The Work of Harish-Chandra

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Contents

1 Parabolic and Cartan Subgroups

We begin by the definition of parabolic subalgebras and subgroups. The setup is the following: $\mathfrak{g}$ is a real reductive Lie algebra with Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a}_0 \subseteq \mathfrak{p}$ is a maximal abelian subalgebra, $\mathfrak{n}_0$ is the direct sum of the restricted root spaces corresponding to the positive restricted roots (relative to a choice of positivity) and finally $\mathfrak{m}_0 := Z_{\mathfrak{k}}(\mathfrak{a}_0)$.

Definition 1.1 (Parabolic subalgebra). Let $\mathfrak{g}$ be a real reductive Lie algebra. A *minimal parabolic subalgebra* is a subalgebra $\mathfrak{q}_0 \subseteq \mathfrak{g}$ which is conjugate via $\mathrm{Ad}(G)$ to the subalgebra $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. A *parabolic subalgebra* is a subalgebra of $\mathfrak{q} \subseteq \mathfrak{g}$ which contains a minimal parabolic subalgebra.

We will always let subscript 0 indicate that we are dealing with a minimal parabolic subalgebra.

Let Σ denote the set of restricted roots w.r.t. $\mathfrak{a}_0$ and let Π denote the set of simple roots w.r.t. the choice of positivity mentioned above.

Theorem 1.2. *The set of conjugacy classes of parabolic subalgebras are parametrized by subsets Γ satisfying $\Sigma^+ \subseteq \Gamma \subseteq \Sigma$, the correspondence being that Γ is related to the parabolic subalgebra*

$$\mathfrak{q}_\Gamma := \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha .$$

Consider the parabolic subalgebra given by Γ. If we define

$$\begin{aligned}\mathfrak{a}_\mathfrak{q} &:= \bigcap_{\alpha\in\Gamma\cap-\Gamma} \ker\alpha \\ \mathfrak{m}_\mathfrak{q} &:= \mathfrak{a}_\mathfrak{q}^\perp \oplus \mathfrak{m}_0 \oplus \bigoplus_{\alpha\in\Gamma\cap-\Gamma} \mathfrak{g}_\alpha \\ \mathfrak{n}_\mathfrak{q} &:= \bigoplus_{\substack{\alpha\in\Gamma\\ \alpha\notin-\Gamma}} \mathfrak{g}_\alpha,\end{aligned}$$

then we have the so-called *Langlands decomposition* of the parabolic subalgebra

$$\mathfrak{q} = \mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q} \oplus \mathfrak{n}_\mathfrak{q}$$

in analogy with the definition of the minimal parabolic subalgebra. The difference is that $\mathfrak{a}_\mathfrak{q}$ and $\mathfrak{n}_\mathfrak{q}$ have become smaller, whereas $\mathfrak{m}_\mathfrak{q}$ has become considerably bigger.

Definition 1.3 (**Parabolic subgroup**)**.** Let G be a real reductive Lie group and let $\mathfrak{q}$ be a parabolic subalgebra of its Lie algebra $\mathfrak{g}$. The normalizer $Q := N_G(\mathfrak{q})$ of the parabolic subalgebra is called a *parabolic subgroup* of G.

Fortunately, this allows a more explicit description: namely consider the Langlands decomposition $\mathfrak{q} = \mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q} \oplus \mathfrak{n}_\mathfrak{q}$ of $\mathfrak{q}$ and let A and N be analytic subgroups of G corresponding to $\mathfrak{a}_\mathfrak{q}$ and $\mathfrak{n}_\mathfrak{q}$ and put $M := {}^0Z_G(\mathfrak{a}_\mathfrak{q})$ [1] then it turns out that Q can be decomposed as

$$Q = MAN$$

and this is called the *Langlands decomposition* of the parabolic subgroup.

Parabolic subalgebras and subgroups are closely related to Cartan subalgebras and Cartan subgroups. In the case where $\mathfrak{g}$ is a complex semisimple Lie algebra, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called a *Cartan subalgebra* if it is maximally abelian and $\operatorname{ad}(H)$ is diagonalizable for every $H \in \mathfrak{h}$. In the real case we make the following definition:

Definition 1.4 (**Cartan subalgebra**)**.** Let $\mathfrak{g}$ be a real semisimple Lie algebra. A subalgebra of $\mathfrak{g}$ is called a *Cartan subalgebra* if the complexification $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$.

For a real reductive Lie algebra $\mathfrak{g}$, a subalgebra is called a *Cartan subalgebra* if it is of the form $\mathfrak{h} \oplus Z_\mathfrak{g}$ where $\mathfrak{h}$ is a Cartan subalgebra of the semisimple algebra $[\mathfrak{g},\mathfrak{g}]$.

In the complex case, all Cartan subalgebras are mutually conjugate. This is no longer true in the real case, but it still holds that the dimensions of the Cartan subalgebras are the same (simply because the complexify to complex Cartan subalgebras which all have the same dimensions). This common dimension is called the *rank* of the Lie algebra $\mathfrak{g}$.

A Cartan subalgebra is called *θ-stable* if $\theta(\mathfrak{h}) \subseteq \mathfrak{h}$. This implies a decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ are the compact and non-compact

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[1] Recall that for a reductive Lie group G, the Lie algebra decomposes as $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus Z_\mathfrak{g}$ where $[\mathfrak{g},\mathfrak{g}]$ is semisimple. Letting G_{ss} denote the (semisimple) analytic subgroup of G with Lie algebra $[\mathfrak{g},\mathfrak{g}]$, then we define ${}^0G := G_{ss}K$ where K is the maximal compact subgroup of G from the Cartan decomposition.

components of $\mathfrak{h}$ respectively (note that $\mathfrak{a}$ need not be related to the $\mathfrak{a}$ above). The Cartan subalgebra is called *maximally compact* resp. *maximally non-compact* if $\mathfrak{t}$ resp. $\mathfrak{a}$ are as large as possible (for instance if either $\mathfrak{t}$ or $\mathfrak{a}$ were maximally abelian).

Proposition 1.5. *The following properties hold*:

1) *Every Cartan subalgebra is conjugate to a θ-stable Cartan subalgebra.*
2) *There exist at most finitely many conjugacy classes of Cartan subalgebras.*
3) *If $\mathfrak{t} \subseteq \mathfrak{k}$ is maximally abelian, then $Z_{\mathfrak{g}}(\mathfrak{t})$ is a Cartan subalgebra of the form $\mathfrak{t} \oplus \mathfrak{a}$ for some $\mathfrak{a} \subseteq \mathfrak{p}$, in particular it is maximally compact.*
4) *If $\mathfrak{a}_0 \subseteq \mathfrak{p}$ is maximally abelian and $\mathfrak{t} \subseteq Z_{\mathfrak{k}}(\mathfrak{a}_0)$ is maximally abelian, then $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}_0$ is a maximally non-compact Cartan subalgebra.*

The last property shows, in particular, that maximally compact resp. non-compact Cartan subalgebras always exist.

Definition 1.6 (Cartan subgroup). For a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ the corresponding *Cartan subgroup* is the centralizer $H := Z_G(\mathfrak{h})$ of $\mathfrak{h}$ in G.

Since the Lie algebra of $Z_G(\mathfrak{h})$ equals $Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, we see that $\mathfrak{h}$ is actually the Lie algebra of H. Although a Cartan subgroup need not be abelian it should still be thought of as an analogy of a maximal torus inside a compact group: the union of the Cartan subgroups exhaust almost all of the group G, where "almost" is in the following sense: Define the polynomial

$$D(g, \lambda) := \det((\lambda + 1)\,\mathrm{id} - \mathrm{Ad}(g)) = \lambda^n + \sum_{j=0}^{n-1} D_j(x)\lambda^j$$

with coefficients being real analytic functions on G. Define

$$\ell := \min_{0 \le j \le n-1} \{ j \mid D_j \text{ not identically } 0 \},$$

so for this ℓ there exist $g \in G$ such that $D_\ell(g) \neq 0$, and these elements are called *regular elements* of G and the set of regular elements is denoted G'. Then the main result of Cartan subgroups is that their conjugacy classes exhaust all the regular elements:

Theorem 1.7. *For a reductive Lie group, let $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$ be a maximal set of non-conjugate θ-stable Cartan subalgebras and let $H_1, \ldots, H_k$ be the corresponding Cartan subgroups, then*

$$G' \subseteq \bigcup_{j=1}^{k} \bigcup_{x \in G} x H_j x^{-1}.$$

The relation of Cartan subalgebras to parabolic subgroups is via so-called *cuspidal parabolic subalgebras* which are the parabolic subalgebras $\mathfrak{q} = \mathfrak{m}_{\mathfrak{q}} \oplus \mathfrak{a}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}$ (not necessarily minimal) for which $\mathfrak{m}_{\mathfrak{q}}$ has a θ-stable maximally compact Cartan subalgebra. Similarly, a parabolic subgroup is called *cuspidal* if its Lie algebra is a cuspidal parabolic subalgebra. Another way of phrasing this is that the rank of

$\mathfrak{m}_\mathfrak{q} \cap \mathfrak{k}$ equals the rank of $\mathfrak{m}_\mathfrak{q}$. In this case, if $\mathfrak{t} \subseteq \mathfrak{m}_\mathfrak{q}$ is a Cartan subalgebra, then $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}_0$ (where $\mathfrak{a}_0 \subseteq \mathfrak{p}$ is maximally abelian) is a θ-stable Cartan subalgebra of $\mathfrak{g}$, cf. Proposition 1.5 4).

Conversely, if $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a θ-stable Cartan subalgebra and if we put on $\Delta(\mathfrak{g}, \mathfrak{h})$ the lexicographic ordering taking $\mathfrak{a}$ before $i\mathfrak{t}$ and define

$$\mathfrak{m} := \mathfrak{g} \cap \Big(\mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \\ \alpha|_\mathfrak{a} = 0}} \mathfrak{g}_\alpha \Big),$$

$$\mathfrak{n} := \mathfrak{g} \cap \Big(\bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}) \\ \alpha|_\mathfrak{a} \neq 0}} \mathfrak{g}_\alpha \Big),$$

then $\mathfrak{q} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Langlands decomposition of a cuspidal parabolic subalgebra. One can show that these constructions are each others inverse and thus

Proposition 1.8. *There is a* 1-1 *correspondence between cuspidal parabolic subalgebras and Cartan subalgebras.*

2 Admissible Representations

We continue to let G denote a real reductive Lie group with maximally compact subgroup K. We assume moreover that G is connected. Let furthermore π be a representation on a Hilbert space V. A vector $v \in V$ is called *K-finite* if

$$\operatorname{span}_{k \in K}\{\pi(k)v\}$$

is finite-dimensional. We denote the space of K-finite vectors by V^K. Let $\delta \in \widehat{K}$ and let V_δ be the corresponding representation space, then we say that v is *K-finite of type δ* if $\operatorname{span}_{k \in K}\{\pi(k)v\}$ is equivalent, as a K-module, to a finite direct sum $V_\delta \oplus \cdots \oplus V_\delta$ and the space of K-finite vectors of δ-type is denoted $V(\delta)$. Obviously, $V^K = \bigoplus_{\delta \in \widehat{K}} V(\delta)$.

Definition 2.1 (Admissible representation). A representation π on a Hilbert space V is called *admissible* $\pi(k)$ is unitary for all $k \in K$ and if $V(\delta)$ is finite-dimensional for all $\delta \in \widehat{K}$.

Equivalently, we can describe it in the following way: By the Peter-Weyl Theorem the restriction of the representation π to K is completely reducible

$$\pi|_K = \bigoplus_{\delta \in \widehat{K}} \delta^{\oplus n_\delta} \qquad \text{and} \qquad V = \bigoplus_{\delta \in \widehat{K}} V_\delta^{\oplus n_\delta}$$

where $n_\delta \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then clearly $V(\delta) = V_\delta^{\oplus n_\delta}$ and hence π is admissible if and only if $n_\delta < \infty$ for all $\delta \in \widehat{K}$. From this we easily deduce

Proposition 2.2. *The set V^K of K-finite vectors is dense in V.*

In 1953 Harish-Chandra proved the following

Theorem 2.3. *An irreducible unitary representation is admissible.*

In 1988 Soergel proved, by giving a counterexample, that irreducible non-unitary representations need not be admissible.

One very nice feature of admissible representations is that they are well-tailored for an infinitesimal description. Recall that for an infinite-dimensional representation π on V, we say that $v \in V$ is a C^∞*-vector* if the map

$$G \ni g \longmapsto \pi(g)v$$

is a smooth map $G \longrightarrow V$. Let V^∞ denote the set of C^∞-vectors. This is known to be a dense and $\pi(G)$-invariant subset of V and we can define a Lie algebra representation π_* of $\mathfrak{g}$ on V^∞ by

$$\pi_*(X)v := \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tX))v$$

C^∞-vectors are usually quite hard to get a hold on, but for admissible representations we can do a lot better (also a result by Harish-Chandra)

Proposition 2.4. *If π is an admissible representation then every K-finite vector is a C^∞-vector, and V^K is $\pi_*(\mathfrak{g})$-invariant* [2].

Thus it makes sense to restrict the representation π_* to a representation on V^K (which was dense in V as remarked above). This leads to the notion of *infinitesimal equivalence*: two admissible representations are said to be *infinitesimally equivalent* if the associated representations of $\mathfrak{g}$ are algebraically equivalent. To be more precise, assume π_1 and π_2 are admissible, then they are infinitesimally equivalent if there exists a linear isomorphism $\varphi : V_1^K \xrightarrow{\sim} V_2^K$ such that $\varphi \circ (\pi_1)_*(X) = (\pi_2)_*(X) \circ \varphi$. The following theorem illustrates how much information about the representation π one can retrieve from the associated representation on K-finite vectors:

Theorem 2.5. *Let G be a connected reductive Lie group.*

1) *A representation π of G on V is irreducible if and only if its associated $\mathfrak{g}$-representation on V^K is irreducible.*
2) *Two irreducible unitary representations are equivalent if and only if they are infinitesimally equivalent.*

Note that for point 1) a $\mathfrak{g}$-representation is called irreducible if it has no nontrivial subspaces whereas a continuous G-representation is called irreducible if it has no nontrivial *closed* subspaces.

Next we turn to the definition of some *very* important admissible representations, namely the so-called *principal series representations.* Let $Q = MAN$ be a parabolic subgroup (not necessarily a minimal one), let σ be an irreducible unitary representation of M on some Hilbert space V_σ and let $\nu \in \mathfrak{a}'_{\mathbb{C}}$. Then we define a representation $\sigma \otimes \exp\nu \otimes 1$ of Q on V_σ by

$$(\sigma \otimes \exp\nu \otimes 1)(man) := e^{(\nu+\rho)\log(a)}\sigma(m)$$

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[2] But it is not true that every C^∞-vector is K-finite.

and we induce from this a representation $U_{Q,\sigma,\nu}$ of G. To be more specific this representation is constructed as follows: Consider the space of continuous functions $f : G \longrightarrow V_\sigma$ with the following invariance $f(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}f(x)$. Equip it with the norm

$$\|f\| := \int_K |f(k)|^2 dk,$$

let $V_{Q,\sigma,\nu}$ denote the norm closure and define the representation $U_{Q,\sigma,\nu}$ on this space by

$$(U_{Q,\sigma,\nu}(g)f)(x) := f(g^{-1}x).$$

If Q is a minimal parabolic subgroup, these representations are called the *principal series representations*, and if Q is non-minimal they are called the *generalized principal series representations.* If ν is (non)imaginary they are called (*non*)*unitary principal series representations* and finally if σ is the trivial representation, they are called *spherical series representations.*

Proposition 2.6. *For the principal series representations the following hold*:

1) *The representation $U_{Q,\sigma,\nu}$ is a continuous representation of G.*
2) *$U_{Q,\sigma,\nu}$ is admissible.*
3) *If ν is imaginary, $U_{Q,\sigma,\nu}$ is a unitary representation.*

In the above proposition Q is a parabolic subgroup which can be minimal or not, but in any case σ is supposed to be an irreducible unitary representation of M.

We will elaborate more on point 2) since it turns out that the principal series representations (in some sense to be described) exhaust all the irreducible admissible representations.

Definition 2.7 (Tempered representation). A representation of a reductive Lie group G on a Hilbert space is called *tempered* if all its K-finite matrix coefficients satisfy

$$|\langle \pi(g)v, w\rangle| \leq c_{v,w}\varphi_0(g)$$

where $\varphi_0(g) := \int_K e^{\rho(H(g^{-1}k))}dk$ is a spherical function on G.

Some simple properties of tempered representations are collected in the following proposition

Proposition 2.8. *If π is a tempered representation, then all its K-finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$.*

If $Q = MAN$ is a parabolic subgroup and σ is a tempered representation of M and $\nu \in \mathfrak{a}'_{\mathbb{C}}$ has $\operatorname{Re}\nu = 0$ then $U(Q, \sigma, \nu)$ is also tempered.

The first claim is a simple consequence of the fact that the spherical functions are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. If it holds also for $\varepsilon = 0$, then the representation is said to be in the *discrete series.* We discuss that case in a later section.

Tempered representations lead us to so-called *Langlands quotients* whose construction is given by the following theorem

Theorem 2.9 (**Langlands quotients**). *Let $Q = MAN$ be a parabolic subgroup, σ a tempered irreducible unitary representation of M and let $\nu \in \mathfrak{a}'_{\mathbb{C}}$ be such that $\operatorname{Re}\nu$ is in the open positive Weyl chamber of $\mathfrak{a}$. Then $U_{Q,\sigma,\nu}$ has a unique irreducible quotient, and this quotient is given by mod'ing out the kernel of the formal intertwining operator $A_{\overline{Q},Q,\sigma,\nu}$.*

Quotient in this case simply means that we take as representation space the quotient $V_{Q,\sigma,\nu}/\ker(A_{\overline{Q},Q,\sigma,\nu})$ and define a G-action by

$$g \cdot [v] := [U_{Q,\sigma,\nu}(g)v].$$

This is well-defined by the intertwining property of $A_{\overline{Q},Q,\sigma,\nu}$. This G-representation is called the *Langlands quotient*, denoted $J(Q,\sigma,\nu)$. It is equivalent to a subrepresentation of $U_{\overline{Q},\sigma,\nu}$, namely $U_{\overline{Q},\sigma,\nu}$ restricted to the image space $A_{\overline{Q},Q,\sigma,\nu}V_{Q,\sigma,\nu} \subseteq V_{\overline{Q},\sigma,\nu}$.

With the construction of the Langlands quotient we can state a classification theorem for irreducible admissible representations

Theorem 2.10 (**Langlands classification**). *Fix a minimal parabolic subgroup Q_0 of the connected reductive group G. The equivalence classes of irreducible admissible representations of G are in 1-1-correspondence with triples of the form $(Q,[\sigma],\nu)$ where*

1) *Q is a parabolic subgroup of G containing Q_0.*
2) *σ is a tempered, irreducible, unitary representation of M and $[\sigma]$ is its equivalence class.*
3) *$\nu \in \mathfrak{a}'_{\mathbb{C}}$ is such that $\operatorname{Re}\nu$ is in the open Weyl chamber.*

The correspondence is given by mapping $(Q,[\sigma],\nu)$ to the Langlands quotient $J(Q,\sigma,\nu)$.

3 Infinitesimal Characters

Let π be an irreducible admissible representation of a connected reductive Lie group G. This induces a representation π_* of the Lie algebra $\mathfrak{g}$ on the space of K-finite vectors. This we can complexify and extend to an algebra representation, also denoted π_*, of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ on V^K. Let $Z(\mathfrak{g}_{\mathbb{C}})$ denote the center of $U(\mathfrak{g}_{\mathbb{C}})$. For $X \in Z(\mathfrak{g})$ it turns out that $\pi_*(X)$ is just a scalar multiple of the identity map id_{V^K}. By the homomorphism property of π_* this implies the existence of a homomorphism $\chi_\pi : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}$ such that

$$\pi_*(X) = \chi_\pi(X)\,\mathrm{id}_{V^K}$$

for all $X \in Z(\mathfrak{g}_{\mathbb{C}})$. Here χ_π is called the *infinitesimal character* of π. Thus we are interested in classifying the characters on the algebra $Z(\mathfrak{g}_{\mathbb{C}})$. This was yet another of Harish-Chandras great achievements.

The strategy is to identify $Z(\mathfrak{g}_{\mathbb{C}})$ with a more manageable algebra whose characters are easier to classify. This identification is via the *Harish-Chandra homomorphism* which we now define. We fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ and let $\Delta^+ = \Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ denote the set of positive roots w.r.t. $\mathfrak{h}_{\mathbb{C}}$ and some choice of positivity. In this case we know that $(\mathfrak{g}_\mathbb{C})_\mathfrak{a}$ is 1-dimensional so pick a basis E_α for each of the root spaces as well as a basis $\{H_1, \dots, H_k\}$ for the Cartan subalgebra. Inside $U(\mathfrak{g}_\mathbb{C})$ we consider the two subspaces $\mathscr{H} := U(\mathfrak{h}_\mathbb{C})$ (which is actually a commutative subalgebra) and

$$\mathscr{P} := \bigoplus_{\alpha \in \Delta^+} U(\mathfrak{g}_\mathbb{C}) E_\alpha. \tag{3.1}$$

Then one can show that $\mathscr{H} \cap \mathscr{P} = \{0\}$ and that $Z(\mathfrak{g}_\mathbb{C}) \subseteq \mathscr{H} \oplus \mathscr{P}$. What does this mean? It means the following: By the PBW-theorem a basis for $U(\mathfrak{g}_\mathbb{C})$ is given by elements of the form

$$E_{-\alpha_1}^{q_1} \cdots E_{-\alpha_l}^{q_l} H_1^{m_1} \cdots H_k^{m_k} E_{\alpha_1}^{p_1} \cdots E_{\alpha_l}^{p_l},$$

and if this belongs to the center, then $p_1 = \cdots = p_l = 0$ implies $q_1 = \cdots = q_l = 0$. In other words, assume v to be a highest weight vector in a $U(\mathfrak{g}_\mathbb{C})$-module of highest weight λ, then a basis element of the form (3.1) would act on v either by 0 (if there exists $p_j \neq 0$) or by the scalar

$$\lambda(H_1)^{m_1} \cdots \lambda(H_k)^{m_k}$$

(if $p_1 = \cdots = p_l = 0$ in which case also $q_1 = \cdots = q_l = 0$).

Thus letting $\gamma'_{\Delta^+} : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow \mathscr{H}$ denote the projection onto the first component we see that

$$X \cdot v = \lambda(\gamma_{\Delta^+}(X))v \tag{3.2}$$

for all $X \in Z(\mathfrak{g}_\mathbb{C})$.

Remembering that $\delta := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ we define a map $\sigma_{\Delta^+} : \mathfrak{h}_\mathbb{C} \longrightarrow \mathscr{H}$ by

$$\sigma_{\Delta^+}(H) := H - \delta(H)1,$$

we more or less translate by δ. This map extends to a map $\sigma_{\Delta^+} : \mathscr{H} \longrightarrow \mathscr{H}$. Now define the *Harish-Chandra homomorphism* $\gamma : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow \mathscr{H}$ by

$$\gamma := \sigma_{\Delta^+} \circ \gamma'_{\Delta^+}.$$

Then, bearing (3.2) in mind, it should come as no surprise that in the highest weight $U(\mathfrak{g}_\mathbb{C})$-module of highest weight λ considered before, we have

$$X \cdot v = (\lambda - \delta)(\gamma(X))v. \tag{3.3}$$

Even though σ_{Δ^+} and γ'_{Δ^+} depend on the choice of positivity, the Harish-Chandra homomorphism does not. Finally, recall that the Weyl group acts on $\mathfrak{h}_\mathbb{C}$ (it acts by the adjoint action of certain equivalence classes of elements in K) and this action extends to an algebra action on $\mathscr{H}$. Let $\mathscr{H}^W$ denote the subalgebra of elements fixed by the Weyl group action.

Theorem 3.1. *The Harish-Chandra homomorphism is an isomorphism*

$$\gamma : Z(\mathfrak{g}_\mathbb{C}) \xrightarrow{\sim} \mathscr{H}^W.$$

If $\Lambda \in \mathfrak{h}_{\mathbb{C}}'$, then since $\mathfrak{h}_{\mathbb{C}}$ is abelian we have $\Lambda([X,Y]) = 0 = [\Lambda(X), \Lambda(Y)]$ (the last bracket being the commutator bracket in $\mathbb{C}$). Consequently Λ extends to an algebra homomorphism $\Lambda : U(\mathfrak{h}_{\mathbb{C}}) \longrightarrow \mathbb{C}$. Composing with the Harish-Chandra homomorphism gives us

$$\chi_\Lambda := \Lambda \circ \gamma : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}.$$

Not all of these maps are different, since $\gamma(X) \in \mathscr{H}^W$ we see

$$\chi_{w\cdot\Lambda}(X) = (w \cdot \Lambda)(\gamma(X)) = \Lambda(\mathrm{Ad}(w^{-1})\gamma(X)) = \Lambda(\gamma(X)) = \chi_\Lambda(X)$$

and hence $\chi_{w\cdot\Lambda} = \chi_\Lambda$. Actually the converse statement also holds: if $\chi_\Lambda = \chi_{\Lambda'}$ then $\Lambda' = w \cdot \Lambda$ for some w in the Weyl group.

Theorem 3.2. *Every homomorphism* $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}$ *is of the form* χ_Λ *for some* $\Lambda \in \mathfrak{h}_{\mathbb{C}}'$.

The Weyl invariance taken into consideration, one might say that the characters of $Z(\mathfrak{g}_{\mathbb{C}})$ are in 1-1 correspondence with a half-open Weyl chamber.

The content of (3.3) is simply that the infinitesimal character of the finite-dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight λ is $\chi_{\lambda+\delta}$.

4 The Discrete Series

In the 30s and 40s when Gelfand and Naimark worked on representation theory for complex semisimple groups, they found that the principal series (defined almost as we have done above) where sufficient for a direct integral decomposition of the left regular representation. This is no longer true in the real case. Here there are additional irreducible representations known as the *discrete series*. In 1966 Harish-Chandra managed to characterize the groups which admit discrete series representations.

For this section we will assume G to be semisimple with finite center.

Definition 4.1 (**Discrete Series Representation**)**.** An irreducible unitary representation of G is called a *discrete series representation* if there exists a K-finite matrix element which is square integrable over G.

In particular, if G is compact, all irreducible representations are in the discrete series.

Proposition 4.2. *For a discrete series representation all matrix elements* (K*-finite or not*) *are in* $L^2(G)$.

So in a sense irreducible unitary tempered representations are those which are "infinitely close" to being in the discrete series.

For a discrete series representation there exists a constant d_π such that the following Schur orthogonality relation hold

$$\int_G \langle \pi(g)u_1, v_1\rangle \overline{\langle \pi(g)u_2, v_2\rangle} dg = d_\pi^{-1} \langle u_1, u_2\rangle \overline{\langle v_1, v_2\rangle}$$

and inspired by the compact case, we call d_π the *formal degree* of π.

Theorem 4.3 (Harish-Chandra). *A semisimple Lie group G admits discrete series representations if and only if the rank of G equals the rank of its maximally compact subgroup K.*

Another way to phrase this requirement is to say that any maximal torus $T \subseteq K$ is a Cartan subgroup of G. Recalling the definition of cuspidal parabolic subgroups we see that a parabolic subgroup $Q = MAN$ is cuspidal if and only if M admits discrete series representations!

The classical real simple groups: First $\mathrm{SL}(n, \mathbb{R})$, this has rank $n-1$, whereas the maximal compact subgroup $\mathrm{SO}(n)$ has rank $\lfloor \frac{n}{2} \rfloor$ and these are identical if and only if $n = 2$, thus among the special linear groups only $\mathrm{SL}(2, \mathbb{R})$ has a discrete series. Second $\mathrm{SU}(p, q)$ which has maximal compact subgroup is $S(\mathrm{U}(p) \times \mathrm{U}(q))$ and both of these have rank $p+q-1$ and thus these groups always have a discrete series. For $\mathrm{SO}(p, g)$ which has rank $\lfloor \frac{p+q}{2} \rfloor$ we have a maximal compact subgroup $S(\mathrm{O}(p) \times \mathrm{O}(q))$ which has rank $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$ and these are equal if and only if pq is even. Finally, $\mathrm{Sp}(n, \mathbb{R})$ has the maximal compact subgroup $\mathrm{U}(n)$ and both have rank n, and thus the real symplectic groups always have discrete series.

The theorem also accounts for the phenomenon encountered before: complex semisimple Lie groups never admit discrete series representations. This is for the following reason: If G is a complex semisimple Lie group, then it has a compact real form K, i.e. a compact subgroup whose complexification is all of G. Note that K is also maximally compact. But the complexification of G, viewed as a real Lie group (the complex structure forgotten) is just $G \times G$ and hence

$$\frac{\operatorname{rank} G}{\operatorname{rank} K} = \frac{\operatorname{rank}(G \times G)}{\operatorname{rank} G} = 2,$$

thus in the complex case it can never happen that $\operatorname{rank} K = \operatorname{rank} G$ and hence groups like $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Sp}(n, \mathbb{C})$ have no discrete series representations.

For now let's assume $\operatorname{rank} G = \operatorname{rank} K$ and let's state the classification theorem for the discrete series representations we now know exist. The assumption implies that a Cartan subalgebra $\mathfrak{h}_\mathbb{C} \subseteq \mathfrak{k}_\mathbb{C}$ is also a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, consequently we can consider the two root systems

$$\Delta_G := \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \qquad \text{and} \qquad \Delta_K := \Delta(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C}).$$

The root spaces of the roots in Δ_G lie either inside $\mathfrak{k}_\mathbb{C}$ or inside $\mathfrak{p}_\mathbb{C}$ and a root $\alpha \in \Delta_G$ is called *compact* resp. *non-compact* if $(\mathfrak{g}_\mathbb{C})_\alpha \subseteq \mathfrak{k}_\mathbb{C}$ resp. $(\mathfrak{g}_\mathbb{C})_\alpha \subseteq \mathfrak{p}_\mathbb{C}$. Then Δ_K is precisely the set of compact roots. Let W_G and W_K denote the Weyl groups of the two root systems. If we have picked a notion of positivity on Δ_G, it induces a positivity on Δ_K by $\Delta_K^+ = \Delta_K \cap \Delta_G^+$. With respect to these choices of positivity let δ_G and δ_K denote half the sum of the positive roots of Δ_G resp. Δ_K.

Before we state the classification theorem, let's remind ourselves about some representation theory of compact groups. Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Since compactness implies that $\mathfrak{k}$ is reductive, we know that $\mathfrak{k} = Z_\mathfrak{k} \oplus [\mathfrak{k}, \mathfrak{k}]$ where $[\mathfrak{k}, \mathfrak{k}]$ is semisimple. Let $\mathfrak{h} \subseteq \mathfrak{k}$ be a maximal abelian subalgebra (which is then automatically a Cartan subalgebra) and let $T \subseteq K$ be its associated analytic subgroup (which is then a maximal torus of K). A functional $\lambda \in \mathfrak{h}_\mathbb{C}'$ is called an *analytic integral element* if it lifts to a character on T, more precisely, if there exists a group homomorphism $\xi_\lambda : T \longrightarrow \mathrm{U}(1)$ such that

$$\xi_\lambda(\exp(H)) = e^{\lambda(H)}$$

for all $H \in \mathfrak{h}$. The set of analytic integral elements form a *lattice* in $\mathfrak{h}'_{\mathbb{C}}$ i.e. a discrete subgroup with compact quotient. So do the $\mathbb{Z}$-linear combination of roots and the algebraic integral elements (those who satisfy $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$) and we have the following inclusions

$$\begin{aligned}\{\mathbb{Z}\text{-combination of roots}\} &\subseteq \{\text{analytic integral elements}\}\\ &\subseteq \{\text{algebraic integral elements}\},\end{aligned}$$

and if K happens to be semisimple, the index of the first lattice in the second equals the determinant of the Cartan matrix, and the index of the second in the third equals the order of the fundamental group $\pi_1(K)$ (which is finite by Weyl's Theorem).

The irreducible representations of K are in 1-1 correspondence with the *dominant* analytic integral elements (dominant meaning that $\langle\alpha,\lambda\rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$) the correspondence being that the representation π of K corresponds to the highest weight of $\pi_*^{\mathbb{C}}|_{[\mathfrak{k}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}]}$, the restriction to $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$ of the complexification of the associated Lie algebra representation.

Theorem 4.4 (**Classification of the Discrete Series**). *Under the assumption that G is semisimple with finite center and that* $\operatorname{rank} G = \operatorname{rank} K$ *and $\mathfrak{h}$ is the common Cartan subalgebra, let $\lambda \in (i\mathfrak{h})'$ be a Δ_G-dominant element for all such that $\lambda + \delta_G$ is an analytic integral element. Then there exists a discrete series representation π_λ with the following properties*

1) *π_λ has infinitesimal character χ_λ.*

2) *$\pi_\lambda|_K$ contains with multiplicity* 1 *the K-type with highest weight $\Lambda = \lambda + \delta_G - 2\delta_K$.*

3) *If Λ' is a highest weight for a K-type appearing as a summand in $\pi_\lambda|_K$, then Λ' is of the form $\Lambda' = \Lambda + \sum_{\alpha\in\Delta_G^+} n_\alpha \alpha$ for $n_\alpha \in \mathbb{Z}_{\geq 0}$.*

Two such representations π_λ and $\pi_{\lambda'}$ are equivalent if and only if $\lambda' = w \cdot \lambda$ for $w \in W_K$, and each discrete series representation is of the form π_λ for some λ.

5 Holomorphic Discrete Series

In some cases the discrete series contains certain representations defined on spaces of holomorphic functions called the *holomorphic discrete series*. The way to describe these is via the *Harish-Chandra decomposition* which we now briefly delineate.

The Harish-Chandra decomposition originates in the question: for which semisimple groups G will G/K admit a complex manifold structure such that G acts by holomorphic maps? In the affirmative case, G/K is called *Hermitian* or a *bounded symmetric domain*. The answer to this question was provided by Harish-Chandra. In the following we will assume that G has a complexification $G^{\mathbb{C}}$.

If we let $\mathfrak{c} \subseteq \mathfrak{k}$ denote the center of $\mathfrak{k}$ the first part of the answer is given by

Theorem 5.1. *If G/K is hermitian then $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$.*

Harish-Chandra showed the opposite implication and this is where the Harish-Chandra decomposition comes into play. So assume $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. Let $\mathfrak{h} \subseteq \mathfrak{k}$ be a maximal abelian subalgebra (hence a Cartan subalgebra of $\mathfrak{k}$). Then we have $\mathfrak{c} \subseteq \mathfrak{h}$ and consequently

$$Z_{\mathfrak{g}}(\mathfrak{h}) \subseteq Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k},$$

i.e. $\mathfrak{h}$ is also maximal abelian in $\mathfrak{g}$ and hence a Cartan subalgebra of $\mathfrak{g}$. In particular $\operatorname{rank} G = \operatorname{rank} K$, so that G admits a discrete series! As before we say that roots $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ are *compact* resp. *non-compact* $(\mathfrak{g}_{\mathbb{C}})_\alpha \subseteq \mathfrak{k}_{\mathbb{C}}$ resp. $(\mathfrak{g}_{\mathbb{C}})_\alpha \subseteq \mathfrak{p}_{\mathbb{C}}$ and we let Δ_K and Δ_N denote the sets of compact resp. non-compact roots. Then by the root space decomposition

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_K} (\mathfrak{g}_{\mathbb{C}})_\alpha \qquad \text{and} \qquad \mathfrak{p}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta_N} (\mathfrak{g}_{\mathbb{C}})_\alpha.$$

Now pick a so-called *good ordering* on Δ, i.e. a system of positivity so that any non-compact positive root is greater that any compact root. W.r.t. such a good ordering let Δ^+, Δ_K^+ and Δ_N^+ denote the corresponding sets of positive roots and define

$$\mathfrak{p}_{\mathbb{C}}^+ := \bigoplus_{\alpha \in \Delta_N^+} (\mathfrak{g}_{\mathbb{C}})_\alpha \qquad \text{and} \qquad \mathfrak{p}_{\mathbb{C}}^- := \bigoplus_{\alpha \in \Delta_N^+} (\mathfrak{g}_{\mathbb{C}})_{-\alpha}.$$

These turn out to be abelian, in particular they are subalgebras, thus we have split $\mathfrak{p}_{\mathbb{C}}$ onto two abelian subalgebras whose *vector space* direct sum equals $\mathfrak{p}_{\mathbb{C}}$. Finally we define

$$\mathfrak{b}_{\mathbb{C}} := \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_{\mathbb{C}})_{-\alpha}$$

and let P^+, $K^{\mathbb{C}}$, P^- and B be the analytic subgroups of $G^{\mathbb{C}}$ with Lie algebras $\mathfrak{p}_{\mathbb{C}}^+$, $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{C}}^-$ and $\mathfrak{b}_{\mathbb{C}}$ respectively. These subgroups are all complex.

Theorem 5.2 (Harish-Chandra decomposition). *Assume that $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$ then multiplication*

$$P^+ \times K^{\mathbb{C}} \times P^- \longrightarrow G^{\mathbb{C}} \tag{5.1}$$

is an injective, holomorphic, regular map with open image. Furthermore $GB \subseteq G^{\mathbb{C}}$ is open and there exists a bounded open subset $\Omega \subseteq P^+$ such that

$$GB = GK^{\mathbb{C}}P^- = \Omega K^{\mathbb{C}} P^-. \tag{5.2}$$

The projection $p^+ : GB \longrightarrow \Omega$ induces a diffeomorphism $G/K \xrightarrow{\sim} \Omega$ and G acts holomorphically on Ω by

$$g \cdot \omega = p^+(g\omega),$$

in other words, G/K is hermitian.

We can view the Harish-Chandra decomposition (5.1) as some kind of refined Cartan decomposition, where we have split $\exp \mathfrak{p}$ into smaller pieces which turn out to be groups. The price is that we don't hit all of $G^{\mathbb{C}}$. Note also in (5.2) that the first equation is just the definition of B, the real content is in the second equation.

Let G be a semisimple Lie group with finite center and assume the condition for the existence of the Harish-Chandra decomposition $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. Let $\lambda \in \mathfrak{h}'_{\mathbb{C}}$ be an analytic integral element. The assumption implies, as we noted above, the existence of a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$. Let $T \subseteq K$ be the maximal torus corresponding to $\mathfrak{h}$ and ξ_λ be the homomorphism $T \longrightarrow \mathrm{U}(1)$ corresponding to λ. We can extend this to a holomorphic map $T^{\mathbb{C}} \longrightarrow \mathbb{C}$ and we can extend it further to $B = T^{\mathbb{C}} N^-$ be defining it to be 1 on N^-.

Now put

$$\Gamma(\lambda) := \{F : GB \longrightarrow \mathbb{C} \mid F \in \mathcal{O}(GB), \forall x \in GB, b \in B : \ F(xb) = \xi_\lambda(b)^{-1} F(x)\}$$

and endow it with the "norm"

$$\|F\|^2 := \int_G |F(g)|^2 dg \in [0, \infty].$$

Let V_λ be the subset of $\Gamma(\lambda)$ consisting of those F with $\|F\| < \infty$. Finally consider the representation $L : G \longrightarrow \mathrm{Aut}(V_\lambda)$ given by

$$(L(g)F)(x) := F(g^{-1}x), \qquad g \in G, \ x \in GB.$$

Note the strong similarities with the Borel-Weil construction of irreducible unitary representations of compact groups, indeed, all irreducible representations of a compact group are in the discrete series and the Borel-Weil theorem states that they are in fact all of this form.

Theorem 5.3 (**Holomorphic discrete series**). *Assume, in addition to the previous requirements, that λ satisfies $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \Delta_K^+$, then it holds that V_λ is a Hilbert space (i.e. complete) and that L is a continuous unitary representation of G.*

If furthermore $\langle \lambda + \delta_G, \alpha \rangle < 0$ for all Δ_N^+, then $V_\lambda \neq \{0\}$ and L is an irreducible discrete series representation.

The representations in this theorem constitute what is known as the *holomorphic discrete series*. The condition $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$ is a necessary and sufficient condition for the existence of holomorphic discrete series representations, and these equivalent conditions imply the existence of discrete series representations. The converse implication does not hold. For instance, $\mathrm{SO}(p, q)$ possesses a discrete series if and only if pq is even, but only the groups $\mathrm{SO}(p, 2)$ have holomorphic discrete series representations.