Sketch of solutions to the exam

Stochastic processes, June 2009

Problem 1

1. Transition matrix:

$$\mathbf{P} = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & \cdot & \cdot & 1/5 & 1/10 & \cdot & 7/10 & \cdot \\ 2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 5 & 4/5 & 1/5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & 1/4 & \cdot & \cdot & 3/4 & \cdot & \cdot \\ 7 & \cdot & \cdot & 3/4 & 1/4 & \cdot & \cdot & \cdot \end{array}$$

(elements in the transition matrix that are 0 have been represented by a dot).

Communication classes:

$$\begin{aligned} C_1 &= \{1,5,6\} && (1 \to 6 \to 5 \to 1) \\ C_2 &= \{2,4\} && (2 \to 4 \to 2) \\ C_3 &= \{3\} && (\text{a closed set: } p_{33} = 1) \\ C_4 &= \{7\} && (\text{the chain cannot enter state 7}) \end{aligned}$$

Period:

$$\begin{aligned} d_1 = d_5 = d_6 &= 3 && \text{since } p_{11}(3k) > 0 \text{ but } p_{11}(3k+1) = p_{11}(3k+2) = 0 \\ d_2 = d_4 &= 2 && \text{since } p_{22}(2k) > 0 \text{ but } p_{22}(2k+1) = 0 \\ d_3 &= 1 && \text{since } p_{33} > 0 \\ d_7 &= \infty && \text{since } p_{77}(n) = 0 \text{ for all } n \in \mathbb{N} \end{aligned}$$

2. Classification

   C_1 is transient as $1 \to 4$ but $4 \not\to 1$ (since C_2 is closed)

   C_2 and C_3 are closed and finite and therefore positive recurrent

   C_4 is transient as $7 \to 4$ but $4 \not\to 7$ (since C_2 is closed)

3. Starting in state 7, in the first step the chain will be absorped in C_3 with probability $3/4$ and in C_2 with probability $1/4$. If it is absorped in C_2 it will be in state 4 for all odd n, and it will be in state 2 for all even n. Thus

$$P(X_n = 2 \,|\, X_0 = 7) \quad = \quad \begin{cases} 0 & \text{for } n \text{ odd} \\ 1/4 & \text{for } n \text{ even} \end{cases}$$

4. By first-step analysis

$$\begin{aligned}
u(1) &= \frac{1}{5}u(3) + \frac{1}{10}u(4) + \frac{7}{10}u(6) \\
u(2) &= 0 \\
u(3) &= 1 \\
u(4) &= 0 \\
u(5) &= \frac{4}{5}u(1) + \frac{1}{5}u(2) \\
u(6) &= \frac{1}{4}u(2) + \frac{3}{4}u(5) \\
u(7) &= \frac{3}{4}u(3) + \frac{1}{4}u(4)
\end{aligned}$$

with solution

$$u(1) = \frac{10}{29}\,;\; u(2) = 0\,;\; u(3) = 1\,;\; u(4) = 0\,;\; u(5) = \frac{8}{29}\,;\; u(6) = \frac{6}{29}\,;\; u(7) = \frac{3}{4}$$

Problem 2

1. Let Z_n be 1 if the n'th fish is untagged and 0 otherwise. Then

$$\begin{aligned}
P(Z_{n+1} = 0 \,|\, X_0, \ldots, X_n, Z_1, \ldots, Z_n) &= \frac{X_n + 1}{N} \\
P(Z_{n+1} = 1 \,|\, X_0, \ldots, X_n, Z_1, \ldots, Z_n) &= \frac{N - X_n - 1}{N}
\end{aligned}$$

Then Z_{n+1} is conditionally independent of $Z_n, \ldots, Z_1, X_{n-1}, \ldots, X_0$ given X_n and

$$X_{n+1} = (X_n + 1)Z_{n+1}.$$

Thus, $\{X_n\}_{n\geq 0}$ is a HMC by Thm 2.2.2.

Alternatively, order all N fish such that the tagged fish come first in the sequence. Let Z_n be the number of the drawn fish, which are iid uniformly distributed on $\{1, \ldots, N\}$. Then

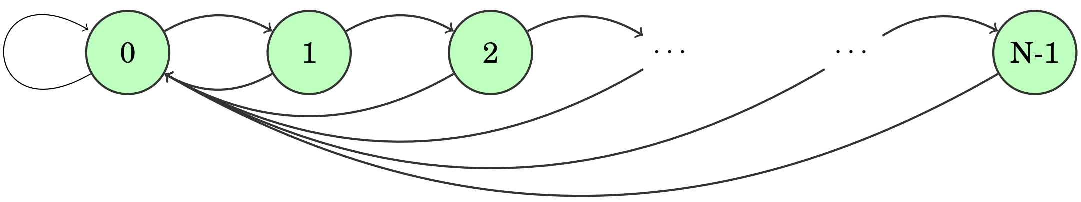
$$X_{n+1} = (X_n + 1)1_{\{Z_{n+1} > X_n + 1\}}.$$

and $\{X_n\}_{n\geq 0}$ is a HMC by Thm 2.2.1.

It has transition matrix

$$\mathbf{P} = \begin{array}{c|cccccc|}
 & 0 & 1 & 2 & \cdots & N-2 & N-1 \\
\hline
0 & 1/N & (N-1)/N & \cdot & \cdots & \cdot & \cdot \\
1 & 2/N & \cdot & (N-2)/N & \cdots & \cdot & \cdot \\
2 & 3/N & \cdot & \cdot & \cdots & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdot \\
N-2 & (N-1)/N & \cdot & \cdot & \cdots & \cdot & 1/N \\
N-1 & 1 & \cdot & \cdot & \cdots & \cdot & \cdot
\end{array}$$

and transition graph



All states communicate, thus it is irreducible. Since it is finite, it is positive recurrent. All states are aperiodic since it is irreducible (all states have the same period) and $d_0 = 1$ because $p_{00} > 0$.

2. If $T = n$ then necessarily the chain has followed the path

$$0 \to 1 \to 2 \to \cdots \to n-1 \to 0$$

   and thus

$$P(T = n \,|\, X_0 = 0) \;=\; p_{01}p_{12}\cdots p_{n-2,n-1}p_{n-1,0} \;=\; \frac{N-1}{N}\frac{N-2}{N}\cdots\frac{N-n+1}{N}\frac{n}{N}$$

3. For $N = 3$ we get

$$E[T \,|\, X_0 = 0] \;=\; \sum_{n=1}^{3} nP(T = n \,|\, X_0 = 0) \;=\; \frac{1}{3} + 2\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\cdot 1 \;=\; \frac{17}{9}$$

   Alternatively, use first-step analysis:

$$\begin{aligned} m(0) &= 1 + \frac{2}{3}m(1) \\ m(1) &= 1 + \frac{1}{3}m(2) \\ m(2) &= 1 \end{aligned}$$

   We obtain

$$m(0) \;=\; 1 + \frac{2}{3}\left(1 + \frac{1}{3}m(2)\right) \;=\; 1 + \frac{2}{3}\left(1 + \frac{1}{3}\right) \;=\; \frac{9+6+2}{9} \;=\; \frac{17}{9}$$

4. Global balance equations:

$$\begin{aligned} \pi(0) &= \frac{1}{3}\pi(0) + \frac{2}{3}\pi(1) + \pi(2) \\ \pi(1) &= \frac{2}{3}\pi(0) \\ \pi(2) &= \frac{1}{3}\pi(1) \end{aligned}$$

   and $\pi(0) + \pi(1) + \pi(2) = 1$. We obtain $\pi = \frac{1}{17}(9, 6, 2)$.

5.

$$\frac{1}{n}\sum_{j=0}^{n-1} X_n \quad \longrightarrow \quad \sum_{n=0}^{2} n\pi(n) \;=\; \frac{6}{17} + 2\frac{2}{17} \;=\; \frac{10}{17}$$

Problem 3

1. Reuters criterion for birth-and-death processes (Thm. 33): $(X(t))_{t\geq 0}$ is non-explosive since
$$\sum_{i=1}^{\infty}\left(\frac{1}{\beta_i}+\cdots+\frac{\delta_1\cdots\delta_i}{\beta_0\cdots\beta_i}\right) > \sum_{i=1}^{\infty}\frac{1}{\beta_i} = \sum_{i=1}^{\infty}\frac{\sqrt{i+1}}{\lambda} = \infty.$$

2. Since the clock of a uniform chain has a bounded jump rate, the jump rates of $(X(t))_{t\geq 0}$ given by its generator Q are bounded. But the generator uniquely determines the distribution of a HMC and in our case Q is unbounded ($q_{j,j-1} = \delta_j := \mu\cdot\sqrt{j}$). Hence, $(X(t))_{t\geq 0}$ is not uniformisable.

3. Kolmogorov backward equation for $(X(t))_{t\geq 0}$:
$$\begin{aligned} p'_{i,j}(t) &= -q_i p_{i,j}(t) + \sum_{k\neq i}^{\infty} q_{i,k}p_{k,j}(t) \\ &= q_{i,i}p_{i,j}(t) + q_{i,i-1}p_{i-1,j}(t) + q_{i,i+1}p_{i+1,j}(t) \\ &= \begin{cases} -\left(\frac{\lambda}{\sqrt{i+1}} + \mu\cdot\sqrt{i}\right)p_{i,j}(t) + \mu\cdot\sqrt{i}\cdot p_{i-1,j}(t) + \frac{\lambda}{\sqrt{i+1}}\cdot p_{i+1,j}(t) & \text{if } i>0 \\ -\lambda p_{0,j}(t) + \lambda\cdot p_{1,j}(t) & \text{if } i=0 \end{cases} \end{aligned}$$

4. Since the birth-and-death process is recurrent it is non-explosive and hence a regular jump Process. For this reason the assumptions of Thm. 29 are fulfilled and we only have to check if $\pi * Q = 0$. Let $i \geq 1$, then
$$\begin{aligned} (\pi * Q)_i &= \sum_{j=0}^{\infty}\pi_j q_{j,i} = \pi_{i-1}q_{i-1,i} + \pi_i q_{i,i} + \pi_{i+1}q_{i+1,i} + 0 \\ &= \beta_{i-1}\pi(0)\cdot\prod_{k=1}^{i-1}\frac{\beta_{k-1}}{\delta_k} - (\delta_i+\beta_i)\pi(0)\cdot\prod_{k=1}^{i}\frac{\beta_{k-1}}{\delta_k} + \delta_{i+1}\pi(0)\cdot\prod_{k=1}^{i+1}\frac{\beta_{k-1}}{\delta_k}. \end{aligned}$$
If we divide the last term by $\pi(0)\cdot\prod_{k=1}^{i-1}\frac{\beta_{k-1}}{\delta_k} > 0$ we get
$$\beta_{i-1} - (\delta_i+\beta_i)\frac{\beta_{i-1}}{\delta_i} + \delta_{i+1}\cdot\frac{\beta_{i-1}}{\delta_i}\cdot\frac{\beta_i}{\delta_{i+1}} = 0.$$
Hence $(\pi * Q)_i = 0$ for all $i \geq 1$. But we also have
$$(\pi * Q)_0 = \pi_0 q_{0,0} + \pi_1 q_{1,0} = -\pi(0)\beta_0 + \pi(0)\cdot\frac{\beta_0}{\delta_1}\cdot\delta_1 = 0.$$

5. Since the process is positive recurrent we know from Thm. 29 that the invariant measure π which has the expression from above is finite. Hence we choose $\pi(0) > 0$ such that $\sum_{i=0}^{\infty} \pi(i) = 1$. By definition, π is strictly positive. Since Q is the generator of a birth-and-death process, $q_{i,j} = 0 = \frac{\pi(j)}{\pi(i)} q_{j,i}$ for all $i \geq 0$ and $j > i+1$ or $j < i-1$. Moreover,
$$q_{i,i+1} = \beta_i = \frac{\pi(0) \prod_{k=1}^{i+1} \frac{\beta_{k-1}}{\delta_k}}{\pi(0) \prod_{k=1}^{i} \frac{\beta_{k-1}}{\delta_k}} \delta_{i+1} = \frac{\pi(i+1)}{\pi(i)} q_{i+1,i}.$$
Hence $q_{i,j} = \frac{\pi(j)}{\pi(i)} q_{j,i}$ for all $i, j \geq 0$ which shows the reversibility.

6. Evidently, $(X(t))_{t \geq 0}$ is irreducible and π is satisfying $\pi * Q = 0$ as computed above and given by
$$\pi(i) = \pi(0) \cdot \prod_{j=1}^{i} \frac{\frac{\lambda}{\sqrt{j}}}{\mu \cdot \sqrt{j}} = \pi(0) \prod_{j=1}^{i} \frac{\lambda/\mu}{j} = \pi(0) \frac{(\lambda/\mu)^i}{i!}.$$
Since $\sum_{i=0}^{\infty} \frac{(\lambda/\mu)^i}{i!} = \exp(\lambda/\mu) < \infty$ we can just put $\pi(0) = \exp(-\lambda/\mu)$. Hence
$$\pi(i) = \exp(-\lambda/\mu) \cdot \frac{(\lambda/\mu)^i}{i!}, \quad i \geq 0$$
is the Poisson distribution with parameter (λ/μ). Thus Thm. 30 yields that $(X(t))_{t \geq 0}$ is positive recurrent and Thm 29 shows that π is the unique invariant distribution of $(X(t))_{t \geq 0}$. (Alternatively, one could argue with the reversibility equation shown above and use Thm. 31.)