Stochastic models

What & Why

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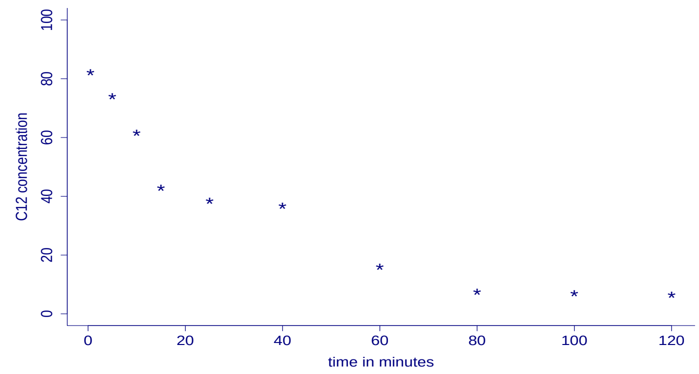
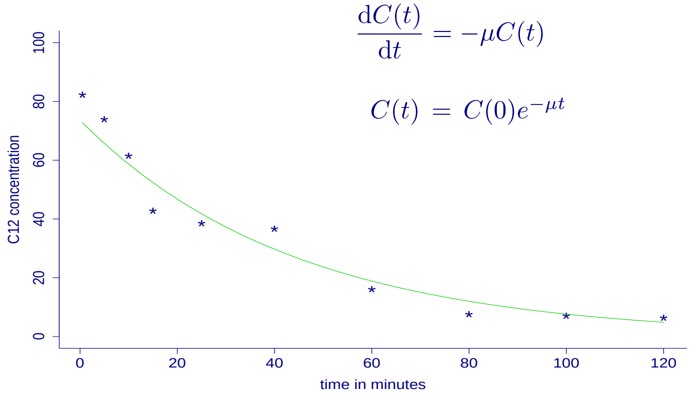
The concentration of a drug in blood



Exponential decay

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = -\mu C(t)$$

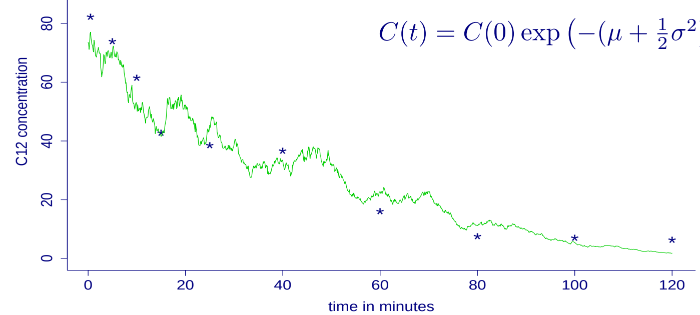
$$C(t) = C(0)e^{-\mu t}$$



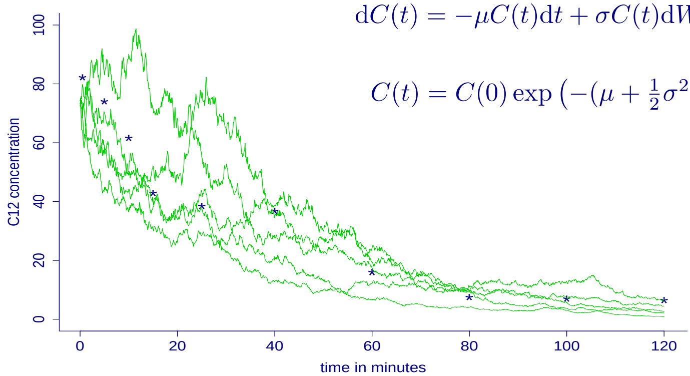
Exponential decay with noise

$$\mathrm{d}C(t) = -\mu C(t)\mathrm{d}t + \sigma C(t)\mathrm{d}W(t)$$

$$C(t) = C(0)\exp\left(-(\mu + \tfrac{1}{2}\sigma^2)t + \sigma W(t)\right)$$



Different realizations



$$dC(t) = -\mu C(t)dt + \sigma C(t)dW(t)$$

$$C(t) = C(0)\exp\left(-(\mu + \tfrac{1}{2}\sigma^2)t + \sigma W(t)\right)$$

Example: Population Dynamics

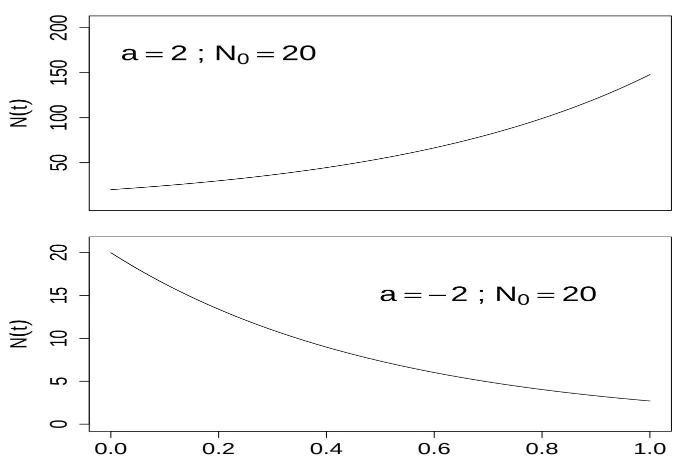
A simple population growth model:

$$\frac{dN(t)}{dt} \quad = \quad a(t)N(t) \;\; ; \;\; N(0) = N_0$$

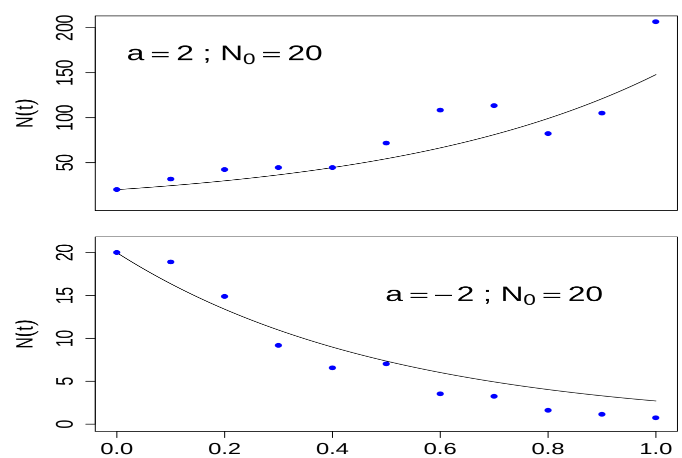
$N(t)$: size of population at time t
(e.g. size of a tumor or concentration of a drug in blood)

$a(t)$: relative rate of growth (or decay) at time t

If $a(t) = a$ is constant: $\quad N(t) = N_0\, e^{at}$



If $a(t) = a$ is constant: $\quad N(t) = N_0\, e^{at}$



Stochastic extension

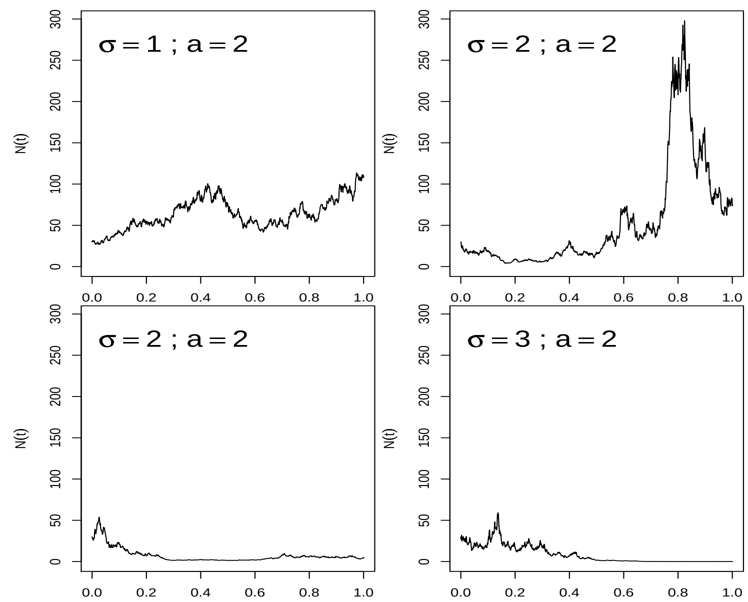
Maybe $a(t)$ is not completely known, but subject to some random environmental effects:

$$a(t) \quad \longrightarrow \quad a(t) + \text{"noise"}$$

E.g. "noise" $= \sigma W(t), W(t) =$ white noise, σ constant.

If $a(t) = a + \sigma W(t)$:

$$dN(t) \quad = \quad aN(t)dt + \sigma N(t) dW(t)$$



$$\begin{aligned} \frac{dN(t)}{dt} &= a(t)N(t) \\ &= (a + \sigma W(t))N(t) \\ &= aN(t) + \sigma N(t)W(t) \end{aligned}$$

We can write

$$N(t) \quad = \quad N_0 + \int_0^t aN(s)ds + \underbrace{\int_0^t \sigma N(s) dW(s)}_{??}$$

Randomization of parameters

Random variation in a parameter a:

$$a \quad \longrightarrow \quad a + \sigma \cdot \text{ "noise"}$$

for a zero mean noise process, $\xi(t)$.

A stochastic process:

$$\frac{dX(t)}{dt} \quad = \quad b(t, X(t)) + \sigma(t, X(t)) \cdot \xi(t)$$

Natural requirements:

- $\xi(t_1)$ and $\xi(t_2)$ are independent for $t_1 \neq t_2$
- $\xi(t)$ is a stationary process
- $E[\xi(t)] = 0$ for all t

This leads us to a *white noise process*

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t)) \cdot \xi(t)$$

Let $0 = t_0 < t_1 < \cdots < t_n = t$. Discretization of above equation:

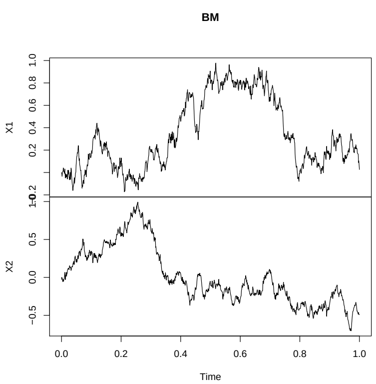
$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)\xi_k \Delta t_k$$

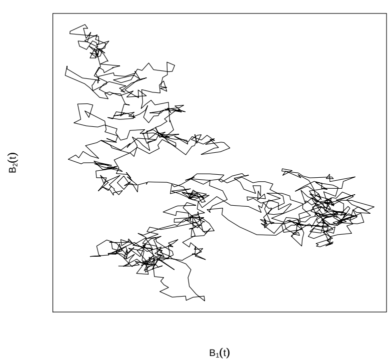
where $X_j = X(t_j), \;\; \xi_k = \xi(t_k), \;\; \Delta t_k = t_{k+1} - t_k$

Replace $\quad \xi_k \Delta t_k \quad$ by $\quad \Delta W_k = W(k+1) - W(k)$

where $W(t)$ is a suitable stochastic process.

$W(t)$ should have stationary, independent increments with mean 0. If we require $W(t)$ to be continuous it turns out that only one solution exists: **Brownian Motion** $B(t)$. Thus $W(t) = B(t)$.





Our discretized version becomes:

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j$$

Is there a limit when $\Delta t_j \longrightarrow 0$?

If so:

$$X(t) = X_0 + \int_0^t b(s, X(s))ds + \underbrace{\int_0^t \sigma(s, X(s))dB(s)}_{??}$$

Basic properties I

$B(t)$ is a Gaussian process:

For all $0 \leq t_1 \leq \cdots \leq t_k$ the random variable $Z = (B(t_1), \ldots, B(t_k))$ has a multinormal distribution, and

$E[B(t)] = B_0 \quad ; \quad \mathrm{Var}[B(t)] = t.$

Basic properties II

$B(t)$ has *independent increments*:

$B(t_1),\ B(t_2) - B(t_1),\ \ldots\ ,\ B(t_k) - B(t_{k-1})$

are independent for all $0 \leq t_1 \leq \cdots \leq t_k$.

Basic properties III

There exists a continuous version, so we simply assume that $B(t)$ is such a continuous version.

We also call it a *standard Wiener process*:

$$W \quad = \quad \{W(t)\}_{t \geq 0},$$

a Gaussian process with independent increments for which

$$W_0 = 0, \quad E[W(t)] = 0, \quad \mathrm{Var}[W(t) - W(s)] = t - s$$

for all $0 \leq s \leq t$.

It can be shown that <u>any</u> continuous time stochastic process with independent increments and finite second moments $E[X(t)^2]$ for all t, is a Gaussian process if $X(t_0)$ is Gaussian.

Construction of the Itô integral

We will define

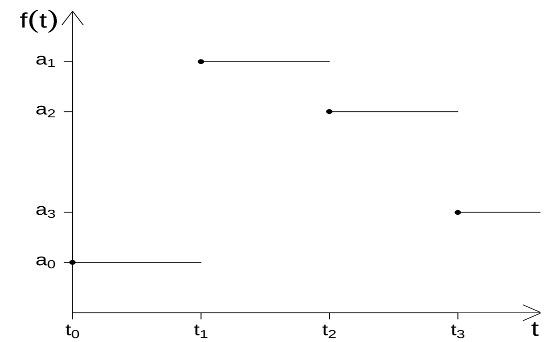
$$\int_0^T f(t)\,dW(t)$$

Let us try the usual tricks from ordinary calculus:

- define the integral for a simple class of functions
- extend by some approximation procedure to a larger class of functions

Assume f is a step-function of the form:

$$f(t) = \sum_{j\geq 0} a_j I_{\left\{\frac{j}{2^n}, \frac{(j+1)}{2^n}\right\}}(t) \quad \text{where} \quad I_{\{a,b\}}(t) = \begin{cases} 1 & \text{if } t \in [a,b) \\ 0 & \text{otherwise} \end{cases}$$



Then it will be natural to define

$$\int_0^T f(t)\,dW(t) = \sum_{j\geq 0} a_j [W(t_{j+1}) - W(t_j)]$$

where

$$t_j = \begin{cases} \frac{j}{2^n} & \text{if} \quad 0 \leq \frac{j}{2^n} \leq T \\ T & \text{if} \quad \frac{j}{2^n} > T \end{cases}$$

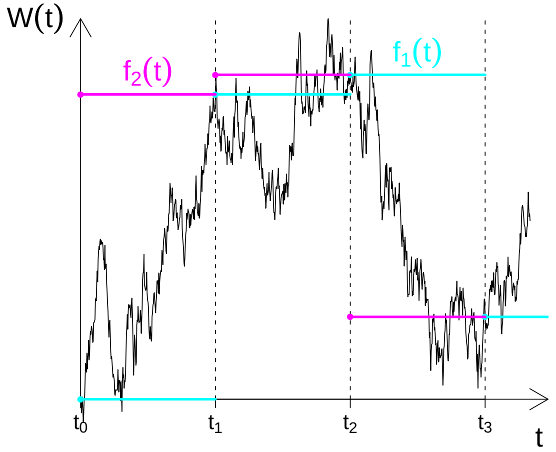
Problems!!!

Example: We want to calculate

$$\int_0^T W(t)\,dW(t)$$

Choose two different, but reasonable approximations:

$$\begin{aligned} f_1(t) &= \sum_{j\geq 0} W(t_j)\, I_{\left\{\frac{j}{2^n}, \frac{(j+1)}{2^n}\right\}}(t) \quad \text{(Left end point)} \\ f_2(t) &= \sum_{j\geq 0} W(t_{j+1})\, I_{\left\{\frac{j}{2^n}, \frac{(j+1)}{2^n}\right\}}(t) \quad \text{(Right end point)} \end{aligned}$$



Then

$$E\left[\int_0^T f_1(t)dW(t)\right] = \sum_{j\geq 0} E[W(t_j)(W(t_{j+1}) - W(t_j))]$$
$$= \sum_{j\geq 0} E[W(t_j)]E[(W(t_{j+1}) - W(t_j))]$$
$$= 0$$

since $W(t)$ has independent increments.

But

$$E\left[\int_0^T f_2(t)dW(t)\right] = \sum_{j\geq 0} E[W(t_{j+1})(W(t_{j+1}) - W(t_j))]$$
$$= \sum_{j\geq 0} E[W(t_{j+1})(W(t_{j+1}) - W(t_j))] - \sum_{j\geq 0} E[W(t_j)(W(t_{j+1}) - W(t_j))]$$
$$= \sum_{j\geq 0} E[(W(t_{j+1}) - W(t_j))^2]$$
$$= \sum_{j\geq 0}(t_{j+1} - t_j)$$
$$= T$$

The variations of the paths of $W(t)$ are too big to define the integral in the ordinary sense.

The problem is that a Wiener process $W(t)$ is nowhere differentiable.

Worse still: the sample paths have unbounded variation on any bounded time interval.

It is natural to approximate a given function $f(t)$ by a step-function of the form:

$$f(t) \quad \approx \quad \sum_{j\geq 0} f(t_j^*)\, I_{\{t_j, t_{j+1}\}}(t)$$

where the points t_j^* belong to the interval $[t_j, t_{j+1}]$.

Define

$$\int_S^T f(t)dW(t) \quad = \quad \lim_{n\to\infty} \sum_{j\geq 0} f(t_j^*)\, [W(t_{j+1}) - W(t_j)]$$

We just saw - unlike ordinary integrals - that

it makes a difference what t_j^* we choose!!!

Two useful and common choices:

- *The Itô integral*: $t_j^* = t_j$, the left end point.
- *The Stratonovich integral*: $t_j^* = (t_j + t_{j+1})/2$, the mid point.

Properties of the Itô integral

Let $0 \leq S < U < T$. Then

$$\begin{aligned}
\int_S^T f dW \quad &= \quad \int_S^U f dW + \int_U^T f dW \\
\int_S^T (cf+g)dW \quad &= \quad c\int_S^T f dW + \int_S^T g dW, c \text{ constant} \\
E[\int_S^T f dW] \quad &= \quad 0 \\
E\left[\left(\int_S^T f dW\right)^2\right] \quad &= \quad E\left[\int_S^T f^2 dt\right] \qquad \text{(The Itô isometry)}
\end{aligned}$$

Some names

We call a stochastic process $X(t)$ for:

An Itô integral if

$$X(t) = X_0 + \int_0^t \sigma(X(t))dW(s) \quad \text{or} \quad dX(t) = \sigma dW(t)$$

An Itô process or a stochastic integral if

$$X(t) \quad = \quad X_0 + \underbrace{\int_0^t b(X(t))ds}_{\text{drift}} + \underbrace{\int_0^t \sigma(X(t))dW(s)}_{\text{diffusion}}$$

or

$$dX(t) \quad = \quad \underbrace{b(X(t))dt}_{\text{drift}} + \underbrace{\sigma(X(t))dW(t)}_{\text{diffusion}}$$

The Itô formula

Let $X(t)$ be an Itô process given by

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t)$$

Let $g(t, x)$ be twice continuously differentiable on $\mathbf{R}_+ \times \mathbf{R}$. Then

$$Y_t = g(t, X(t))$$

is again an Itô process, and

$$dY_t = \left\{ \frac{\partial g}{\partial t}(t, X(t)) + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2}(t, X(t)) \right\} dt + \frac{\partial g}{\partial x}(t, X(t))dX(t)$$

Example:

$$\text{Calculate} = \int_0^t W(s)dW(s)$$

Choose $X(t) = W(t)$ and $g(t, x) = \frac{1}{2}x^2$. Then

$$Y_t = g(t, W(t)) = \frac{1}{2}W(t)^2$$

Apply Itô's formula:

$$\begin{aligned} dY_t &= \left\{ \frac{\partial g}{\partial t}(t, X(t)) + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2}(t, X(t)) \right\} dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) \\ &= \left\{ 0 + \frac{1}{2} \right\} dt + W(t)dW(t) \end{aligned}$$

Hence

$$dY_t = d\left(\frac{1}{2}W(t)^2\right) = \frac{1}{2}dt + W(t)dW(t)$$

or

$$\frac{1}{2}W(t)^2 = \frac{1}{2}t + \int_0^t W(s)dW(s).$$

Finally

$$\int_0^t W(s)dW(s) = \frac{1}{2}W(t)^2 - \frac{1}{2}t.$$

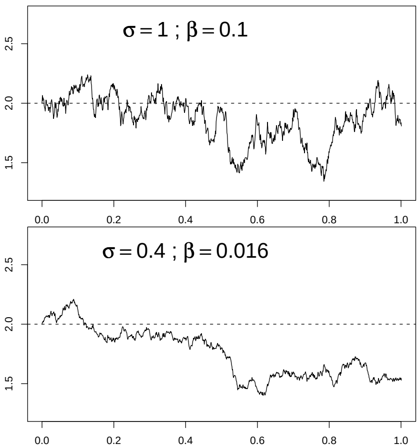
Example: the Ornstein-Uhlenbeck process

$$dX(t) = -\beta(X(t) - \alpha)dt + \sigma dW(t)$$

Solution:

$$X(t) = X_0 e^{-\beta t} + \alpha(1 - e^{-\beta t})\sigma \int_0^t e^{-\beta(t-s)}dW(s)$$

Parameter interpretation in the OU-process



β: how "strongly" the system reacts to perturbations (the "decay-rate" or "growth-rate")

Also:
$\tau = 1/\beta$ is the time constant of the system

α: the asymptotic mean

σ: the variation or the size of the noise

Example: population growth model

$$dN(t) = aN(t)dt + \sigma N(t)dW(t)$$

The Itô solution:

$$N(t) = N_0 \exp\left\{(a - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}$$

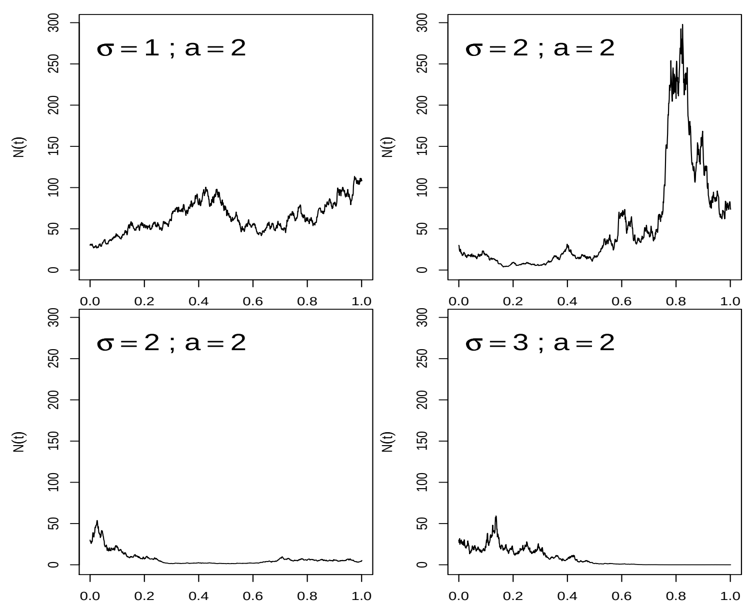
The Stratonovich solution:

$$N(t) = N_0 \exp\{at + \sigma W(t)\}$$

Qualitative behavior of the Itô solution

$$N(t) = N_0 \exp\left\{(a - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}$$

- If $a > \frac{1}{2}\sigma^2$ then $N(t) \to \infty$ when $t \to \infty$.
- If $a < \frac{1}{2}\sigma^2$ then $N(t) \to 0$ when $t \to \infty$.
- If $a = \frac{1}{2}\sigma^2$ then $N(t)$ will fluctuate between arbitrary large and arbitrary small values as $t \to \infty$.



Whereas for the Stratonovich solution we have

$$N(t) = N_0 \exp\{at + \sigma W(t)\}$$

- If $a > 0$ then $N(t) \to \infty$ when $t \to \infty$.
- If $a < 0$ then $N(t) \to 0$ when $t \to \infty$.

... just like in the deterministic case.

Numeric solutions

When no explicit solution is available we can approximate different characteristics of the process by simulation. (Realizations, moments, qualitative behavior etc)

- Different schemes (Euler, Milstein, higher order schemes...)
- Rate of convergence (Weak and strong)

Consider the Itô stochastic differential equation

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t)$$

and a time discretization

$$0 = t_0 < t_1 < \cdots < t_j < \cdots < t_N = T$$

Put

$$\Delta_j = t_{j+1} - t_j$$
$$\Delta W_j = W(t_{j+1}) - W(t_j)$$

Then

$$\Delta W_j \sim N(0, \Delta_j)$$

The Euler-Maruyama scheme

We approximate the process $X(t)$ given by

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t) \ ; \ X(0) = x_0$$

at the discrete time-points $t_j, 1 \leq j \leq N$ by

$$Y_{t_{j+1}} = Y_{t_j} + a(Y_{t_j})\Delta_j + b(Y_{t_j})\Delta W_j \ ; \ Y_{t_0} = x_0$$

where $\Delta W_j = \sqrt{\Delta_j} \cdot Z_j$, with $Z_j \sim N(0,1)$ for all j.

The Euler-Maruyama scheme

Let us consider the expectation of the absolute error at the final time instant T:

There exist constants $K > 0$ and $\delta_0 > 0$ such that

$$E(|X_T - Y_{t_N}|) \quad \leq \quad K\delta^{0.5}$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$.

We say that the approximating process Y *converges in the strong sense* with order 0.5.

(Compare with the Euler scheme for an ODE which has order 1).

The Euler-Maruyama scheme

Sometimes we do not need a close *pathwise* approximation, but only some function of the value at a given final time T (e.g. $E(X_T)$, $E(X_T^2)$ or generally $E(g(X_T))$):

There exist constants $K > 0$ and $\delta_0 > 0$ such that for any polynomial g

$$|E(g(X_T) - E(g(Y_{t_N}))| \quad \leq \quad K\delta$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$.

We say that the approximating process Y *converges in the weak sense* with order 1.

The Milstein scheme

We can even do better!

We approximate $X(t)$ by

$$\begin{aligned} Y_{t_{j+1}} &= Y_{t_j} + a(Y_{t_j})\Delta_j + b(Y_{t_j})\Delta W_j \\ &\quad + \frac{1}{2} b(Y_{t_j}) b'(Y_{t_j}) \{(\Delta W_j)^2 - \Delta_j\} \quad \text{(now Milstein...)} \end{aligned}$$

where the prime $'$ denotes the derivative.

The Milstein scheme

The Milstein scheme converges in the strong sense with order 1:

$$E(|X_T - Y_{t_N}|) \quad \leq \quad K\delta$$

We could regard the Milstein scheme as the proper generalization of the deterministic Euler-scheme.

If $b(X(t))$ does not depend on $X(t)$ the Euler-Maruyama and the Milstein scheme coincide.