A Short Account of Classifying Spaces and Characteristic Classes

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1 Fiber Bundles

The basic notion penetrating these notes will be that of a fiber bundle

Definition 1.1. *By a* fiber bundle *or a* locally trivial bundle *we understand a triple of topological spaces E, X and F (the* total space*, the* base space *and the* fiber *respectively) and a surjective continuous map $\pi : E \longrightarrow X$ which satisfies the following: Around each point $x \in X$ there exists a neighborhood U and a homeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times F$ (called a* local trivialization*) rendering the following diagram commutative* (pr_1 *is projection onto U*)

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

The set of trivializations $\{(U_i, \Phi_i)\}$ is called a *trivialization cover* of the bundle.

In short we write $F \to E \to X$ for a fiber bundle. Even though this is not a real short exact sequence it is nonetheless true that a fiber bundle gives a long exact sequence - in the homotopy groups:

$$\cdots \longrightarrow \pi_n(F, p_0) \xrightarrow{\iota_*} \pi_n(E, p_0) \xrightarrow{\pi_*} \pi_n(X, x_0) \xrightarrow{\partial_n} \pi_{n-1}(F, p_0) \longrightarrow \cdots \quad (1.1)$$

where p_0 and x_0 are base points and $\pi(p_0) = x_0$. This is (not surprisingly) known as the *homotopy long exact sequence*. A special kind of fiber bundle is well-known and well-studied: the covering spaces. These are just fiber bundles with discrete fiber. A crucial property of coverings is that they admit lifts of homotopies: they posses the so-called *homotopy lifting property* HLP w.r.t. any space: A map $\pi : E \longrightarrow X$ is said to have the HLP w.r.t. a space Y if given two maps $F : I \times Y \longrightarrow X$ and $\widetilde{f} : Y \longrightarrow E$ lifting $F|_{\{0\} \times Y}$, there exists a lift $\widetilde{F} : I \times Y \longrightarrow E$ of F with $\widetilde{F}(0, y) = \widetilde{f}(y)$. The situation is depicted in the commuting diagram below

$$\begin{array}{ccc} \{0\}\times Y & \xrightarrow{\widetilde{f}} & E \\ \big\downarrow & \overset{\widetilde{F}}{\nearrow} & \big\downarrow \pi \\ I\times Y & \xrightarrow[F]{} & X \end{array}$$

A pair of spaces with a map $\pi : E \longrightarrow X$ satisfying HLP for all spaces is called a *fibration*. Thus covering spaces are fibrations. In fact it can be shown that in general fiber bundles posses the HLP for all spaces, hence they are also fibrations. For a general fibration $\pi : E \longrightarrow X$ we cannot exercise as strict control over the fibers as we could for fiber bundles. But, even though the fibers need no longer be homeomorphic, one can show that the fibers are always *homotopy equivalent* to each other.

Prominent examples of fiber bundles are *vector bundles* and *principal G-bundles*:

Definition 1.2. *A real/complex* vector bundle *of rank k over X is a pair (E, π) where E is a topological space and $\pi : E \longrightarrow M$ is a continuous surjective map satisfying*:

1) *Each fiber $E_x := \pi^{-1}(x) \subseteq E$ is a real/complex k-dimensional vector space.*

2) *For every $x_0 \in X$ there is a neighborhood U of x_0 and a homeomorphism (called a* local trivialization*) $\Phi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{K}^k$ such that the following diagram commutes*:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U\times\mathbb{K}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

and such that for each $x \in U$, the map Φ_U restricted to E_x is a vector space isomorphism $E_x \xrightarrow{\sim} \{x\} \times \mathbb{K}^k$.

Definition 1.3. *Let G be a topological group and X a topological Hausdorff space. A* principal G-bundle *over X is a triple (P, π, σ) where P is a topological space, $\pi : P \longrightarrow X$ is a continuous surjective map and $\sigma : P \times G \longrightarrow P$ is a right action of G on P ($\sigma(p, g)$ will also just be denoted $p \cdot g$), satisfying*

1) *σ preserves π-fibers in the sense that $\pi(p \cdot g) = \pi(p)$.*

2) *For all $x_0 \in X$ there is an open neighborhood U of x_0 and a homeomorphism $\Phi_U : \pi^{-1}(U) \longrightarrow U \times G$ such that*

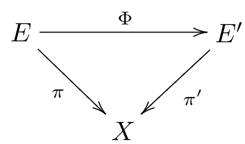
$$\Phi_U(p) = (\pi(p), \varphi_U(p))$$

where $\varphi_U : \pi^{-1}(U) \longrightarrow G$ satisfies $\varphi_U(p \cdot g) = \varphi_U(p) \cdot g$.

It should be fairly easy to check that vector bundles and principal G-bundles are indeed fiber bundles. For principal G-bundles one can show that the G-action σ is free, and that for any $p \in P$ the G-orbit of p is actually equal to the fiber $\pi^{-1}(\pi(p))$.

The easiest example of a fiber bundle is the *product bundle* over X with fiber F which has total space equal to $X \times F$ and π equal to the projection onto X.

Definition 1.4. *Consider two fiber bundles $F \to E \xrightarrow{\pi} X$ and $F' \to E' \xrightarrow{\pi'} X$ over the same base space. A* bundle map *from E to E' is a continuous map $\Phi : E \longrightarrow E'$ such that the following diagram is commutative*



i.e. Φ maps fibers of E to fibers of E'.

By a bundle isomorphism *we understand a bundle map $\Phi : E \longrightarrow E'$ which is also a homeomorphism.*

It follows immediately that fibers are mapped homeomorphically into fibers by a bundle isomorphism.

If the fibers carry some more structure, as is the case for vector bundles for instance, then one can define more sophisticated bundle maps which preserve the extra structure given. But we will not need that. The set of isomorphism classes of real/complex vector bundles of rank n over a space X is denoted $\mathrm{Vect}^n_{\mathbb{K}}(X)$.

Definition 1.5. *A* trivial bundle *is a fiber bundle which is isomorphic to a product bundle.*

2 Structure groups

As Christian Berg once said: Behind all beautiful mathematics there is a group. This is also true for fiber bundles, in that we from now on will restrict focus to those fiber bundles who admit a so-called *structure group*. The need for a structure group arises when we want to investigate how we can pass from one local trivialization to another. We motivate the general definition by a proposition from vector bundle theory stating that the structure group of a rank n vector bundle is $\mathrm{GL}(n, \mathbb{K})$:

Proposition 2.1. *Let $\pi : E \longrightarrow X$ be a vector bundle and consider two local trivializations $\Phi : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{K}^n$ and $\Psi : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{K}^n$. Then there exists a continuous map (called a* transition function*) $\tau_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(n, \mathbb{K})$ such that $\Phi \circ \Psi^{-1} : (U_i \cap U_j) \times \mathbb{K}^n \longrightarrow (U_i \cap U_j) \times \mathbb{K}^n$ has the form*

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v).$$

In general we make the following definition

Definition 2.2. *Let $F \to E \to X$ be a fiber bundle and let G be a topological group acting continuously on the fiber F. By a G-*atlas *for the fiber bundle we understand a trivialization cover (U_i, Φ_i) such that for each pair of trivializations (U_i, Φ_i) and (U_j, Φ_j) there exists a continuous map $g_{ij} : U_i \cap U_j \longrightarrow G$ such that $\Phi_i \circ \Phi_j^{-1} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$ is of the form*

$$\Phi_i \circ \Phi_j^{-1}(p, x) = (p, g_{ij}(p) \cdot x).$$

A fiber bundle equipped with a G-atlas is referred to as a fiber bundle with structure group G *or simply as a* G-bundle (*not to be confused with a* principal *G-bundle*).

Thus in this new terminology, a vector bundle is a fiber bundle with structure group $\mathrm{GL}(n, \mathbb{K})$. We denote by $\mathrm{Bun}(X, G)$ the set of isomorphism classes of G-bundles over X. The set of isomorphism classes of vector bundles over X is denoted by $\mathrm{Vect}^n_{\mathbb{K}}(X)$.

It is not hard to verify that the transition functions for a G-bundle satisfy the following elementary conditions:

1) $\tau_{ii}(p) = p$ for all $p \in U_i$,
2) $\tau_{ij}(p)^{-1} = \tau_{ji}(p)$ for all $p \in U_i \cap U_j$,
3) $\tau_{ij}(p)\tau_{jk}(p) = \tau_{ik}(p)$ for all $p \in U_i \cap U_j \cap U_k$.

The last one is known as the *cocycle condition*.

Now one asks, is a principal G-bundle also a G-bundle? Yes, of course, for if not the choice of names would be rather stupid. We construct transition functions as follows: Consider two trivialization neighborhoods U_i, U_j with homeomorphisms $\Phi_i = (\pi, \varphi_i)$ and $\Phi_j = (\pi, \varphi_j)$. Let $x \in U_i \cap U_j$ and let p be an element of the fiber above x. Then we define $g_{ij} : U_i \cap U_j \longrightarrow G$ by $g_{ij}(x) = \varphi_i(p)\varphi_j(p)^{-1}$. It is not hard to verify that this is well-defined (independent of the choice of element of the fiber) and continuous. Thus a principal G-bundle is really a G-bundle.

An important operation on G-bundles is that of *reduction of the structure group*:

Definition 2.3. *Let $F \to E \to X$ be a fiber bundle with structure group G and let H be a subgroup of G. We say that the fiber bundle admits a* reduction of its structure group to H *if there is an H-bundle $F \to E' \to X$ (with the same fiber F) which is isomorphic to E.*

The H-action on the fiber F is just restriction of the G-action on F to H. This is not just of academic interest. In fact reduction of the structure group often corresponds to some property of the bundle, for example orientability or existence of a fiber metric as we shall see below. But first a 'trivial' example:

Proposition 2.4. *A fiber bundle $F \to E \to X$ with structure group G is trivial iff its structure group can be reduced to the trivial group.*

Proof. Assume E to be trivial. Then it is isomorphic to a product bundle. But it is immediately seen that a product bundle has the trivial group as structure group, hence the structure group of E reduces to the trivial group.

Now, assume $F \to E' \to X$ to be a fiber bundle with trivial structure group. This means that for two trivializations (U_i, Φ_i) and (U_j, Φ_j) we have $\Phi_i \circ \Phi_j^{-1}(p,x) = (p,x)$ for $(p,x) \in (U_i \cap U_j) \times F$. This ensures that the following map $\Phi : \pi^{-1}(U_i \cup U_j) \longrightarrow (U_i \cup U_j) \times F$ is well-defined

$$\Phi(y) = \begin{cases} \Phi_i(y), & y \in \pi^{-1}(U_i) \\ \Phi_j(y), & y \in \pi^{-1}(U_j) \end{cases}.$$

We can continue gluing the homeomorphisms together to obtain a global trivialization. Hence the fiber bundle is trivial.

If the structure group of the fiber bundle $F \to E \to X$ admits a reduction of its structure group to the trivial group it is, by the above, isomorphic to a trivial bundle, hence is itself trivial. □

>From now on, we will lose nothing in assuming trivial bundles to have trivial structure group.

The following two examples concern vector bundles.

Definition 2.5. *A* pointwise orientation *of a vector bundle E is a choice of orientation for each fiber E_p.*

A trivialization (U, Φ) of a pointwise oriented vector bundle is said to be oriented *if the restriction $\Phi|_{E_p} : E_p \longrightarrow \mathbb{K}^n$ is orientation preserving, i.e. has positive determinant (provided $\mathbb{K}^n$ comes equipped with its usual orientation).*

A vector bundle is oriented *if it is pointwise oriented and there exists a trivialization cover consisting of oriented trivializations.*

Assume E to be an oriented vector bundle and consider two trivializations (U_i, Φ_i) and (U_j, Φ_j) and a point $p \in U_i \cap U_j$. Then since both $\Phi_i|_{E_p}$ and $\Phi_j|_{E_p}$ preserve orientation we have that τ_{ij} maps $U_i \cap U_j$ into $\mathrm{GL}^+(n, \mathbb{K})$, hence the structure group of E is $\mathrm{GL}^+(n, \mathbb{K})$. Conversely, let E be a vector bundle with structure group $\mathrm{GL}^+(n, \mathbb{K})$. Equip E with the pointwise orientation rendering $\Phi|_{E_p} : E_p \longrightarrow \mathbb{K}^n$ an orientation preserving linear map. Since the transition functions map into $\mathrm{GL}^+(n, \mathbb{K})$ this is a well-defined pointwise orientation turning E into an oriented vector bundle. Since orientation of vector bundles is preserved by vector bundle isomorphism we have obtained the following

Proposition 2.6. *Let E be a vector bundle with structure group $\mathrm{GL}(n, \mathbb{K})$. Then E is oriented iff it admits a reduction of its structure group to $\mathrm{GL}^+(n, \mathbb{K})$.*

Likewise one can prove the following

Proposition 2.7. *A real vector bundle admits a reduction of its structure group to* $\mathrm{O}(n)$ *iff it has a continuous fiber metric.*

A complex vector bundle admits a reduction of its structure group to $\mathrm{U}(n)$ *iff it has a continuous Hermitian fiber metric.*

It is a partition-of-unity argument to show that any vector bundle over a paracompact base space has a fiber metric. Most of the base spaces we encounter are paracompact, examples being compact Hausdorff spaces, manifolds, metric spaces or CW-complexes. Thus there is usually no loss of generality in assuming the structure group of a real/complex vector bundle to be $\mathrm{O}(n)$ resp. $\mathrm{U}(n)$.

3 Classifying Spaces and Characteristic Classes

This section is heavily based on the notion of a pullback bundle or induced bundle. The idea is, that if we have a bundle over Y and a continuous map $f : X \longrightarrow Y$ we get a bundle over X whose fiber is the fiber over $f(x)$. The formalization of this is the following:

Definition 3.1. *Let* $\pi : E \longrightarrow Y$ *be a fiber with fiber* F *bundle and* $f : X \longrightarrow Y$ *a continuous map. Defining the space* f^*E *by*

$$f^*E := \{(x,p) \in X \times E \mid f(x) = \pi(p)\}$$

and a map $\widetilde{\pi} : f^*E \longrightarrow X$ *by* $(x,p) \longmapsto x$ *we get a fiber bundle with fiber* F. *Indeed, if* (U_i, Φ_i) *is a trivialization cover of* E *then* $(f^{-1}U_i, \Psi_i)$ *where*

$$\Psi_i : \widetilde{\pi}^{-1}(f^{-1}U_i) \longrightarrow (f^{-1}U_i) \times F$$

is given by $\Psi_i(x,p) = (x, \mathrm{pr}_2(\Phi_i(p)))$*, is a trivialization cover of* f^*E*.*

If a fiber bundle carries some extra structure, for instance if it is a vector bundle or a principal G-bundle then this structure is preserved by pullbacks, i.e. the pullback of a vector bundle/principal G-bundle is a vector bundle resp. principal G-bundle. If the fiber bundle has structure group G, then the pullback also has structure group G, in that the transition functions are all of the form $f^*g_{ij} = g_{ij} \circ f$.

If $f, g : X \longrightarrow Y$ are two homotopic maps and X and Y are paracompact spaces, then the two pullbacks f^*E and g^*E of a bundle over Y can be shown to be isomorphic (this is *not* trivial). Thus the pullback only depends on the homotopy classes of maps $X \longrightarrow Y$. Formally, an element $[X, Y]$ (the set of homotopy classes of maps $X \longrightarrow Y$) gives a map $\mathrm{Bun}(Y,G) \longrightarrow \mathrm{Bun}(X,G)$. If X and Y are homotopy equivalent and f happens to be a homotopy equivalence, we have a homotopy inverse $g : Y \longrightarrow X$. This implies that f^* and g^* are inverses of each other, meaning that there is a bijective correspondence $\mathrm{Bun}(X,G) \longrightarrow \mathrm{Bun}(Y,G)$. In other words homotopy equivalent spaces carry the same set of bundles. Since a one-point space admits trivial bundles only, we have in particular that any bundle over a contractible space is trivial.

To classify bundles over more complicated spaces we need:

Definition 3.2. *Let G be a topological group. By a* universal bundle *for G we understand a G-bundle $p : EG \longrightarrow BG$ over a space BG, called the* classifying space *for G, such that for each G-bundle $\pi : E \longrightarrow X$ over some CW-complex X there is a map $f : X \longrightarrow BG$ such that E is isomorphic to the pullback f^*EG of the universal bundle.*

Very good! But such exotic creatures have to be very rare? Well not quite. In the 1950's Milnor showed this remarkable result

Theorem 3.3. *Every topological group has a universal bundle.*

So what do we have? Let G denote a topological group and let $p : EG \longrightarrow BG$ be its universal bundle. Any map $f : X \longrightarrow BG$ gives rise to a G-bundle over X by pulling back the universal bundle along f. This only depends on the homotopy class of f. Conversely any G-bundle over X is by definition, the pullback of the universal bundle. Hence we have the following

Theorem 3.4. *Let X be a CW-complex, G a topological group and $p : EG \longrightarrow BG$ its universal bundle. The map $[X, BG] \longrightarrow \mathrm{Bun}(X, G)$ given by $[f] \longmapsto f^*(EG)$ is a bijection.*

I.e. G-bundles are classified by maps into BG. This accounts for the name classifying space.

Let's consider a CW-complex X and let's try to classify the real vector bundles of rank n over X. Since we can assume the structure group of these vector bundles to be $\mathrm{O}(n)$, what we need is the universal bundle of $\mathrm{O}(n)$. First we construct the classifying space $B\,\mathrm{O}(n)$. The construction is via *Grassmannians*. By the Grassmannian $G_n(\mathbb{R}^k)$ $(n \leq k)$ we understand the set of real n-dimensional subspaces of $\mathbb{R}^n$. This can be topologized so that it becomes a compact manifold. The Grassmannians are generalizations of the real projective spaces, since $G_1(\mathbb{R}^k) = \mathbb{RP}^k$. Observe, that, in an obvious way, we have inclusions $G_n(\mathbb{R}^k) \subseteq G_n(\mathbb{R}^{k+1}) \subseteq \ldots$ Now define the infinite Grassmannian by

$$G_n(\mathbb{R}^\infty) := \bigcup_{k=n}^{\infty} G_n(\mathbb{R}^k)$$

and give this space the so-called weak topology (i.e. a subset $A \subseteq G_n(\mathbb{R}^\infty)$ is open iff all intersections $A \cap G_n(\mathbb{R}^k)$ are open in $G_n(\mathbb{R}^k)$). This turns $G_n(\mathbb{R}^\infty)$ into a paracompact topological space. We see that $G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$.

Theorem 3.5. $B\,\mathrm{O}(n) = G_n(\mathbb{R}^\infty)$*, i.e. the infinite Grassmannian is the classifying space for* $\mathrm{O}(n)$*.*

That was the base space, now we construct the total space of the universal bundle. Define $E_n(\mathbb{R}^k) := \{(A, x) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k \mid x \in A\}$, i.e. E_n is the space that to each point $A \in G_n(\mathbb{R}^k)$ has the subspace of $\mathbb{R}^k$ which A represents. With the obvious projection map $\pi : E_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k)$ this becomes a rank

n vector bundle, called the *canonical vector bundle* over $G_n(\mathbb{R}^k)$. As before we define

$$E_n(\mathbb{R}^\infty) := \bigcup_{k=n}^{\infty} E_n(\mathbb{R}^k)$$

and equip it with the weak topology. Again there is a natural projection map $\pi : E_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty)$ which gives us a rank n vector bundle. Since $G_n(\mathbb{R}^\infty)$ was paracompact, this has structure group $\mathrm{O}(n)$.

Theorem 3.6. *The vector bundle $\pi : E_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty)$ is the universal bundle of $\mathrm{O}(n)$. Hence for every CW-complex X there is a bijection*

$$[X, G_n(\mathbb{R}^\infty)] \xrightarrow{\sim} \mathrm{Vect}^n_{\mathbb{R}}(X)$$

given by $[f] \longmapsto f^*(E_n(\mathbb{R}^\infty))$.

For complex vector bundles the situation is analogous. We assume the vector bundles to have structure group $\mathrm{U}(n)$. Define the complex Grassmannian $G_n(\mathbb{C}^k)$ (for $n \leq k$) to be the set of n-dimensional complex subspaces of $\mathbb{C}^k$ (in particular $G_1(\mathbb{C}^k) = \mathbb{CP}^k$) and put

$$G_n(\mathbb{C}^\infty) := \bigcup_{k=n}^{\infty} G_n(\mathbb{C}^k)$$

equipped with the weak topology (so that $G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$). Similarly define $E_n(\mathbb{C}^k) := \{(A, x) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k \mid x \in A\}$ and put

$$E_n(\mathbb{C}^\infty) := \bigcup_{k=n}^{\infty} E_n(\mathbb{C}^k)$$

again with the weak topology. Once again we have a natural projection map $\pi : E_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$ turning it into a complex vector bundle of rank n.

Theorem 3.7. *The vector bundle $\pi : E_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$ is the universal bundle for $\mathrm{U}(n)$, in particular $B\,\mathrm{U}(n) = G_n(\mathbb{C}^\infty)$. Hence for each CW-complex X there is a bijection $[X, G_n(\mathbb{C}^\infty)] \longrightarrow \mathrm{Vect}^n_{\mathbb{C}}(X)$ by $[f] \longmapsto f^*(E_n(\mathbb{C}^\infty))$.*

To measure twistedness of the vector bundles we introduce systems of cohomology classes called characteristic classes. The cohomology ring of the real infinite Grassmannian with $\mathbb{Z}_2$ coefficients is

$$H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$$

i.e. a polynomial ring on n generators, where w_k is of dimension k. Similarly

$$H^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$$

where c_k is of dimension $2k$.

Definition 3.8. *Let E be a real vector bundle over X and let $f : X \longrightarrow G_n(\mathbb{R}^\infty)$ be the corresponding classifying map. The cohomology classes $w_k(E) := f^*(w_k) \in H^k(X;\mathbb{Z}_2)$, $k = 1,\dots,n$ are called the* Stiefel-Whitney classes *of E. The sum $w(E) = 1+w_1(E)+\cdots+w_n(E) \in H^*(X;\mathbb{Z}_2)$ is called the* total Stiefel-Whitney class. *If M is a real smooth manifold we define the Stiefel-Whitney classes of M to be the Stiefel-Whitney classes of its tangent bundle.*

If E is complex vector bundle and $f : X \longrightarrow G_n(\mathbb{C}^\infty)$ is its classifying map, then the n cohomology classes $c_k(E) := f^*(c_k) \in H^{2k}(X;\mathbb{Z})$ are called the Chern classes *of E. The sum $c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(X;\mathbb{Z})$ is called the* total Chern class.

In general when having a G-bundle $\pi : E \longrightarrow X$ corresponding to some characteristic map $f : X \longrightarrow BG$ then the pullback via f of some cohomology class in $H^*(BG)$ is called a *characteristic class* of the bundle.

The Stiefel-Whitney classes satisfies the following important properties

1) $w_k(E_1 \oplus E_2) = \sum_{j=0}^n w_j(E_1) \smile w_{k-j}(E_2)$ (where $w_0(E) := 1$) or equivalently $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$.

2) If $\pi : E \longrightarrow Y$ is a real vector bundle and $\varphi : X \longrightarrow Y$ is a continuous map then $w_k(\varphi^* E) = \varphi^*(w_k(E))$.

It is not hard to check the last assertion: Assume E to have classifying map $f : Y \longrightarrow G_n(\mathbb{R}^\infty)$ (where n is of course the rank of E). Then $\varphi^*(w_k(E)) = \varphi^*(f^* w_k)$ by definition of w_k. On the other hand $E = f^*(G_n(\mathbb{R}^\infty))$ and hence $\varphi^* E = \varphi^*(f^*(G_n(\mathbb{R}^\infty)))$, i.e. $\varphi^* E$ has classifying map $f \circ \varphi : X \longrightarrow G_n(\mathbb{R}^\infty)$. Therefore $w_k(\varphi^* E) = (f \circ \varphi)^* w_k = \varphi^*(f^* w_k)$, thus the two sides are equal.

In the complex case the situation is similar. The Chern classes satisfy the same properties

1) $c_k(E_1 \oplus E_2) = \sum_{j=0}^n c_j(E_1) \smile c_{k-j}(E_2)$ (again $c_0(E) := 1$) or equivalently $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$.

2) If $\pi : E \longrightarrow Y$ is a complex vector bundle and $\varphi : X \longrightarrow Y$ is a continuous map then $c_k(\varphi^* E) = \varphi^*(c_k(E))$.

If E is a direct sum of line bundles $L_1 \oplus \cdots \oplus L_n$, and $x_i := c_1(L_i)$, then property 2 gives

$$c(E) = \prod_{i=1}^n (1 + x_i) = 1 + \sigma_1(x_1,\dots,x_n) + \cdots + \sigma_n(x_1,\dots,x_n)$$

where σ_i is the i'th elementary symmetric polynomial in n variables. These are given by

$$\begin{aligned}
\sigma_1(x_1,\dots,x_n) &= x_1+\cdots+x_n \\
\sigma_2(x_1,\dots,x_n) &= \sum_{i_1<i_2} x_{i_1}x_{i_2} \\
\sigma_3(x_1,\dots,x_n) &= \sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}x_{i_3} \\
&\vdots \\
\sigma_n(x_1,\dots,x_n) &= x_1\cdots x_n
\end{aligned}$$

Thus we have $c_k(E) = \sigma_k(x_1,\dots,x_n)$ (still under the assumption that E is a sum of line bundles).

Example 3.9. Now we can calculate the Stiefel-Whitney classes in the simplest case possible: the trivial bundle. Since the Stiefel-Whitney classes are invariant under isomorphism we can just as well calculate them for a product bundle $X\times\mathbb{R}^n$. If $\{x_0\}\times\mathbb{R}^n$ denotes the product bundle over a one-point space then we know (or can easily see) that $X\times\mathbb{R}^n$ is just the pullback of $\{x_0\}\times\mathbb{R}^n$ along the trivial map $\varphi : X \longrightarrow \{x_0\}$. Thus the Stiefel-Whitney classes of $X\times\mathbb{R}^n$ are pullbacks of the Stiefel-Whitney classes of $\{x_0\}\times\mathbb{R}^n$. But as $H^k(\{x_0\};\mathbb{Z}_2) = 0$ for $k > 0$ these are just 0, so a trivial bundle has trivial Stiefel-Whitney classes.

So now we have one necessary condition for triviality of a real vector bundle: the Stiefel-Whitney classes have to be trivial. If not, the bundle is non-trivial. A similar calculation gives the same result for Chern classes. The next example shows that this condition is however not sufficient

Example 3.10. We will calculate the Stiefel-Whitney classes of the tangent bundle of the spheres. The key observation is that if we form the direct sum of TS^n with the normal bundle NS^n (which is a trivial line bundle) we obtain the product bundle $S^n\times\mathbb{R}^{n+1}$. Property 2 of Stiefel-Whitney classes gives

$$1 = w(TS^n\oplus NS^n) = w(TS^n)\smile w(NS^n) = w(TS^n)\smile 1 = w(TS^n).$$

Thus the Stiefel-Whitney classes of TS^n are all zero. However, it is a highly non-trivial fact proved by Adams that among the spheres only S^1, S^3 and S^7 have trivial tangent bundles.

We can elaborate a little more on this argument. If we by I_n denote the trivial rank n bundle over X, we say that two vector bundles E and F over X are *stably equivalent* if there exists natural numbers n, m such that $E\oplus I_n$ is isomorphic to $F\oplus I_m$. We claim that stably equivalent vector bundles have the same characteristic classes (we assume the vector bundles to be complex and we do the calculation for Chern classes):

$$c(E) = c(E)\smile 1 = c(E)\smile c(I_n) = c(E\oplus I_n) = c(F\oplus I_m) = c(F).$$

Hence Stiefel-Whitney and Chern classes are invariant under stable equivalence.

4 K-Theory

In this section we define two (contravariant) functors from the category of compact Hausdorff spaces to the category of abelian group known as the K-functors. Before we start off we need the following result which makes it all work out and which explains the restriction to compact spaces.

Proposition 4.1. *Let X be compact Hausdorff and E a vector bundle over X. Then there exists a vector bundle E' over X as well such that $E \oplus E'$ is the trivial vector bundle.*

Henceforth all spaces will be compact.

We consider $\mathrm{Vect}_{\mathbb{C}}(X)$. With the operation $\oplus$ this is an abelian semigroup (recall that $E \oplus F \cong F \oplus E$). By the Grothendieck construction this can be completed to an abelian group and this group is called $K(X)$. To be a little more specific, consider $\mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X)$ and define an equivalence relation $\sim$ on this product by $(E, F) \sim (G, H)$ iff there exists a $J \in \mathrm{Vect}_{\mathbb{C}}(X)$ such that $E \oplus H \oplus J \cong F \oplus G \oplus J$. Then define $K(X) = (\mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X))/\sim$ [1]. By $E \longmapsto [E]$ we denote the Grothendieck map $\mathrm{Vect}_{\mathbb{C}}(X) \longrightarrow K(X)$. Furthermore it is a standard property of the Grothendieck group that it is of the form $K(X) = \{[E] - [F] \mid E, F \in \mathrm{Vect}_{\mathbb{C}}(X)\}$ and that the composition is given by

$$([E] - [F]) + ([G] - [H]) = [E \oplus G] - [F \oplus H].$$

We can sharpen the first observation a bit: Let $[E] - [F]$ be an element of $K(X)$, then there exist H and I_k such that $[E] - [F] = [H] - [I_k]$. To see this use Proposition 4.1 to get a vector bundle F' such that $F \oplus F' \cong I_k$. Put $H = E \oplus F'$, then

$$[E] - [F] = ([E] + [F']) - ([F] + [F']) = [H] - [I_k].$$

If $f : X \longrightarrow Y$ is a continuous map, then we get a map $f^* : K(Y) \longrightarrow K(X)$ simply by pulling back vector bundles

$$f^*([E] - [F]) = [f^*E] - [f^*F].$$

This is well-defined and satisfies $(f \circ g)^* = g^* \circ f^*$, and since $f^*(E \oplus F) \cong f^*(E) \oplus f^*(F)$, it is a group homomorphism. Hence K is a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups. Observe, that if $f : X \longrightarrow Y$ is a homotopy equivalence with homotopy inverse $g : Y \longrightarrow X$, we get $f^* \circ g^* = \mathrm{id}_{K(Y)}$ and $g^* \circ f^* = \mathrm{id}_{K(X)}$ and hence $K(X) \cong K(Y)$. Thus the K-functor can only detect topological properties up to homotopy.

Observe that all we have done so far could equally well have been done for real vector bundles or, for that matter, quaternionic vector bundles. The corresponding real K-group is denoted $KO(X)$ and the quaternionic K-group $KSp(X)$ (in

[1] A good account of the Grothendieck construction can be found in N. J. Laustsen, F. Larsen and M. Rørdam: An Introduction to K-Theory for C*-Algebras.

fact complex K-theory is sometimes denoted $KU(X)$ but as we will be interested in complex K-theory only this is superfluous). The difference will become visible when discussing Bott-periodicity, where if fact complex K-theory is simpler than real K-theory.

For those familiar with K-theory of C*-algebras one can show (due to a theorem by Swan) the following

Theorem 4.2. *Let X be a compact Hausdorff space. Then there is an isomorphism of abelian groups $K(X) \xrightarrow{\sim} K_0(C(X))$.*

Example 4.3. Let's calculate the K-group of a one-point space $\{x_0\}$. As the only vector bundles over $\{x_0\}$ are product bundles, and since these can only be distinguished by their dimension we have a natural identification $\mathrm{Vect}_{\mathbb{C}}(\{x_0\}) = \mathbb{N}_0$. If we Grothendieck this semigroup we get $\mathbb{Z}$, hence $K(\{x_0\}) \cong \mathbb{Z}$. Since K depends only on the homotopy type we have for any compact contractible space X that $K(X) \cong \mathbb{Z}$.

Now we outline the definition of the second K-functor. Recall that two vector bundles E and F are called stably equivalent, noted $E \sim_s F$ if there exist $m, n \in \mathbb{N}_0$ such that $E \oplus I_m \cong F \oplus I_n$. This is an equivalence relation in $\mathrm{Vect}_{\mathbb{C}}(X)$ and we denote the stable equivalence class of E by $[E]_s$. Define the *reduced K-group* of X to be

$$\widetilde{K}(X) = \mathrm{Vect}_{\mathbb{C}}(X)/\sim_s .$$

This is indeed a group. It is easy to see that $[E]_s + [F]_s = [E \oplus F]_s$ is a well-defined composition on $\widetilde{K}(X)$ making it an abelian semigroup with $[I_k]_s$ as zero element (for any k). We only need to cook up inverses. But thanks to Proposition 4.1 there exists to $[E]_s$ a vector bundle E' such that $E \oplus E' \cong I_n$ and hence

$$[E]_s + [E']_s = [E \oplus E']_s = [I_n]_s = 0.$$

Again we see the necessity of the space X being compact. If $f : X \longrightarrow Y$ is a continuous map, we define a map $\widetilde{f}^* : \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$ by $\widetilde{f}^*([E]_s) = [f^*E]_s$. One can check that this is well-defined and satisfies $\widetilde{(f \circ g)}^* = \widetilde{g}^* \circ \widetilde{f}^*$ and hence that $\widetilde{K}$ too is a contravariant functor from topological spaces to abelian groups.

We will show that $\widetilde{K}(X)$ is isomorphic to a subgroup of $K(X)$. To this end introduce the *virtual dimension* of an element $[E] - [F] \in K(X)$ to be the difference $\operatorname{rank} E - \operatorname{rank} F$. It is not hard to check that this is indeed a well-defined integer. The set of elements in $K(X)$ of virtual dimension 0 form a subgroup of $K(X)$.

Proposition 4.4. *The map from $\widetilde{K}(X)$ to the subgroup in $K(X)$ of elements of virtual dimension 0 given by $[E]_s \longmapsto [E] - [I_k]$ (where k is the rank of E) is an isomorphism. Furthermore*

$$K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}.$$

PROOF. Let's first check that it is well-defined. If $[E]_s = [F]_s$ there exist m and n such that $E \oplus I_m \cong F \oplus I_n$ and hence

$$[E \oplus I_m] - [I_{m+n}] = [F \oplus I_n] - [I_{m+n}].$$

But the LHS equals $[E] - [I_n]$ and the RHS equals $[F] - [I_m]$, thus the map is well-defined.

The map is obviously a group homomorphism. It is surjective for any element in $K(X)$ can be written on the form $[E] - [I_n]$, hence any element of virtual dimension 0 is of the form $[E] - [I_k]$ where k is the rank of E.

To show injectivity let E and F be bundles of rank m and n respectively such that $[E] - [I_m] = [F] - [I_n]$. This implies that there exists a bundle K such that

$$E \oplus I_n \oplus K \cong F \oplus I_m \oplus K \tag{4.1}$$

By Proposition 4.1 there is a K' such that $K \oplus K' = I_k$. Hence by adding K' to (4.1) we obtain $E \oplus I_{n+k} \cong F \oplus I_{m+k}$, i.e. $[E]_s = [F]_s$.

To show the second isomorphism we consider the following split exact sequence

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \underset{c^*}{\overset{\iota^*}{\rightleftarrows}} K(\{x_0\}) \longrightarrow 0$$

where the first map is inclusion (injective), and where ι^* and c^* are induced by the maps $\iota : \{x_0\} \longrightarrow X$, $\iota(x_0) = x_0$ and $c : X \longrightarrow \{x_0\}$. We clearly have $c \circ \iota = \mathrm{id}_{\{x_0\}}$ and hence $\iota^* \circ c^* = \mathrm{id}_{K(\{x_0\})}$. Examining the definition of pullback of a vector bundle one sees that the pullback $\iota^*(E)$ is simple the bundle over $\{x_0\}$ which has fiber E_{x_0}. From this it follows that ι^* is surjective. We just need to see that $\ker \iota^* = \widetilde{K}(X)$. For a general element of $K(X)$ we have

$$\iota^*([E] - [F]) = [\iota^* E] - [\iota^* F] = [\{x_0\} \times E_{x_0}] - [\{x_0\} \times F_{x_0}]$$

If this is the zero element we must have $\operatorname{rank} E = \operatorname{rank} F$, and hence $[E] - [F]$ has virtual dimension 0, i.e. is an element of $\widetilde{K}(X)$. Therefore the short exact sequence splits and we have an isomorphism

$$K(X) \cong \widetilde{K}(X) \oplus K(\{x_0\}) \cong \widetilde{K}(X) \oplus \mathbb{Z}. \qquad \square$$

Reduced K-groups are also defined in the real or quaternionic case and are denoted $\widetilde{KO}(X)$ and $\widetilde{KSp}(X)$ respectively.

Example 4.5. Let X be a one-point space, or generally, a compact contractible space. We previously showed that $K(X) \cong \mathbb{Z}$. By the isomorphism $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ from Proposition 4.4 we see that $\widetilde{K}(X) = 0$.

Even though there is a close connection between the K-theories described by vector bundles and C*-algebras, the vector bundle picture does feature some extra structure not shared by its operator algebraic analog, namely a product. Not surprisingly the product comes from tensor product of vector bundles:

$$([E] - [F])([G] - [H]) = ([E \otimes G] + [F \otimes H]) - ([E \otimes H] + [F \otimes G])$$

This product is well-defined and turns $K(X)$ into a commutative ring with unit $[I_1]$. If we have a map $f : X \longrightarrow Y$ one can show that $f^*(E \otimes F) \cong f^*(E) \otimes f^*(F)$ and thus that $f^* : K(Y) \longrightarrow K(X)$ is indeed a ring homomorphism. Since $\widetilde{K}(X)$ equals the kernel of the ring homomorphism ι^*, it is an ideal in $K(X)$ and hence a ring in its own right, albeit not necessarily a unital one.

An additional feature exhibited by K is that of an *external product* (compare to the cross product in ordinary cohomology), a map $\mu : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$ (the tensor product being tensor product of rings with multiplication given by $(a \otimes b)(c \otimes d) = ac \otimes bd$) defined by

$$\mu(a \otimes b) = p_1^*(a)p_2^*(b)$$

where $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are the projection maps. As

$$\begin{aligned}\mu((a \otimes b)(c \otimes d)) &= \mu(ac \otimes bd) = p_1^*(ac)p_2^*(bd) = p_1^*(a)p_1^*(c)p_2^*(b)p_2^*(d) \\ &= p_1^*(a)p_2^*(b)p_1^*(c)p_2^*(d) = \mu(a \otimes b)\mu(c \otimes d),\end{aligned}$$

μ is a ring homomorphism.

Lemma 4.6. *The external product map $\mu : K(X) \otimes K(S^2) \longrightarrow K(X \times S^2)$ is a ring isomorphism.*

This leads immediately to the celebrated *Bott Periodicity Theorem*

Theorem 4.7. *There is an isomorphism $\widetilde{K}(S^2X) \longrightarrow \widetilde{K}(X)$* [2]*.*

This is where real and complex K-theory real go separate ways. For real K-theory Bott periodicity would read

$$\widetilde{KO}(S^8X) \xrightarrow{\sim} \widetilde{KO}(X).$$

Finally a curious fact which relates K-theory to operator theory and index theory.

Definition 4.8. *Let H be a Hilbert space. An operator $T \in B(H)$ is called a* Fredholm operator *if it has closed image and if $\ker T$ and $\ker T^*$ are finite dimensional. The space of Fredholm operators on H is denoted $\mathscr{F}(H)$. The* index *of a Fredholm operator is the integer*

$$\operatorname{index}(T) = \dim\ker T - \dim\ker T^*.$$

As $\mathscr{F}(H)$ is a topological space, we can form $[X, \mathscr{F}(H)]$ for any compact Hausdorff space X. It is in fact a group where the composition originates from pointwise composition: $f \circ g(T) = f(T) \circ g(T)$. If H is a complex infinite-dimensional separable Hilbert space, one can show that there is a group isomorphism

$$\operatorname{index} : [X, \mathscr{F}(H)] \xrightarrow{\sim} K(X).$$

[2] S^2X is the suspension of the suspension of X, the suspension of a space being the quotient $X \times I/\sim$ where $\sim$ is the relation that collapses $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. Since $X \times I$ compact and SX is the image of $X \times I$ under the continuous quotient map, SX is itself compact.

In this sense $\mathscr{F}(H)$ is a classifying space for K-theory. The reason for the name index is the following: If X is just a point then $[X, \mathscr{F}(H)]$ equals the set of path components of $\mathscr{F}(H)$. In this case the map $[X, \mathscr{F}(H)] \longrightarrow K(X) \cong \mathbb{Z}$ is really the index-map which maps path components of Fredholm operators to their common index. Thus, in particular, two Fredholm operators which can be connected by a path in $\mathscr{F}(H)$ have the same index.

5 The Chern Character

The total Chern class of a vector bundle exhibits a strange behavior since it sends direct sums to products (property 1). We would like a map defined on vector bundles which behaves a little nicer, i.e. sends direct sums to sums and tensor products to products. This is achieved by the introduction of the Chern character.

We begin by defining the Chern character on complex vector bundles. This we will do in three steps: first we define it on line bundles (i.e. one-dimensional vector bundles) then on a direct sum of line bundles and finally, thanks to the so-called splitting principle, on an arbitrary vector bundle. Later we will extend the definition to K-theory.

Let X be a topological space for which there is an N such that $H^n(X;\mathbb{Z}) = 0$ for $n > N$. This is for example the case if X is a manifold or a CW complex having cells only in dimension less than N. Let L be a complex line bundle over X and define the *Chern character* $\mathrm{ch}(L)$ on L by

$$\mathrm{ch}(L) = \exp(c_1(L)) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \cdots + \frac{1}{n!}c_1(L)^n + \cdots$$

Since $c_1(L)^k \in H^{2k}(X;\mathbb{Z})$ this series terminates after finitely many terms by the property of X, and is thus well-defined. Since we have no longer integer coefficients $\mathrm{ch}(L)$ is no longer an element of the integer cohomology but rather of the rational cohomology. This is the price we have to pay for the nice properties of the Chern character.

For a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles we simply define the Chern character to be the sum of the Chern character of the line bundles

$$\begin{aligned}\mathrm{ch}(L_1 \oplus \cdots \oplus L_n) &= \sum_{i=1}^n \mathrm{ch}(L_i) = \sum_{i=1}^n \exp(c_1(L_i)) \\ &= n + (x_1 + \cdots + x_n) + \cdots + (x_1^k + \cdots + x_n^k)/k! + \cdots \qquad (5.1)\end{aligned}$$

(again we have put $x_i = c_1(L_i)$). The polynomial $x_1^k + \cdots + x_n^k$ is a symmetric polynomial of degree k. The main result of symmetric polynomials states that any symmetric polynomial p of degree k in the variables $x_1, \ldots, x_n$ can be written as a polynomial $s(\sigma_1, \ldots, \sigma_k)$ in the elementary symmetric polynomials introduced above. In particular the symmetric polynomial $x_1^k + \cdots + x_n^k$ can be written as a polynomial $s_k(\sigma_1, \ldots, \sigma_k)$. These particular polynomials s_k are called the *Newton polynomials*. The first Newton polynomials are

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_1^2 - 2\sigma_2 \\ s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ s_4 &= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4. \end{aligned}$$

In general one has the following recursion formula

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^{k-2}\sigma_{k-1}s_1 + (-1)^{k-1}k\sigma_k.$$

Substituting these expressions into (5.1) and recalling $\sigma_k(x_1, \dots, x_n) = c_k(E)$ we get

$$\mathrm{ch}(E) = n + \sum_{k=1}^{n} s_k(c_1(E), \dots, c_n(E)). \tag{5.2}$$

This formula is derived under the assumption that E is a direct sum of line bundles. But the right hand side is expressed in terms of E only. Therefore we can take (5.2) to be the definition of the Chern character in the general case.

As promised the Chern character has nice properties

1) $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$,

2) $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \smile \mathrm{ch}(F)$.

Now we can immediately extend the definition of the Chern character to K-theory: Simply define $\mathrm{ch} : K(X) \longrightarrow H^*(X; \mathbb{Q})$ (or, as the Chern classes are even dimensional, $\mathrm{ch} : K(X) \longrightarrow H^{\mathrm{ev}}(X; \mathbb{Q})$) by

$$\mathrm{ch}([E] - [F]) = \mathrm{ch}(E) - \mathrm{ch}(F).$$

The properties 1) and 2) above ensure that this is in fact a ring homomorphism. The Chern character provides an important link between K-theory and singular cohomology.