Exercises

The phenomenon of switch-like response is involved in various signaling pathways in living systems. One way of modeling this could be by *stochastic resonance*. Consider the equation of motion of $X(t) = X_t$,

$$dX_t = f(X_t)\,dt + \sigma(X_t)dW_t$$

where the drift term can be expressed in terms of a potential $U(x)$ by $f(x) = -\frac{dU(x)}{dx}$, and $W(t) = W_t$ is Brownian motion. Consider $U(x)$ given by the double well potential

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2}.$$

This leads to the stochastic differential equation

$$dX_t = X_t(1 - X_t^2)\,dt + \sigma(X_t)dW_t\,;\; X_0 = x_0.$$

If $\sigma(X_t) = 0$, it is an ordinary differential equation with solution

$$X_t = \text{sign}(x_0)\,e^t\,(x_0^{-2} + e^{2t} - 1)^{-1/2}$$

In the exercises use a step size of either 0.1 or 0.01.

1. Using the exact solution, plot trajectories of X_t for $\sigma(X_t) = 0$ and different initial conditions: $x_0 = 2$, $x_0 = 0.2$, $x_0 = -0.2$ and $x_0 = -2$. Try for $0 \leq t \leq 5$.
2. Why is it called a *double well potential*? Plot the potential for $-2 < x < 2$.
3. Set $\sigma(X_t) = 0.5$. Simulate trajectories using the Euler-Maruyama scheme for $0 \leq t \leq 500$. Try also with $\sigma(X_t) = 0.1$ and $\sigma(X_t) = 1$. Are the solutions corresponding to different noise levels qualitatively different? If so, why?
4. Set $\sigma(X_t) = 0.5\sqrt{(1 + X_t^2)}$. Simulate trajectories using both the Euler-Maruyama scheme and the Milstein scheme. Compare the trajectories by setting the seed of the random number generator (`randn('state',seed)`, where `seed` is the same number for both schemes), and plot the trajectories on top of one another with two different colors. Try with a step size of both 0.1 and 0.01.
5. Repeat exercise 1 for $0 \leq t \leq 500$. How could we solve the problems that occur for large t?