University of Copenhagen
Faculty of Science

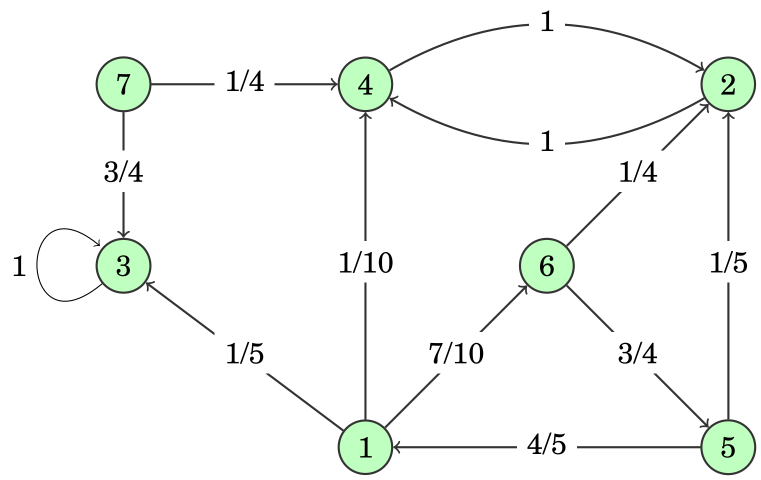
Stochastic processes, June 2009
3 hour written exam

You are allowed to use all kinds of books and notes. You may use any kind of writing utensils, including non-permanent ones such as pencils, and erasers. Computers and electronic calculators are allowed, though wireless modems and other means of communication must be disabled during the exam. You may write in Danish, in English or in a combination of Danish and English as you prefer.

This problem set contains three problems with 4+5+6 questions.

Problem 1

Let $(X_n)_{n\in\mathbb{N}_0}$ be a homogenous Markov chain on $E = \{1, 2, 3, 4, 5, 6, 7\}$ with transition graph



1. Write down the transition matrix and the communication classes. Find the period of each state.
2. Classify each communication class according to whether it is transient, positive recurrent or null recurrent.
3. Find the probability of being in state 2 at time n if the chain is started in state 7.
4. Let τ be the first time $(X_n)_{n\in\mathbb{N}_0}$ visits state 2, 3 or 4, i.e.

$$\tau = \inf\{n \geq 0 : X_n \in \{2, 3, 4\}\}.$$

Find the probability $u(i) = P(X_\tau = 3 \mid X_0 = i)$ for all $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

Problem 2

The following approach to estimate the size of a fish population in a small lake is proposed. At time $n = 0$ a fish is sampled at random, tagged and returned to the lake. Then at time $n = 1$ another fish is sampled at random; if it is not the tagged fish, it will be tagged and returned to the lake. This continues until the randomly sampled fish is a tagged one. In this case a new tag is applied and the entire procedure starts all over with the new tag, and the old tags are ignored. Let X_n be the number of untagged fish observed since last time a new tag was applied, and define $X_0 = 0$. Then $X_n = n$ for $n = 0, 1, 2, \ldots, T-1$, where $T = \min\{n \geq 1 : X_n = 0\}$ is the return time to state 0, and thus $X_T = 0$. Let N be the size of the population.

1. Show that $\{X_n\}_{n\geq 0}$ is a homogeneous Markov chain. Find its transition matrix, specify the communication classes, and classify each communication class according to whether it is transient, positive recurrent or null recurrent. Find the period of each state.
2. Find $P(T = n \mid X_0 = 0)$ for $n = 1, \ldots, N$.

Assume now and for the rest of the problem that $N = 3$. It can be used without proof that in this specific case the transition matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

3. Compute $E[T \mid X_0 = 0]$ when $N = 3$.
4. Find the invariant distribution when $N = 3$.
5. Compute the long-run average value when $N = 3$ of

$$\frac{1}{n} \sum_{j=0}^{n-1} X_n$$

Problem 3

Let $(X(t))_{t\geq 0}$ be a birth-and-death process describing the size of a population, where the birth and death rates are given by

$$\beta_j := \frac{\lambda}{\sqrt{j+1}}, \quad j \geq 0 \qquad \text{and} \qquad \delta_j := \mu \cdot \sqrt{j}, \quad j \geq 1,$$

where $\lambda, \mu > 0$. If the population size is small, there is enough food and space such that the birth rate is high and the death rate is small, vice versa if the population size is high.

1. Show that $(X(t))_{t\geq 0}$ is non-explosive.
2. Show that $(X(t))_{t\geq 0}$ is not uniformisable.
3. Write down the Kolmogorov backward equation for $(X(t))_{t\geq 0}$.
4. Let now in general $\beta_j > 0$, $j \geq 0$ be the birth rates and $\delta_j > 0$, $j \geq 1$ be the death rates of a birth-and-death process. Show that an invariant measure of an irreducible and recurrent birth-and-death process is given by

   $$\pi(i) = \pi(0) \cdot \prod_{j=1}^{i} \frac{\beta_{j-1}}{\delta_j}, \quad i \geq 0$$

   for some $\pi(0) > 0$.
5. Show that an irreducible and positive recurrent birth-and-death process is reversible.
6. Specify the candidate for the invariant distribution of the process $(X(t))_{t\geq 0}$ defined above and prove that $(X(t))_{t\geq 0}$ is positive recurrent.