to a finite set. Combining with Proposition 1.6, where we showed that only finitely many spin$^c$-structures have the same determinant line bundle, we see that only finitely many feasible spin$^c$-structures exist.

The first Chern class of $L$ is represented by the curvature of any connection $\mathcal{A}$ on $L$, thus $c_1(L) = [\frac{i}{2\pi} F_\mathcal{A}]$ and hence (recall that $F_\mathcal{A}$ is imaginary-valued)

$$\begin{aligned} 4\pi^2 c_1(L)^2[M] &= \int_M iF_\mathcal{A} \wedge iF_\mathcal{A} = \int_M (|F_\mathcal{A}^+|^2 - |F_\mathcal{A}^-|^2) dV_g \\ &= \|F_\mathcal{A}^+\|_{L^2}^2 - \|F_\mathcal{A}^-\|_{L^2}^2. \end{aligned}$$

>From this it follows that

$$\|F_\mathcal{A}^-\|_{L^2}^2 = -4\pi^2 c_1(L)^2[M] + \|F_\mathcal{A}\|_{L^2}^2 \leq \|F_\mathcal{A}^+\|_{L^2}^2 - 8\pi^2\chi(M) - 12\pi^2\sigma(M).$$

If $(\psi, \mathcal{A})$ is a solution to the Seiberg-Witten equations perturbed by $\eta$ we get from the second equation

$$|F_{\mathcal{A}}^{+}(x)| \leq |q(\psi)(x)| + |\eta(x)| \leq |\psi(x)|^2 + |\eta(x)|.$$

$\eta$ is at least continuous, so it is bounded, and by the estimate in Lemma 2.26 $\psi$ is bounded by a constant which depends on $\eta$ but not on $\mathcal{A}$. Thus we get a uniform $L^2$-bound on $F_{\mathcal{A}}^{+}$. Combined with the $L^2$-bound on $F_{\mathcal{A}}^{-}$ we obtain an $L^2$-bound on $F_{\mathcal{A}}$ which depends only on $\eta$, but is independent of the line bundle $L$! This implies that the first Chern classes of these line bundles must belong to a bounded (hence finite) subset of $H^2(M;\mathbb{Z})$. Since the first Chern class is an isomorphism $\operatorname{Pic}^{\infty}(M) \xrightarrow{\sim} H^2(M;\mathbb{Z})$, $L$ must belong to a finite set as well. □

Thus as seen from a Seiberg-Witten point of view only finitely many of the possible spin$^c$-structures are interesting. One might then ask, what are the dimensions of the moduli spaces in the interesting cases? This problem has not yet been fully resolved, although the following conjecture is strong believed to be true:

**Conjecture 2.52.** *Given a manifold $M$, then for all $\sigma \in \mathbb{B}_M$ we have $d(\sigma) = 0$.*

**Theorem 2.53.** *Let $M$ be a manifold on which we may choose a metric $g$ whose scalar curvature satisfies $\kappa(x) > 0$ for all $x \in M$, then $\mathbb{B}_M = \emptyset$, i.e. for any choice of spin$^c$-structure, the Seiberg-Witten invariant is 0.*

Proof. Pick an arbitrary spin$^c$-structure $\sigma$. Since $M$ is compact and $\kappa$ is continuous (smooth) we must have $\min_{x\in M} \kappa(x) > 0$. Since $i\Omega_{+}^{2}(M)_{\infty}^{\mathrm{reg}}$ is dense, we can pick an $\eta$ in this space as close to 0 as we like, in particular we can find $\eta$ such that $\|\eta\|_\infty \leq \frac{1}{4}\min_{x\in M} \kappa(x)$. Then it is a consequence of Lemma 2.26 that all solutions in $\mathcal{S}(\eta)$ are reducible. But since $\eta \notin \Xi_{\infty}^{\mathrm{red}}$ there are no reducible solutions, hence no solutions at all. Then $\mathcal{M}(\eta) = \mathcal{M}^*(\eta) = 0$ and thus the Seiberg-Witten invariant must be 0. □

As mentioned earlier, we can not formally define the Seiberg-Witten invariant for $S^4$ since $b_2^+ = 0$, but on the other hand, $S^4$ does comply with the requirement that its scalar curvature should be strictly positive (indeed for the standard metric, the scalar curvature is just 1). We can then use the theorem to simply *define* the Seiberg-Witten invariant of $S^4$ to be 0 (although that might not be of very much interest).

# CHAPTER 3

# Seiberg-Witten Invariants on Symplectic Manifolds

## 3.1 Survey of Symplectic and Complex Structures

Here we will briefly and without too many proofs introduce the notion of a symplectic manifold. Since symplectic manifolds are closely related to (almost) complex manifolds these will naturally enter our discussion. Then we will discuss spin$^c$-structures on these manifolds and finally give an expression of the corresponding Dirac operator in terms of Cauchy-Riemann operators (to be defined).

**Definition 3.1.** A *symplectic vector space* is a pair $(V,\omega)$ of a *real* vector space $V$ and a non-degenerate 2-form $\omega \in \Lambda^2 V^*$.

In the following proposition some elementary properties of symplectic vectors space are stated. The results should be well-known so proofs are omitted.

**Proposition 3.2.** *Let $(V,\omega)$ be a symplectic vector space. Then the following hold*:

1) *The vector space $V$ is of even dimension.*
2) *If $\dim V = 2n$, then the $n$-times wedge product $\omega \wedge \cdots \wedge \omega$ is a nonzero $2n$-form. In other words, a symplectic vector space has a canonical orientation.*
3) *There exists a basis $\{A_1, B_1, \ldots, A_n, B_n\}$ for $V$ with corresponding dual basis $\{\alpha^1, \beta^1, \ldots, \alpha^n, \beta^n\}$ such that*

$$\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i.$$

4) *The map $\Phi_\omega : V \longrightarrow V^*$ given by $\Phi_\omega(v)(w) = \omega(v,w)$ is an isomorphism.*

The last part says that one can "raise" and "lower" indices, just as one can if one has a metric. For brevity we will write $v^\flat$ instead of $\Phi_\omega(v)$ and $\varphi^\sharp$ instead of $\Phi_\omega^{-1}(\varphi)$.

In a geometric setting a symplectic structure is then just a "smooth choice" of symplectic forms on each fiber:

**Definition 3.3** (**Symplectic Manifold**). A *symplectic manifold* is a pair $(M, \omega)$ of a smooth manifold $M$ and a closed 2-form $\omega$ such that for each $x \in M$ the 2-form $\omega_x$ on $T_xM$ is non-degenerate.

The properties above translate into similar results in the manifold case. The only non-trivial part is the translation of point 3, which is by no means trivial, in fact it is one of the corner stones of symplectic geometry, known as the *Darboux Theorem*.

**Proposition 3.4.** *Let $(M, \omega)$ be a symplectic manifold. Then the following hold:*

1) *The manifold $M$ is of even dimension.*
2) *If $\dim V = 2n$, then the $n$-times wedge product $\omega \wedge \cdots \wedge \omega$ is a nowhere vanishing $2n$-form. Thus the manifold is orientable and carries a canonical orientation.*
3) *Around each point $x \in M$ there exists coordinates (called* Darboux coordinates*) $(U, x_1, y_1, \ldots, x_n, y_n)$ such that $\omega$ on $U$ takes the local form*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

4) *The map $\mathfrak{X}(M) \longrightarrow \Omega^1(M)$ given by $X \longmapsto X^\flat$ where $X^\flat(Y) = \omega(X, Y)$ is an isomorphism.*

To introduce spin and spin$^c$-structures we need a metric. But it turns out that a natural setting in which to discuss *simultaneously* symplectic and metric structures is the setting of almost complex manifolds. Therefore we will take some time to familiarize ourselves with such structures. First the linear structure:

**Definition 3.5.** Let $V$ be a real finite-dimensional vector space. $J \in \mathrm{Aut}_{\mathbb{R}}(V)$ is called a *complex structure* on $V$ provided $J^2 = -\mathrm{id}_V$.

Vi may view $(V, J)$ as a complex vector space upon defining complex scalar multiplication $\mathbb{C} \times V \longrightarrow V$ by $(a+ib)v := av + bJv$. Thus we might contemplate that the dimension of $V$ is even, and this is indeed true: taking the determinant of the defining relation $J^2 = -\mathrm{id}_V$ gives

$$(\det J)^2 = (-1)^{\dim V}$$

and since the number on the left hand side is positive ($J$ was a real linear map) we must have $\dim V$ even.

Now let us consider the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus iV$ of $V$. This is a complex vector space when given the complex scalar multiplication

$$z(v \otimes z') = v \otimes (zz').$$

If $V$ has a complex structure $J$ we can extend it to $V_{\mathbb{C}}$, simply by defining

$$J(v + iw) = Jv + iJw.$$

Obviously, this extension is a complex linear map and it satisfies $J^2 = -\mathrm{id}_V$:

$$J^2(v + iw) = J^2v + iJ^2w = -v - iw = -(v + iw).$$

It is easy to check that $\pm i$ are the only possible eigenvalues for $J$ on $V_{\mathbb{C}}$. Let $V_{\mathbb{C}}^{+} := \ker(I + iJ)$ and $V_{\mathbb{C}}^{-} := \ker(I - iJ)$ denote the eigenspaces for $i$ resp. $-i$. It is a consequence of the following lemma that both $i$ and $-i$ are in fact eigenvalues:

**Lemma 3.6.** *Given the situation above, the following hold*:

1) $V_{\mathbb{C}}^{\pm} = \operatorname{im}(I \mp iJ)$.
2) $V_{\mathbb{C}}$ *decomposes as a direct sum of the eigenspaces*: $V_{\mathbb{C}} = V_{\mathbb{C}}^{+} \oplus V_{\mathbb{C}}^{-}$.
3) $\dim_{\mathbb{R}} V_{\mathbb{C}}^{\pm} = n/2$, *thus both* $i$ *and* $-i$ *are eigenvalues.*

Proof. For 1) we need to show $\ker(I + iJ) = \operatorname{im}(I - iJ)$. Observe that

$$(I + iJ)(I - iJ) = I + iJ - iJ - i^2 J^2 = 0,$$

which gives $\operatorname{im}(I - iJ) \subseteq \ker(I + iJ)$. Conversely, if $v \in \ker(I + iJ)$, then

$$(I - iJ)(\tfrac{1}{2}v) = (I - iJ)(\tfrac{1}{2}v) + (I + iJ)(\tfrac{1}{2}v) = v,$$

i.e. $v \in \operatorname{im}(I - iJ)$. Thus $\ker(I + iJ) = \operatorname{im}(I - iJ)$.

For 2) we see that

$$v = (I + iJ)(\tfrac{1}{2}v) + (I - iJ)(\tfrac{1}{2}v)$$

hence we may write $v$ as a sum of elements from $\operatorname{im}(I + iJ) = V_{\mathbb{C}}^{-}$ and from $\operatorname{im}(I - iJ) = V_{\mathbb{C}}^{+}$.

To show 3) we observe that $V_{\mathbb{C}}^{+} \cap V_{\mathbb{C}}^{-} = \{0\}$. Now let $v = v_1 + iv_2 \in V_{\mathbb{C}}$. We define the conjugate of $\overline{v}$ of $v$ to be $\overline{v} = v_1 - iv_2$. The map $v \longmapsto \overline{v}$ is clearly linear and injective. It is easily seen that conjugation maps $V_{\mathbb{C}}^{\pm} \longrightarrow V_{\mathbb{C}}^{\mp}$ and injectivity assures that the two spaces must have the same dimension. And since the two spaces sum to give $V_{\mathbb{C}}$, their common dimension must be $n$. $\square$

We can take exterior powers of the spaces $V_{\mathbb{C}}^{\pm}$, resulting in the spaces

$$\Lambda^{p,q} V := \Lambda^p V_{\mathbb{C}}^{+} \otimes_{\mathbb{C}} \Lambda^q V_{\mathbb{C}}^{-}.$$

It is not hard to check that

$$\Lambda^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q} V.$$

The complex conjugation mentioned in the proof of the lemma above maps $V_{\mathbb{C}}^{\pm}$ to $V_{\mathbb{C}}^{\mp}$ and gives rise to a conjugation in $\Lambda^{p,q}V$ by requiring

$$\overline{\xi_1 \wedge \cdots \wedge \xi_p \otimes \eta_1 \wedge \cdots \wedge \eta_q} = \overline{\eta}_1 \wedge \cdots \wedge \overline{\eta}_q \otimes \overline{\xi}_1 \wedge \cdots \wedge \overline{\xi}_p.$$

This is an isomorphism $\Lambda^{p,q} V \xrightarrow{\sim} \Lambda^{q,p} V$.

Now, let's introduce metrics. A real inner product $g$ on $V$ is said to be *compatible* with the complex structure $J$ if

$$g(v, Jw) = -g(Jv, w),$$

i.e. if $J$ is skew-adjoint w.r.t. $g$. From this we can define a symplectic form by $\omega(v, w) := g(Jv, w)$ and it is easily checked that this is alternating and non-degenerate. It satisfies $\omega(Jv, Jw) = \omega(v, w)$ - we say that $\omega$ is *compatible* with the complex structure. Conversely, a symplectic form $\omega$ which is compatible with the complex structure gives rise to a metric by $g(v, w) := \omega(v, Jw)$. Again this is easily checked to be a (real) metric compatible with $J$. In other words picking a $J$-compatible metric or a $J$-compatible symplectic form are just two ways of doing the same.

**Definition 3.7 (Kähler Vector Space).** A real vector space $V$ carrying a complex structure $J$ and a compatible metric $g$ is called a *Kähler vector space*.

By our remarks above, a Kähler vector space is automatically a symplectic vector space.

Our first goal is two-fold: assuming $V$ is 4-dimensional we want (1) to produce an orthogonal basis for $\Lambda^2_+ V_{\mathbb{C}}$ and (2) to show that $\omega$ is a self-dual 2-form. To this end we are lead to some considerations on bases for the Kähler vector spaces. The first part can be carried out in general, so we might just as well assume that $V$ is of dimension $2n$. Later we will restrict to dimension 4.

First, we note that $v$ is orthogonal to $Jv$, simply because

$$g(Jv,v) = g(v,Jv) = -g(Jv,v)$$

(first we applied symmetry and second we applied compatibility with $J$). Furthermore $J$ preserves lengths, since

$$g(Jv,Jv) = -g(v,J^2v) = g(v,v).$$

That means that we can find a real orthonormal basis $\{v_1,w_1,\ldots,v_n,w_n\}$ for $V$ where $w_k = Jv_k$. Let $\{\alpha^1,\beta^1,\ldots,\alpha^n,\beta^n\}$ be the dual basis. Upon extending $J$ to the dual by defining

$$(J\alpha)(v) := \alpha(Jv)$$

it follows that $\beta^k = -J\alpha^k$. Now, we complexify $V$ and define (for $1 \leq k \leq n$)

$$z_k := \frac{1}{2}(v_k - iw_k) \qquad \text{and} \qquad \overline{z}_k := \frac{1}{2}(v_k + iw_k).$$

This is an orthogonal basis for $V_{\mathbb{C}}$ w.r.t. the complex extension $g_{\mathbb{C}}$ of the metric $g$. Similarly we define

$$\xi_k := \alpha^k + i\beta^k \qquad \text{and} \qquad \overline{\xi}_k := \alpha^k - i\beta^k$$

and this is a complex basis for $V^*_{\mathbb{C}}$. We see that

$$J\xi_k = J(\alpha^k + i\beta^k) = -\beta^k + iJ\beta^k = -\beta^k + i\alpha^k = i(\alpha^k + i\beta^k) = i\xi_k,$$

i.e. $\xi_k \in V^+_{\mathbb{C}}$. Similarly one calculates that $\overline{\xi}_k \in V^-_{\mathbb{C}}$. Thus we see that $\Lambda^{p,q}V^*$ is spanned by elements of the form

$$\xi_{i_1} \wedge \cdots \wedge \xi_{i_p} \otimes \overline{\xi}_{j_1} \wedge \cdots \wedge \overline{\xi}_{j_q}$$

where $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$. Granted this state of affairs, it is very easy to see that we can write $\omega$ as

$$\omega = \alpha^1 \wedge \beta^1 + \cdots + \alpha^n \wedge \beta^n, \tag{3.1}$$

simply apply both sides to some arbitrary vector, which is written in terms of the orthonormal basis $\{v_1,w_1,\ldots,v_n,w_n\}$.

The Hodge star $* : \Lambda^k V^* \longrightarrow \Lambda^{2n-k}V^*$ can be extended by complex linearity to $\Lambda^k V^*_{\mathbb{C}}$. It satisfies the following formula

$$\alpha \wedge *\overline{\beta} = g_{\mathbb{C}}(\alpha,\beta)dV_g, \tag{3.2}$$

and is given as the unique element satisfying this.

Now, let us restrict to dimension 4 where we have bases $\{v_1,w_1,v_2,w_2\}$ and $\{\alpha^1,\beta^1,\alpha^2,\beta^2\}$ as well as $\{z_1,\overline{z}_1,z_2,\overline{z}_2\}$ and $\{\xi_1,\overline{\xi}_1,\xi_2,\overline{\xi}_2\}$. Then it is not hard to calculate that

$$\xi^1 \wedge \xi^2 = \alpha^1 \wedge \alpha^2 - \beta^1 \wedge \beta^2 + i(\alpha^1 \wedge \beta^2 + \beta^1 \wedge \alpha^2)$$

and

$$\overline{\xi}^1 \wedge \overline{\xi}^2 = \alpha^1 \wedge \alpha^2 - \beta^1 \wedge \beta^2 - i(\alpha^1 \wedge \beta^2 + \beta^1 \wedge \alpha^2)$$

and that both 2-forms are self-dual w.r.t. $*$. Then remains the symplectic form. From its appearance in terms of the basis (3.1) we get by formula (3.2) that $\omega \wedge \omega = 2dV_g$. Since furthermore $\overline{\omega} = \omega$ and $g_{\mathbb{C}}(\omega, \omega) = 2$ we get that $*\omega = \omega$, i.e. that $\omega$ is self-dual. Since $\Lambda^2_+ V^*_{\mathbb{C}}$ has dimension 3, we conclude that $\{\xi_1 \wedge \xi_2, \overline{\xi}_1 \wedge \overline{\xi}_2, \omega\}$ is a $g_{\mathbb{C}}$-orthogonal basis for $\Lambda^2_+ V^*_{\mathbb{C}}$ or, in other words, that we have an orthogonal splitting

$$\Lambda^2_+ V^*_{\mathbb{C}} = \Lambda^{2,0} V^* \oplus \Lambda^{0,2} V^* \oplus \mathbb{C}\omega. \tag{3.3}$$

This completes our treatment of the linear theory. Now we "just" have to transform all this into a manifold setting by applying our linear theory to the tangent spaces.

**Definition 3.8** (**Almost Complex Manifold**)**.** Let $M$ be a smooth manifold. An *almost complex structure* on $M$ is a bundle endomorphism on the tangent bundle $J : TM \longrightarrow TM$ satisfying $J^2 = -\operatorname{id}_{TM}$. The pair $(M, J)$ is called an *almost complex manifold.*

The last requirement pertains to each fiber, so $J_x^2 = -\operatorname{id}_{T_xM}$, thus $T_xM$ must be of even dimension as mentioned above. Since each tangent space is given a complex structure by $J$, the tangent spaces have *canonical* orientations, and hence $M$ is orientable.

Extend $J$ to the complexification of the tangent bundle: $J : TM \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C}$ [1] simply by defining $J$ on each fiber $T_xM \otimes \mathbb{C}$ by $J_x(v \otimes z) = J_x(v) \otimes z$. It can be checked that this is again a smooth bundle map and that $J^2 = -\operatorname{id}_{TM\otimes\mathbb{C}}$. As above, $J_x : T_xM \otimes \mathbb{C} \longrightarrow T_xM \otimes \mathbb{C}$ has two eigenvalues, namely $\pm i$ and we can decompose the complexified tangent space into eigenspaces:

$$T_xM \otimes \mathbb{C} = (T_xM)^{1,0} \oplus (T_xM)^{0,1}$$

where $(T_xM)^{1,0}$ is the $i$-eigenspace and $(T_xM)^{0,1}$ is the $-i$-eigenspace. Define the bundles

$$(TM)^{1,0} := \coprod_{x \in M} (T_xM)^{1,0} \qquad \text{and} \qquad (TM)^{0,1} := \coprod_{x \in M} (T_xM)^{0,1}.$$

It is not hard to check that these are smooth complex vector bundles over $M$ of complex rank $\frac{1}{2} \dim M$ and that

$$TM \otimes \mathbb{C} = (TM)^{1,0} \oplus (TM)^{0,1}.$$

An almost complex structure $J$ on $M$ induces a dual bundle map $J^* : T^*M \longrightarrow T^*M$: it maps a covector $\omega \in T_x^*M$ to the covector $J^*\omega \in T_x^*M$ given by $(J^*\omega)(v) = \omega(Jv)$. This is again a smooth bundle map satisfying $(J^*)^2 = -\operatorname{id}_{T^*M}$ and it extends to the complexification $J^* : T^*M \otimes \mathbb{C} \longrightarrow T^*M \otimes \mathbb{C}$. Thus we have a decomposition

$$T^*M \otimes \mathbb{C} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$$

of the complexified cotangent bundle of $J^*$.

Sections of $T^*M \otimes \mathbb{C}$ are complex-valued 1-forms on $M$, and sections of the $k$'th exterior power over $T^*M \otimes \mathbb{C}$ are complex-valued $k$-forms on $M$. The usual exterior derivative $d_k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ extends by complex linearity to a

---

[1] Formally, it should have been noted that this is the tensor product over $\mathbb{R}$, but carrying this notation along all the way would be too cumbersome.

map $\Omega^k(M,\mathbb{C}) \longrightarrow \Omega^{k+1}(M,\mathbb{C})$. On the other hand, we have according to the eigenbundle decomposition:

$$\Lambda^k(T^*M\otimes\mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p(T^*M)^{1,0}\otimes_{\mathbb{C}}\Lambda^q(T^*M)^{0,1}$$

and upon denoting

$$\Lambda^{p,q}T^*M := \Lambda^p(T^*M)^{1,0}\otimes_{\mathbb{C}}\Lambda^q(T^*M)^{0,1} \quad \text{and} \quad \Omega^{p,q}(M) := \Gamma(\Lambda^{p,q}T^*M)$$

we get

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k}\Omega^{p,q}(M).$$

The bundle $\Lambda^{n,0}T^*M$ plays an important role in algebraic geometry where it is usually denoted $K_M$. $\Lambda^{0,n}T^*M$ is usually denoted $K_M^{-1}$ (the reason for this inverse-notation is that the bundles $\Lambda^{n,0}T^*M$ and $\Lambda^{0,n}T^*M$ are dual of each other). Any form in $\Omega^{p,q}(M)$ can be written as

$$\zeta = \xi_1\wedge\cdots\wedge\xi_p\otimes\eta_1\wedge\cdots\wedge\eta_q$$

where $\xi_i$ are $(1,0)$-forms and $\eta_j$ are $(0,1)$-forms. The $(p,0)$-part $\xi_1\wedge\cdots\wedge\xi_p$ is traditionally called the *holomorphic part* of $\zeta$ whereas the $(0,q)$-part is called the *anti-holomorphic part* of $\zeta$.

When restricting the exterior derivative $d_k : \Omega^k(M,\mathbb{C}) \longrightarrow \Omega^{k+1}(M,\mathbb{C})$ to $\Omega^{p,q}(M)\subseteq\Omega^k(M)$ (when $p+q=k$), it turns out that $d_k$ maps into at most four of the subspaces of $\Omega^{k+1}$, namely

$$d_k(\Omega^{p,q}(M)) \subseteq \Omega^{p+2,q-1}(M)\oplus\Omega^{p+1,q}(M)\oplus\Omega^{p,q+1}(M)\oplus\Omega^{p-1,q+2}(M)$$

(provided of course that $p,q\geq 1$). Thus we may write

$$d_k|_{\Omega^{p,q}(M)} = d_k^{2,-1}\oplus d_k^{1,0}\oplus d_k^{0,1}\oplus d_k^{-1,2}.$$

Having done so we quickly rename two of the terms, namely

$$\partial_k := d_k^{1,0} \qquad \text{and} \qquad \overline{\partial}_k := d_k^{0,1}, \tag{3.4}$$

thus $\partial_k$ increases the degree of the holomorphic part by 1 whereas $\overline{\partial}_k$ increases the degree of the anti-holomorphic part by 1.

**Proposition 3.9.** *Given an almost complex manifold* $(M,J)$ *the following are equivalent*:

1) *For all* $p,q\geq 1$*, the maps* $d_{p+q}^{2,-1}$ *and* $d_{p+q}^{-1,2}$ *above are identically* $0$.
2) $\partial_0^2 f = \overline{\partial}_0^2 f = 0$ *for all* $f\in C^\infty(M)$.
3) *The* Nijenhuis tensor $N_J\in\Omega^2(M)$ *given by*

$$N_J(X,Y) := \frac{1}{4}\big([JX,JY]-[X,Y]-J[X,JY]-J[JX,Y]\big)$$

*vanishes identically.*

**Definition 3.10 (Complex Structure).** An almost complex structure $J$ on $M$ is called *integrable* or a *complex structure* if one (hence all) of the above conditions is satisfied.

The reason for the name "complex structure" is that a manifold with a complex structure is in fact a complex manifold in the sense that it carries an atlas of holomorphic charts, i.e. a collection of homeomorphism $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$, where $V_\alpha \subseteq \mathbb{C}^n$ is some open set, such that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic where defined. This result is known as the *Newlander-Nirenberg Theorem*.

Next we will discuss metric structures on almost-complex and symplectic manifolds.

**Definition 3.11** (**Almost Hermitian Structure**)**.** Let $(M, J)$ be an almost complex manifold. By an *almost hermitian structure* we understand a Riemannian metric $g$ on $M$ which is compatible with $J$ in the sense that

$$g(JX, Y) = -g(X, JY) \tag{3.5}$$

for $X, Y \in \mathfrak{X}(M)$.

Given an almost hermitian structure on $M$, we can define a 2-form $\omega$ by $\omega(X, Y) := g(JX, Y)$. Just as above, this is an anti-symmetric, non-degenerate 2-form, satisfying $\omega(JX, JY) = \omega(X, Y)$, i.e. $\omega$ is compatible with $J$. But note that $\omega$ is not guaranteed to be closed, so $\omega$ is not necessarily a symplectic form!

Conversely, given a non-degenerate 2-form compatible with $J$ we define a real metric $g$ by

$$g(X, Y) := \omega(X, JY).$$

A triple $(J, g, \omega)$ where $g$ and $\omega$ are related in this way is called a *compatible triple*. Given such a triple, we can define a hermitian metric (i.e. a fiberwise sesquilinear, conjugate-symmetric, positive definit form) $h$ on $TM$ by

$$h(X, Y) := g(X, Y) - i\omega(X, Y).$$

This is easily checked to be a hermitian metric.

**Definition 3.12** (**Kähler Manifold**)**.** Let $(J, g, \omega)$ be a compatible triple on a manifold $M$. This triple is called an *almost Kähler structure* and $M$ an *almost Kähler manifold* if $\omega$ is closed, i.e. if $\omega$ is a symplectic form.

An almost Kähler structure is called a *Kähler structure* and $M$ a *Kähler manifold* if the almost complex structure $J$ is integrable.

Thus an almost Kähler manifold is both a symplectic manifold, a (real) Riemannian manifold and a hermitian manifold, and a Kähler manifold is, additionally, a complex manifold. This is such a rich structure that one might worry if such objects exist at all. But they do: the complex spaces $\mathbb{C}^n$ (or rather $\mathbb{R}^{2n}$ when viewed as real manifolds) with the obvious complex structure and the usual metric are Kähler manifolds. Also the complex projective spaces $\mathbb{CP}^n$ with the Fubini-Study metric are (compact) Kähler manifolds.

To be able to discuss Dirac operators we first have to say something about connections. It turns out that they are related to what is known as Cauchy-Riemann operators.

For a vector bundle $E$ we define $\Omega^{p,q}(M, E)$ or just $\Omega^{p,q}(E)$ to be the space of $E$-valued $(p, q)$-forms, i.e. smooth sections of the bundle $(\Lambda^{p,q}T^*M) \otimes E$.

**Definition 3.13** (**Cauchy-Riemann Operator**)**.** A *holomorphic Cauchy-Riemann operator* on a complex Riemannian [2] vector bundle $E$ is a first order differential operator $D : \Gamma(E) \longrightarrow \Omega^{1,0}(E)$ satisfying

$$D(f\psi) = (\partial f)\psi + fD\psi$$

---

[2] By hermitian vector bundle we mean a complex vector bundle equipped with a hermitian (i.e. sesquilinear, conjugate symmetric) metric.

for all $f \in C^\infty(M)$ and $\psi \in \Gamma(E)$ where $\partial$ is the holomorphic part of $d$.

Similarly, an *anti-holomorphic Cauchy-Riemann operator* is a first order differential operator $\overline{D} : \Gamma(E) \longrightarrow \Omega^{0,1}(E)$ such that

$$\overline{D}(f\psi) = (\overline{\partial} f)\psi + f\overline{D}\psi$$

for all $f \in C^\infty(M)$ and $\psi \in \Gamma(E)$ where $\overline{\partial}$ is the anti-holomorphic part of $d$ as defined in (3.4).

A holomorphic Cauchy-Riemann operator $D : \Gamma(E) \longrightarrow \Omega^{1,0}(E)$ can be extended to $\Omega^*(E)$ by defining $D : \Omega^{p,q}(E) \longrightarrow \Omega^{p+1,q}(E)$ in the following way:

$$D(\eta \otimes \psi) := (\partial\eta) \otimes \psi + (-1)^{p+q}\eta \wedge D\psi$$

where $\eta \in \Omega^{p,q}(M)$ and $\psi \in \Gamma(E)$. Similarly an anti-holomorphic Cauchy-Riemann operator $\overline{D} : \Gamma(E) \longrightarrow \Omega^{0,1}(E)$ can be extended to $\Omega^*(E)$ upon defining $\overline{D} : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$ by

$$\overline{D}(\eta \otimes \psi) := (\overline{\partial}\eta) \otimes \psi + (-1)^{p+q}\eta \wedge \overline{D}\psi.$$

So what has this got to do with connections? Assuming $\nabla$ to be a metric connection on $E$, we know that $\nabla$ is a differential operator $\Gamma(E) \longrightarrow \Omega^1(E)$. Projecting to the $(1,0)$'s component of $\Omega^1(E)$ yields a new differential operator

$$\partial_\nabla := \pi^{1,0} \circ \nabla$$

and similarly, projecting to the $(0,1)$'s component of $\Omega^1(E)$ gives

$$\overline{\partial}_\nabla := \pi^{0,1} \circ \nabla.$$

If $E = M \times \mathbb{C}$ is the trivial complex line bundle on $M$, then $\Gamma(E) = C^\infty(M)$ and $\Omega^1(M,E) = \Omega^1(M,\mathbb{C})$ and picking the flat connection $d$ on $E$ would give us exactly the operators $\partial$ and $\overline{\partial}$ defined in (3.4).

It's easy to show that $\partial_\nabla$ is in fact a holomorphic Cauchy-Riemann operator:

$$\begin{aligned}\partial_\nabla(f\psi) &= \pi^{1,0}(\nabla(f\psi)) = \pi^{1,0}(df \otimes \psi) + \pi^{1,0}(f\nabla\psi)\\ &= \pi^{1,0}(df) \otimes \psi + f\pi^{1,0}(\nabla\psi) = \partial f \otimes \psi + f\partial_\nabla\psi\end{aligned}$$

and similarly, $\overline{\partial}_\nabla$ is an anti-holomorphic Cauchy-Riemann operator. One can show (cf. [18] Proposition 1.4.13) that any Cauchy-Riemann operator happens to be on one of these forms for some connection, but we will not make use of this fact.

Finally we need to introduce the so-called Chern-connection. It is a fundamental fact of Riemannian geometry that the Levi-Civita connection is the unique metric connection on $TM$ which is torsion-free. Recall that we defined the *torsion* of a connection $\nabla$ on $TM$ by

$$\tau_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Analogous to the existence of the Levi-Civita connection in real Riemannian geometry, we have the following result in the complex case:

**Proposition 3.14.** *Let $(M,J,g)$ be an almost Kähler manifold and let $N_J$ denote its Nijenhuis tensor. There exists a unique metric connection, called the* Chern connection *on $TM$ whose torsion is equal to $N_J$.*

For a proof of this fact see [18] Proposition 1.4.4. Since $M$ is, in particular, a Riemannian manifold, it carries the Levi-Civita connection. If $M$ happens to be Kähler, i.e. if $N_J = 0$, then the Chern connection and the Levi-Civita connection coincide.

## 3.2 Spin Structures on Almost Kähler Manifolds

Let $(M, J, g)$ be an almost Kähler manifold. This implies, as we have seen, that the tangent bundle $TM$ carries a hermitian metric $h$. We can therefore form the unitary frame bundle $P_{\mathrm{U}}(M)$ of $M$. If $\dim_{\mathbb{R}} M = 2n$, this is a principal $\mathrm{U}(n)$-bundle. From the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we obtain a natural inclusion homomorphism $\iota : \mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$

**Lemma 3.15.** *There exists a unique homomorphism* $f : \mathrm{U}(n) \longrightarrow \mathrm{Spin}^c(2n)$ *which renders the following diagram commutative*

$$\begin{tikzcd}
 & \mathrm{Spin}^c(2n) \arrow[d, "\Lambda^c"] \\
\mathrm{U}(n) \arrow[ru, "f"] \arrow[r, "\iota\times\det"'] & \mathrm{SO}(2n)\times\mathrm{U}(1)
\end{tikzcd}$$

Proof. Let $A \in \mathrm{U}(n)$ and let $\gamma : [0,1] \longrightarrow \mathrm{U}(n)$ be a continuous curve with $\gamma(0) = 1$ and $\gamma(1) = A$ (exists since $\mathrm{U}(n)$ is connected). Then consider the curve $t \longmapsto (\iota(\gamma(t)), \det\gamma(t))$ in $\mathrm{SO}(2n) \times \mathrm{U}(1)$. Since $\Lambda^c$ is a double covering we have a unique lift of this curve to a curve $\widetilde{\gamma} : I \longrightarrow \mathrm{Spin}^c(2n)$ which satisfies $\widetilde{\gamma}(0) = 1$. Now define $f(A) := \widetilde{\gamma}(1)$, then one can check that this is a well-defined homomorphism which lifts $\iota \times \det$. ☐

We can use this lemma to show that the almost Kähler manifold always has a spin$^c$-structure: We will define the principal $\mathrm{Spin}^c(2n)$-bundle of such a spin$^c$-structure as the associated bundle of $P_{\mathrm{U}}(M)$ relative to $f$:

$$P^c_{\mathrm{Spin}}(M) := P_{\mathrm{U}}(M) \times_f \mathrm{Spin}^c(2n),$$

then observe that $P_{\mathrm{SO}}(M) = P_{\mathrm{U}}(M) \times_\iota \mathrm{SO}(2n)$ and finally define a bundle map $\Phi : P^c_{\mathrm{Spin}}(M) \longrightarrow P_{\mathrm{SO}}(M)$ by

$$\Phi([\xi, g]) = [\xi, \rho^c(g)]$$

where we recall that $\rho^c : \mathrm{Spin}^c(2n) \longrightarrow \mathrm{SO}(2n)$ is given by $\rho^c([g, z]) = \Lambda(g)$. $\Phi$ is well-defined, for if $[\xi, g] = [\xi', g']$, i.e. if $(\xi', g') = (\xi \cdot h, f(h^{-1})g)$ for some $h \in \mathrm{U}(n)$ then

$$\begin{aligned}\Phi[\xi', g'] &= [\xi', \rho^c(g')] = [\xi \cdot h, \rho^c(f(h^{-1})g)] = [\xi \cdot h, \iota(h^{-1})\rho^c(g)] \\ &= [\xi, \rho^c(g)] = \Phi[\xi, g].\end{aligned}$$

Hence $(P^c_{\mathrm{Spin}}(M), \Phi)$ is a spin$^c$-structure on $M$. Thus we have shown

**Proposition 3.16.** *An almost Kähler manifold is a spin$^c$-manifold.*

The explicit spin$^c$-structure constructed above is called the *canonical spin$^c$-structure* on the almost Kähler manifold.

Notice, that if the principal $\mathrm{U}(n)$-bundle is given by the gluing cocycle $(g_{\alpha\beta})$, then the canonical spin$^c$-structure is given by the gluing cocycle $f(g_{\alpha\beta})$. This is of the form $[h_{\alpha\beta}, z_{\alpha\beta}]$ where $z_{\alpha\beta} = \det g_{\alpha\beta}$, in other words the determinant line bundle for the canonical spin$^c$-structure is given by the cocycle $(\det g_{\alpha\beta})^2$.

Associated with a choice of spin$^c$-structure $\sigma$ is the spinor bundle $S = S^+ \oplus S^-$. The symplectic form $\omega$ acts by Clifford multiplication on the spinor bundle $S$ thus giving a bundle map $\omega\cdot : S \longrightarrow S$. The following proposition is a special feature of symplectic manifolds and absolutely fundamental for the rest of this chapter (for a proof the reader is referred to [9] Section 3.4)

**Proposition 3.17.** *Let $\sigma = \sigma_0 \otimes \mathfrak{L}$ be a spin$^c$-structure on an almost Kähler manifold $(M, J, g)$ of dimension $2n$ (where $\sigma_0$ is the canonical spin$^c$-structure), and let $S$ be the associated spinor bundle. Then $S$ splits into eigenbundles $S = S_0 \oplus \cdots \oplus S_n$ for the map $\omega\cdot : S \longrightarrow S$, where $S_k$, $0 \leq k \leq n$ is the eigenbundle corresponding to the eigenvalue $i(n-2k)$.*

*Moreover the bundle map $\Lambda^{0,k}T^*M \otimes S_n \longrightarrow S_{n-k}$ given by $\eta \otimes \psi \longmapsto \frac{1}{2^{k/2}}\eta\cdot\psi$ (where the dot is Clifford multiplication) is an isometric bundle isomorphism, and hence*

$$S \cong \Big(\bigoplus_{k=0}^{n} \Lambda^{0,k}T^*M\Big) \otimes S_n. \tag{3.6}$$

*Finally, $\mathfrak{L} = S_n$, in particular for the canonical spin$^c$-structure on $M$, the bundle $S_n$ is the trivial complex line bundle, and therefore*

$$S \cong \bigoplus_{k=0}^{n} \Lambda^{0,k}T^*M. \tag{3.7}$$

*This decomposition pertains to the plus/minus spinor bundles, i.e. for the canonical spin$^c$-structure we have*

$$S^+ \cong \bigoplus_{k=0}^{\lfloor n/2\rfloor} \Lambda^{0,2k}T^*M \qquad \text{and} \qquad S^- \cong \bigoplus_{k=1}^{\lfloor n/2\rfloor} \Lambda^{0,2k-1}T^*M. \tag{3.8}$$

>From now on and throughout this chapter we will restrict out attention to 4-manifolds, i.e. $n = 2$ in the notation of the above proposition. In this case the formulas for the spinor bundles for a spin$^c$-structure $\sigma_0 \otimes \mathfrak{L}$ read

$$S^+ \cong \big((M \times \mathbb{C}) \oplus \Lambda^{0,2}T^*M\big) \otimes \mathfrak{L} \qquad \text{and} \qquad S^- \cong (\Lambda^{0,1}T^*M) \otimes \mathfrak{L}. \tag{3.9}$$

Associated to a spin$^c$-structure $\sigma$ we have the determinant line bundle $L$. In case of the canonical spin$^c$-structure, combining (1.3) (in this formula $k$ should be 4) with (3.8) gives

$$L = \Lambda^2(S^-) = \Lambda^2(\Lambda^{0,1}T^*M) = \Lambda^{0,2}T^*M = K_M^{-1}.$$

Any connection on $TM$ will, by duality, give rise to a connection on $T^*M$ and hence also on $\Lambda^{p,q}T^*M$ for all $p$ and $q$. In particular it gives rise to a connection $\mathcal{A}$ on $K_M^{-1} = L$ and thus to a Dirac operator $\not{D}_{\mathcal{A}}$! On an almost complex manifold $M$ we have, as explained earlier, a "canonical" connection, namely the Chern connection. The associated connection on $L$ then determines a Dirac operator, called the *canonical Dirac operator* on the almost complex manifold, denoted $\not{D}_c$. We are fortunate that this Dirac operator has a particularly simple appearance (see [18] Proposition 1.4.23):

**Proposition 3.18.** *Let $\not{D}_c$ be the canonical Dirac operator for the canonical spin$^c$-structure on the almost complex manifold $(M, J, g)$. Then under the isomorphisms (3.9) the Dirac operator equals*

$$\not{D}_c = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$$

*as a map $\not{D}_c : \Gamma(S^+) \longrightarrow \Gamma(S^-)$ where $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$.*

Let's be a bit more precise: As we have seen above, (3.8) we have

$$\Gamma(S^+) \cong C^\infty(M) \oplus \Omega^{0,2}(M) \tag{3.10}$$

and

$$\Gamma(S^-) = \Omega^{0,1}(M).$$

We know that $\overline{\partial}$ maps $\Omega^{0,0}(M) = C^\infty(M)$ to $\Omega^{0,1}(M)$ and $\overline{\partial}^*$ maps $\Omega^{0,2}(M) \longrightarrow \Omega^{0,1}(M)$, so what the formula really means is the following

$$\not{D}_c(\psi) = \not{D}_c(\psi_1, \psi_2) = \sqrt{2}(\overline{\partial}\psi_1 + \overline{\partial}^*\psi_2) \tag{3.11}$$

where $\psi = (\psi_1, \psi_2)$ is the decomposition of the positive spinor field $\psi$ according to (3.10).

How does this look in general? Well assume we have a general spin$^c$-structure $\sigma = \sigma_0 \otimes \mathcal{L}$. Then $\det(\sigma) = \det(\sigma_0) \otimes \mathcal{L}^{\otimes 2}$. On $\det(\sigma_0) = K_M^{-1}$ we still have the (connection induced by the) Chern connection, $\mathcal{A}_0$. If we pick a connection $\mathcal{B}$ on $\mathcal{L}$ we get a connection $\mathcal{A} := \mathcal{A}_0 \otimes \mathcal{B}^{\otimes 2}$ on $\det(\sigma)$ [3] and any connection is of this form. Hence we get a Dirac operator $\not{D}_{\mathcal{A}}$. Fortunately we are able to express this Dirac operator in terms of the anti-holomorphic Cauchy-Riemann operators $\partial_{\mathcal{B}}$ and $\overline{\partial}_{\mathcal{B}}$ as before. Recall that these were defined on $\Omega^{0,k}(M, \mathcal{L}) \cong \Omega^{0,k}(M) \otimes \Gamma(\mathcal{L})$ mapping into $\Omega^{0,k+1}(M, \mathcal{L}) \cong \Omega^{0,k+1}(M) \otimes \Gamma(\mathcal{L})$. From Proposition 3.18 one can derive:

**Proposition 3.19.** *Let* $\sigma = \sigma_0 \otimes \mathcal{L}$ *be a spin*$^c$*-structure on* $M$ *and* $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{B}^{\otimes 2}$ *be a connection on* $\det(\sigma)$. *Under the isomorphism* (3.6) *the Dirac operator* $\not{D}_{\mathcal{A}}$ *is given by*

$$\not{D}_{\mathcal{A}} = \sqrt{2}(\overline{\partial}_{\mathcal{B}} + \overline{\partial}_{\mathcal{B}}^*).$$

Again this formula should be interpreted properly as in (3.11).

In addition to the Dirac operator, also the curvature 2-form $F_{\mathcal{A}}$ enters the Seiberg-Witten equations. With $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{L}^{\otimes 2}$ what does this curvature 2-form look like? The answer is surprisingly simple:

$$F_{\mathcal{A}} = F_{\mathcal{A}_0} + 2F_{\mathcal{B}}. \tag{3.12}$$

This is a general result: if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two connections on line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $M$ we can form the tensor product connection

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \mathcal{A}_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{A}_2$$

on $\mathcal{L}_1 \otimes \mathcal{L}_2$ and ask what its curvature is. The tricky part is to express the local connection 1-forms of the tensor product in terms of the local connection 1-forms of $\mathcal{A}_1$ and $\mathcal{A}_2$. After some fuzzy calculations they turn out to be just the sum of the connection 1-forms of $\mathcal{A}_1$ and $\mathcal{A}_2$. Then the curvature 2-forms of the tensor product are just sums of the curvature 2-forms of $\mathcal{A}_1$ and $\mathcal{A}_2$ and they patch together to form $F_{\mathcal{A}_1 \otimes \mathcal{A}_2}$ which is then just the sum $F_{\mathcal{A}_1} + F_{\mathcal{A}_2}$.

## 3.3 Taubes' Theorem

The theorem by Taubes we are going to prove, is about the canonical spin$^c$-structure, $\sigma_0$. On the determinant line bundle $\det(\sigma_0)$ we have the Chern connection $\mathcal{A}_0$ and any connection on $\det(\sigma_0)$ is of the form $\mathcal{A}_0 + \alpha$ where $\alpha \in i\Omega^1(M)$. To bring this in line with our previous discussion we note that we can write the determinant line bundle of $\sigma_0$ as

$$\det(\sigma_0) \cong \det(\sigma_0) \otimes (M \times \mathbb{C})^{\otimes 2} \tag{3.13}$$

corresponding to $\sigma_0 = \sigma_0 \otimes (M \times \mathbb{C})$ (tensoring with the trivial line bundle yields nothing new). Any connection on the trivial line bundle $M \times \mathbb{C}$ is of the form $\mathcal{B} = d + \beta$ for $\beta \in i\Omega^1(M)$. One can easily check that the connection

---

[3] In the article by Michael Hutchings and Clifford Henry Taubes in [7] they write this connection as $\mathcal{A} + 2\mathcal{B}$.

$\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{B}^{\otimes 2}$ corresponds to $\mathcal{A}_0 + 2\beta$ under the isomorphism (3.13) above. Furthermore, the curvature of $\mathcal{B}$ is just $F_{\mathcal{B}} = d\beta$ and hence

$$F_{\mathcal{A}_0 \otimes \mathcal{B}^{\otimes 2}} = F_{\mathcal{A}_0} + 2d\beta$$

which is in accordance with (1.24).

We are now ready to state and prove the promised result:

**Theorem 3.20 (Taubes).** *Let $(M, \omega)$ be a smooth, compact, symplectic four-manifold with $\omega$-compatible complex structure $J$ and which satisfies $b_2^+ > 1$. Then*

$$\mathbf{SW}(M, \sigma_0) = \pm 1,$$

*i.e. the Seiberg-Witten invariant of the manifold $M$ and the canonical spin$^c$-structure $\sigma_0$ equals $\pm 1$.*

Proof. We will simply show that the moduli space consists of only one element and so we investigate the Seiberg-Witten equations for the pair $(\psi, \mathcal{A})$ where $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{B}^{\otimes 2} = \mathcal{A} + 2\beta$ is an arbitrary connection. We are free to use any perturbation parameter that suit our needs, and we will choose $\eta := -ir\omega - F_{\mathcal{A}_0}^+$ (this makes sense, since $\omega$ is a self-dual 2-form as shown above) where $r$ is some fixed, sufficiently great real number (to be specified later) and where $\mathcal{A}_0$ is our reference connection. For $\psi \in \Gamma(S^+)$ we let $\psi_1 \in C^\infty(M)$ and $\psi_2 \in \Omega^{0,2}(M)$ be such that $\psi = \sqrt{r}(\psi_1, \psi_2)$ (note the rescaling by $\sqrt{r}$!). Then by the formula for the Dirac operator above, the first Seiberg-Witten equation can be written as

$$\overline{\partial}_{\mathcal{B}} \psi_1 = -\overline{\partial}_{\mathcal{B}}^* \psi_2. \tag{3.14}$$

>From the local expressions in (2.7) we can deduce the following alternative *global* version of the second Seiberg-Witten equation:

$$F_{\mathcal{A}}^+ = ir(|\psi_1|^2 - |\psi_2|^2)\omega + 2(\overline{\psi_1}\psi_2 - \psi_1\overline{\psi_2}) - \eta.$$

With the $\eta$ chosen above the second Seiberg-Witten equation reads

$$F_{\mathcal{A}}^+ - F_{\mathcal{A}_0}^+ = -ir(1 - |\psi_1|^2 + |\psi_2|^2)\omega - 2r(\psi_1\overline{\psi_2} - \overline{\psi_1}\psi_2)$$

and since $F_{\mathcal{A}}^+ - F_{\mathcal{A}_0}^+ = 2F_{\mathcal{B}}^+ = 2(d\beta)^+$ we get

$$F_{\mathcal{B}}^+ = (d\beta)^+ = -\frac{ir}{2}(1 - |\psi_1|^2 + |\psi_2|^2)\omega - r(\psi_1\overline{\psi_2} - \overline{\psi_1}\psi_2) \tag{3.15}$$

By (3.3) we have the orthogonal decomposition

$$\Lambda_+^2 T^*M = \mathbb{C}\omega \oplus K_M \oplus K_M^{-1} = \mathbb{C}\omega \oplus \Lambda^{2,0}T^*M \oplus \Lambda^{0,2}T^*M$$

where $\mathbb{C}\omega$ is the trivial complex line bundle whose fiber over $x \in M$ is just $\mathbb{C}\omega_x$. Therefore by (3.15) we have

$$F_{\mathcal{B}}^{0,2} = r\overline{\psi_1}\psi_2 \qquad \text{and} \qquad \langle \omega, F_{\mathcal{B}}^+ \rangle = -ir(1 - |\psi_1|^2 + |\psi_2|^2)$$

where $\langle\,,\,\rangle$ refers to the fiber metric on $\Lambda^2 T^*M$ induced by $g$ (in verifying the second formula one recalls that $\langle \omega, \omega \rangle = 2$). We have at least one obvious solution to the equations (3.14) and (3.15), namely $\mathcal{A} \equiv \mathcal{A}_0$ and $\psi_1 \equiv 1$ and $\psi_2 \equiv 0$. We will show that, up to gauge equivalence, this is the only solution, provided $r$ is great enough.

Define the map $\mathcal{N} : \Omega^{1,0}(M) \longrightarrow \Omega^{0,2}(M)$ in the following way: the exterior derivative maps

$$d : \Omega^{1,0}(M) \longrightarrow \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$$

and therefore we can define $\mathcal{N} := \pi^{0,2} \circ d$ (note that for a Kähler manifold this is identically 0). This is a tensor field, i.e. for any smooth (complex-valued) function $f$ we have $\mathcal{N}(f\alpha) = f\mathcal{N}(\alpha)$. This can be shown in the following way: For $\alpha \in \Omega^{1,0}(M)$ we get

$$d(f\alpha) = df \wedge \alpha + f d\alpha.$$

Since $\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$, we can split $df$ accordingly: $df = (df)^{1,0} + (df)^{0,1}$. Now since the wedge product maps $\Omega^{p,q}(M) \times \Omega^{r,s}(M) \longrightarrow \Omega^{p+r,q+s}(M)$ we see that $df \wedge \alpha \in \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$ and thus this term is killed by the projection $\pi^{0,2}$. Therefore

$$\mathcal{N}(f\alpha) = (d\alpha)^{0,2} = (fd\alpha)^{0,2} = f(d\alpha)^{0,2} = f\mathcal{N}(\alpha).$$

Thus $\mathcal{N}$ is induced by a bundle map $\Lambda^{1,0}T^*M \longrightarrow \Lambda^{0,2}T^*M$ and hence over each $x$ we have a linear map $\mathcal{N}_x : \Lambda^{1,0}T^*_xM \longrightarrow \Lambda^{0,2}T^*_xM$. This is bounded, i.e. $|\mathcal{N}_x(\alpha_x)| \leq C_x|\alpha_x|$. Since $M$ is compact and $C_x$ depends continuously on $x$ we can find a maximum $\widetilde{C}$ such that

$$|\mathcal{N}_x(\alpha_x)| \leq \widetilde{C}|\alpha_x|$$

for all $x$.

Additionally we need the following complex version of the Bochner-Lichnerowicz identity (in this formula $\mathcal{B}$ maps $C^\infty(M) \longrightarrow \Omega^1(M)$ and $\mathcal{B}^*$ maps opposite)):

$$2\overline{\partial}^*_{\mathcal{B}}\overline{\partial}_{\mathcal{B}} = \mathcal{B}^*\mathcal{B} - i\langle\omega, F_{\mathcal{B}}\rangle \tag{3.16}$$

(a proof may be found in [6], p. 212).

Now take the first Seiberg-Witten equation (3.14) and act with $\overline{\partial}_{\mathcal{B}}$ on both sides to obtain (see below for explanations)

$$\begin{aligned}\overline{\partial}_{\mathcal{B}}\overline{\partial}^*_{\mathcal{B}}\psi_2 &= -\overline{\partial}_{\mathcal{B}}\overline{\partial}_{\mathcal{B}}\psi_1 = -F^{0,2}_{\mathcal{B}}\psi_1 + \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\\ &= -r|\psi_1|^2\psi_2 + \mathcal{N}(\partial_{\mathcal{B}}\psi_1).\end{aligned} \tag{3.17}$$

The second identity here is not quite obvious, so let's spend some time proving it. This is an instructive exercise in the rules of calculation for Cauchy-Riemann operators. We calculate each side separately and compare:

$$\begin{aligned}F^{0,2}_{\mathcal{B}}\psi_1 &= (d\beta)^{0,2}\psi_1 = [d(\beta^{1,0}) + d(\beta^{0,1})]^{0,2}\psi_1\\ &= \overline{\partial}(\beta^{0,1})\psi_1 + \psi_1[d(\beta^{1,0})]^{0,2}.\end{aligned}$$

Next we calculate $\mathcal{N}(\partial_{\mathcal{B}}\psi_1)$:

$$\begin{aligned}\mathcal{N}(\partial_{\mathcal{A}}\psi) &= (d \circ \partial_{\mathcal{B}}\psi_1)^{0,2} = (d(d\psi_1 + \psi_1\beta)^{1,0})^{0,2}\\ &= (d(\partial\psi_1))^{0,2} + (d(\psi_1\beta^{1,0}))^{0,2}\\ &= -\overline{\partial}^2\psi_1 + (d\psi_1 \wedge \beta^{1,0})^{0,2} + [\psi_1 d(\beta^{1,0})]^{0,2}\\ &= -\overline{\partial}^2\psi_1 + [\psi_1 d(\beta^{1,0})]^{0,2}.\end{aligned}$$

Finally we just need to calculate $\overline{\partial}_{\mathcal{A}}\overline{\partial}_{\mathcal{A}}$, and this is just and application of the extension rule for Cauchy-Riemann operators to higher exterior powers:

$$\begin{aligned}\overline{\partial}_{\mathcal{B}}\overline{\partial}_{\mathcal{B}} &= \overline{\partial}_{\mathcal{B}}(\overline{d}\psi_1 + \psi_1\beta^{0,1}) = \overline{\partial}^2\psi_1 + \overline{\partial}\psi_1 \wedge \beta^{0,1} + \overline{\partial}(\beta^{0,1})\psi_1 - \beta^{0,1} \wedge \partial_{\mathcal{B}}\psi_1\\ &= \overline{\partial}^2\psi_1 + \overline{\partial}\psi_1 \wedge \beta^{0,1} + \overline{\partial}(\beta^{0,1} - \beta^{0,1} \wedge \overline{\partial}\psi_1 - \beta^{0,1} \wedge \psi_1\beta^{0,1}\\ &= \overline{\partial}^2\psi_1 + \overline{\partial}(\beta^{0,1})\psi_1.\end{aligned}$$

Comparing these terms we obtain the desired formula: $\overline{\partial}_{\mathcal{B}}\overline{\partial}_{\mathcal{B}}\psi_1 = F^{0,2}_{\mathcal{B}}\psi_1 - \mathcal{N}(\partial_{\mathcal{B}}\psi_1)$

Now we return to (3.17) and take the (pointwise) inner product with $\psi_2$ and obtain

$$\langle\psi_2, \overline{\partial}_{\mathcal{B}}\overline{\partial}^*_{\mathcal{B}}\psi_2\rangle = -r|\psi_1|^2|\psi_2|^2 + \langle\psi_2, \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\rangle.$$

The left-hand side is just $|\overline{\partial}^*_{\mathcal{B}}\psi_2|^2$ and by integration we get

$$\int_M |\overline{\partial}^*_{\mathcal{B}}\psi_2|^2 dV_g = \int_M (\langle\psi_2, \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\rangle - r|\psi_1|^2|\psi_2|^2) dV_g. \tag{3.18}$$

Observe the inequality

$$\int_M \langle\psi_2, \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\rangle dV_g \le \frac{r}{2}\int_M |\psi_2|^2 dV_g + \frac{C}{r}\int_M |\mathcal{B}\psi_1|^2 dV_g \tag{3.19}$$

where $C > 0$ is some constant, independent of $r$. It can be proved as follows (recall that $\langle\,,\,\rangle$ and $|\cdot|$ denote *pointwise* inner products and norms):

$$\begin{aligned} 0 &\le \Big\langle \sqrt{\tfrac{r}{2}}\psi_2 - \tfrac{1}{\sqrt{r}}\mathcal{N}(\partial_{\mathcal{B}}\psi_1), \sqrt{\tfrac{r}{2}}\psi_2 - \tfrac{1}{\sqrt{r}}\mathcal{N}(\partial_{\mathcal{B}}\psi_1)\Big\rangle \\ &= \frac{r}{2}|\psi_2|^2 + \frac{1}{r}|\mathcal{N}(\partial_{\mathcal{B}}\psi_1)|^2 - \sqrt{2}\langle\psi_2, \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\rangle \end{aligned}$$

and therefore using the boundedness property of $\mathcal{N}$ we get

$$\begin{aligned} \langle\psi_2, \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\rangle &\le \frac{r}{2\sqrt{2}}|\psi_2|^2 + \frac{1}{\sqrt{2}r}|\mathcal{N}(\partial_{\mathcal{B}}\psi_1)|^2 \le \frac{r}{2}|\psi_2|^2 + \frac{\widetilde{C}^2}{r}|\partial_{\mathcal{B}}\psi_1|^2 \\ &\le \frac{r}{2}|\psi_2|^2 + \frac{\widetilde{C}^2}{r}|\mathcal{B}\psi_1|^2. \end{aligned}$$

Putting $C := \widetilde{C}^2$ and integrating we get (3.19).

Combining (3.19) with (3.18) gives

$$\begin{aligned} \int_M (|\overline{\partial}^*_{\mathcal{B}}\psi_2|^2 + r|\psi_1|^2|\psi_2|^2) dV_g &= \int_M \langle\psi_2, \mathcal{N}(\partial_{\mathcal{B}}\psi_1)\rangle dV_g \\ &\le \frac{r}{2}\int_M |\psi_2|^2 dV_g + \frac{C}{2}\int_M |\mathcal{B}\psi_1|^2 dV_g. \end{aligned}$$

Rearranging yields

$$\int_M (|\overline{\partial}^*_{\mathcal{B}}\psi_2|^2 + r|\psi_1|^2|\psi_2|^2 - \frac{r}{2}|\psi_2|^2) dV_g \le \frac{C}{2}\int_M |\mathcal{B}\psi_1|^2 dV_g.$$

By applying the complex version of the Bochner-Lichnerowicz identity we can estimate this last integral:

$$\begin{aligned} |\mathcal{B}\psi_1|^2 = \langle\mathcal{B}\psi_1, \mathcal{B}\psi_1\rangle &= \langle\psi_1, \mathcal{B}^*\mathcal{B}\psi_1\rangle \\ &= \langle\psi_1, 2\overline{\partial}^*_{\mathcal{B}}\overline{\partial}_{\mathcal{B}}\psi_1\rangle + \langle\alpha, i\langle\omega, F_{\mathcal{B}}\rangle\psi_1\rangle \\ &= 2|\overline{\partial}_{\mathcal{B}}\psi_1|^2 + r(1 - |\psi_1|^2 + |\psi_2|^2)|\psi_2|^2 \end{aligned}$$

so upon integration we get

$$\int_M |\mathcal{B}\psi_1|^2 dV_g = \int_M \big(2|\overline{\partial}_{\mathcal{B}}\psi_1|^2 + r(1 - |\psi_1|^2 + |\psi_2|^2)|\psi_2|^2\big) dV_g. \tag{3.20}$$

Since $\mathcal{B}$ is the connection on the trivial line bundle $L = \det(\sigma_0)$ we have $0 = c_1(L) = \frac{1}{2\pi i}[F_{\mathcal{B}}]$, i.e. $F_{\mathcal{B}}$ is cohomological to 0. Thus

$$0 = 2\pi i[\omega] \smile c_1(L) = [\omega \wedge F_{\mathcal{B}}]$$

and therefore $\int_M \omega \wedge F_{\mathcal{B}} = 0$. But since $\omega$ is self-dual we get

$$\begin{aligned}\omega \wedge F_{\mathcal{B}} &= F_{\mathcal{B}} \wedge \omega = F_{\mathcal{B}} \wedge *\omega = \langle F_{\mathcal{B}}, \omega\rangle dV_g = \langle \omega, F_{\mathcal{B}}\rangle dV_g = \langle \omega, F_{\mathcal{B}}^+\rangle dV_g \\ &= -ir(1 - |\psi_1|^2 + |\psi_2|^2)dV_g \qquad (3.21)\end{aligned}$$

from which we conclude that $\int_M r(1-|\psi_1|^2+|\psi_2|^2)dV_g = 0$. Adding this to the right-hand side of (3.20) we obtain

$$\int_M |\mathcal{B}\psi_1|^2 dV_g = \int_M 2|\overline{\partial}_{\mathcal{B}}\psi_1|^2 + r(1-|\psi_1|^2+|\psi_2|^2)(|\psi_1|^2-1)dV_g. \qquad (3.22)$$

Let's work on the right-hand side of this equation. First we can, by (3.14) replace $|\overline{\partial}_{\mathcal{B}}\psi_1|^2$ by $|\overline{\partial}^*_{\mathcal{B}}\psi_2|^2$ then we can calculate the second term and estimate by using (3.20) (times 2):

$$\begin{aligned}\text{RHS} &= \int_M 2|\overline{\partial}^*_{\mathcal{B}}\psi_2|^2 - r(1-|\psi_1|^2)^2 + r|\psi_1|^2|\psi_2|^2 - r|\psi_2|^2\, dV_g \\ &\le \frac{2C}{r}\int_M |\mathcal{B}\psi_1|^2 dV_g - r\int_M (1-|\psi_1|^2)^2 dV_g - r\int_M |\psi_1|^2|\psi_2|^2 dV_g \\ &\le \frac{2C}{r}\int_M |\mathcal{B}\psi_1|^2 dV_g - r\int_M (1-|\psi_1|^2)^2 dV_g.\end{aligned}$$

Hence we can rearrange (3.22) to obtain

$$\int_M \Big(1-\frac{2C}{r}\Big)|\mathcal{B}\psi_1|^2 + r(1-|\psi_1|^2)^2 dV_g \le 0.$$

Choose $r$ so large that $1-2C/r > 0$, then for this inequality to be satisfied we must have $|\psi_1| = 1$ and $|\mathcal{B}\psi_1| = 0$. By integrating (3.21) we see that $\psi_2 \equiv 0$. By a gauge transformation we can assume $\psi_1 \equiv 1$ and hence $\mathcal{B}$ must be the trivial connection, i.e. $\mathcal{A} = \mathcal{A}_0$. Thus up to gauge transformation we only have the solution $(1, \mathcal{A}_0)$ and hence the moduli space $\mathcal{M}(\eta)$ consists of only one point. □

# INDEX

# BIBLIOGRAPHY

[1] Poul Aage, *The Construction of the Seiberg-Witten Invariants*, Master's thesis, University of Copenhagen, 1997.

[2] Bernhelm Booß-Bavnbek and Krzysztof P. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Mathematics: Theory and Applications, Birkhäuser, 1993.

[3] V. Bouchard and A. Wijns (ed.), *First Modave Summer School in Mathematical Physics - proceedings*, Modave Schools and Workshops, vol. 1, International Solvay Institute, 2005.

[4] John B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, vol. 96, Springer Verlag, 1990.

[5] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geometry **18** (1983), 279–315.

[6] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Mathematical Monographs, vol. 11, Clarendon Press, 1991.

[7] Yakov Eliasberg and Lisa Traynor (editors), *Symplectic Geometry and Topology*, IAS/Park City Mathematical Series, vol. 7, American Mathematical Society, Institute for Advanced Study, 1999.

[8] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982), 357–453.

[9] Thomas Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, 2000.

[10] Gerd Grubb, *Distributions and Operators*, Lecture Notes in Mathematics, vol. 252, Springer-Verlag, 2009.

[11] Daniel Huybrechts, *Complex Geometry : An Introduction*, Universitext, International Solvay Institute, 2005.

[12] Wen Jiang, *Dirac operators.*

[13] H. Blaine Lawson and Marie-Louise Michelson, *Spin Geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, 1989.

[14] John M. Lee, *Riemannian Manifolds. An Introduction to Curvature*, Graduate Texts in Mathematics, vol. 176, Springer Verlag, 1997.

[15] Dusa McDuff and Dietmar Salamon, *Introduction to Symplectic Topology, 2nd ed.*, Oxford Mathematical Monographs, Oxford Science Publications, 1998.

[16] Reinhold Meise and Dietmat Vogt, *Introduction to Functional Analysis*, Oxford Graduate Texts in Mathematics, vol. 1997, Clarendon Press, 2000.

[17] John W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Mathematical Notes, vol. 44, Princeton University Press, 1996.

[18] Liviu I. Nicolaescu, *Notes on Seiberg-Witten Theory*, Graduate Studies in Mathematics, vol. 28, Americal Mathematical Society, 2000.

[19] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1961), 391–406.

[20] Edwin H. Spanier, *Algebraic Topology*, McGraw-Hill series in higher mathematics, McGraw-Hill, 1966.

[21] Norman Earl Steenrod, *Topology of Fibre Bundles*, Princeton Mathematical Series, vol. 14, Princeton University Press, 1951.

[22] Clifford Henry Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), 809–822.

[23] Edward Witten, *Monopoles and 4-manifolds*, Math. Res. Lett. **1** (1994), 769–796.