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# Malliavin Calculus - a Crash Course

Note Title

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## INTRODUCTION. THE WIENER-ITÔ CHAOS EXPANSION THEOREM.

Let  $f(t_1, \dots, t_n)$  be a deterministic function on  $[0, T]^n$ . Then the iterated Itô integral of  $f$  is defined by

$$J_n(f) = \int_0^T \left( \int_0^{t_n} \left( \int_0^{t_{n-1}} \dots \left( \int_0^{t_2} f(t_1, \dots, t_n) dB(t_1) dB(t_2) \dots \right) dB(t_{n-1}) \right) dB(t_n) \right)$$

This integral converges if  $f \in L^2([0, T]^n)$ .

Similarly, if  $f \in \hat{L}^2([0, T]^n)$ , i.e.  $f$  is symmetric and in  $L^2([0, T]^n)$ , then we define

$$I_n(f) := n! J_n(f) = \int_{[0, T]^n} f(t_1, \dots, t_n) dB^{\otimes n}$$

(Note that if we put

$$G_n = \{(t_1, \dots, t_n); 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T\}$$

then the volume of  $G_n$  is  $\frac{1}{n!} \text{Volum}([0, T]^n)$ )

## THEOREM (Chaos expansion)

Let  $F \in L^2(P)$  be  $\mathcal{F}_T$ -measurable

Then there exist a sequence  $f_n \in \hat{L}^2([0, T]^n)$

such that

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad (\text{convergence in } L^2(P))$$

Moreover,  $E[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2([0,T]^n)}^2$

### DEFINITION (Malliavin derivative)

Let  $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(P)$  be  $\mathcal{F}_T$ -measurable.

Assume that  $F \in \mathcal{D}_{1,2}$ , i.e.

$$\|F\|_{\mathcal{D}_{1,2}}^2 := \sum_{n=0}^{\infty} n n! \|f_n\|_{L^2([0,T]^n)}^2 < \infty$$

Then we define the Malliavin derivative of  $F$  at  $t$ ,  $D_t F$ , by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t))$$

EXAMPLE Suppose  $F = \int_0^T f(t) dB(t)$ ,  $f \in L^2[0,T]$  det.

Then  $F = I_1(f)$ , so

$$D_t F = f(t).$$

### APPLICATION:

The Clark-Ocone theorem:

Let  $F \in \mathcal{D}_{1,2}$ . Then  $F = E[F] + \int_0^T E[D_t F | \mathcal{F}_t] dB(t)$

Sketch of proof. Consider

$$\begin{aligned}
 \int_0^T E[D_t F | \mathcal{F}_t] dB(t) &= \int_0^T E\left[\sum_n I_{n-1}(f_n(\cdot, t)) | \mathcal{F}_t\right] dB(t) \\
 &= \sum_n \int_0^T E[I_{n-1}(f_n(\cdot, t)) | \mathcal{F}_t] dB(t) \\
 &= \sum_n \int_0^T n I_{n-1}(f_n(\cdot, t) \cdot \chi_{[0,t]}^{(n)}(\cdot)) dB(t) \\
 &= \sum_n \int_0^T n(n-1)! J_{n-1}(f_n(\cdot, t) \chi_{[0,t]}^{(n)}(\cdot)) dB(t) \\
 &= \sum_{n=1}^{\infty} n! J_n(f_n) = \sum_{n=1}^{\infty} I_n(f_n) = F - E[F] \quad \square
 \end{aligned}$$

### SOME BASIC PROPERTIES

(I) If  $X$  is  $\mathcal{F}_t$ -measurable, then

$$D_s X = 0 \quad \text{if } s > t$$

(II) (Chain rule)

$$D_t(\varphi(F)) = \varphi'(F) D_t F$$

(III) (Fundamental theorem)

$$D_t\left(\int_0^T g(s, \omega) dB(s)\right) = g(t, \omega) + \int_0^T D_t g(s, \omega) dB(s)$$

(IV) (Duality formula)

$$E \left[ G \int_0^T u(s, \omega) dB(s) \right] = E \left[ \int_0^T u(s, \omega) D_s G ds \right]$$

APPLICATION: Efficient computation of the "greeks".

(Fournie et al, Finance & Stochastic 1999)

Consider the following market

(risk free asset)  $dS_0(t) = \rho S_0(t) dt; S_0(0) = 1$

(risky asset)  $dS_1(t) = S_1(t) [\mu(t) dt + \sigma(t) dB(t)]; S_1(0) > 0$

Assume that  $\mu(t) = \mu(S_1(t)), \sigma(t) = \sigma(S_1(t))$

(Markovian system),  $\rho$  constant.

We want to replicate a payoff at time  $T$  given by  $F = \varphi(S_1(T))$ .

Let  $\theta(t) = (\theta_0(t), \theta_1(t))$  be a portfolio, giving the number of units held at time  $t$  of the two assets, respectively. The corresponding wealth process is given by

$$\textcircled{1} \quad V(t) = \theta_0(t) S_0(t) + \theta_1(t) S_1(t)$$

The portfolio is assumed to be self-financing, in the sense that

$$\textcircled{2} \quad dV(t) = \underline{\theta_0(t)} dS_0(t) + \theta_1(t) dS_1(t)$$

Combining ① and ② we get

$$dV(t) = \frac{V(t) - \theta_1(t) S_1(t)}{S_0(t)} dS_0(t) + \theta_1(t) dS_1(t)$$

or

$$dV(t) = V(t) \rho dt + \theta_1(t) S_1(t) [(\mu - \rho) dt + \sigma dB(t)]$$

or

$$dV(t) - \rho V(t) dt = \underbrace{\sigma(t) \theta_1(t) S_1(t)}_{dB(t)} \left[ \frac{\mu - \rho}{\sigma} dt + dB(t) \right]$$

or

$$\underbrace{e^{-\rho T} V(T)}_F = V(0) + \int_0^T e^{-\rho t} \sigma(t) \theta_1(t) S_1(t) d\tilde{B}(t)$$

$\tilde{B}(t)$  is a B.m. with respect to another probability measure, say  $\mathbb{Q}$  (the Girsanov theorem).

Alternatively, one can show that

$$\theta_1(t) = e^{-\rho(T-t)} \frac{d}{dx} E[\varphi(X^x(T-t))],$$

where

$$dX^x(t) = X^x(t) [\rho dt + \sigma(X(t)) dB(t)]; X^x(0) = x$$

PROBLEM How do we compute  $\theta, (t)$  if  $\varphi$  is not smooth?

We generalize the problem to the following:

Suppose

$$dX(t) = dX^x(t) = b(X(t))dt + \sigma(X(t))dB(t),$$

for some smooth functions  $b, \sigma \neq 0$  ( $X^x(0) = x$ )

Suppose  $\varphi$  is a bounded function.

Compute  $\frac{d}{dx} E[\varphi(X^x(T))]$  !

THEOREM (Fournie et al)

Let  $Y(t)$  be the derivative process, i.e.

$$dY(t) = b'(X^x(t))Y(t)dt + \sigma'(X^x(t))Y(t)dB(t)$$

$$Y(0) = 1.$$

Then

$$\frac{d}{dx} E[\varphi(X^x(T))] = E[\varphi(X^x(T)) \Lambda_a^{(w)}]$$

where

$$\Lambda_a = \int_0^T a(t) \sigma^{-1}(X^x(t)) Y(t) dB(t).$$

Here  $a(t)$  is an arbitrary continuous deterministic function s.t.

$$\int_0^T a(t) dt = 1.$$

Sketch of proof.

LEMMA 1  $D_s X(t) = \Upsilon(t) \Upsilon^{-1}(s) \sigma(X(s)) \mathcal{X}_{[0,t]}(s)$   
 $s < t$

Proof.

$$D_s X(t) = D_s \left( \int_0^t b(X(u)) du + \int_0^t \sigma(X(u)) dB(u) \right)$$

$$= \int_s^t b'(X(u)) D_s X(u) du + \int_s^t \sigma'(X(u)) D_s X(u) dB(u) + \sigma(X(s))$$

If we put  $Z(t) = D_s X(t)$  ( $s$  fixed)

then

$$dZ(t) = b'(X(t)) Z(t) dt + \sigma'(X(t)) Z(t) dB(t); t > s$$

$$Z(s) = \sigma(X(s))$$

Itô formula gives the solution

$$Z(t) = D_s X(t) = \sigma(X(s)) \exp \left( \int_s^t \sigma'(X(u)) dB(u) \right. \\ \left. + \int_s^t (b'(X(u)) - \frac{1}{2} \sigma'^2(X(u))) du \right)$$

$$= \Upsilon(t) \Upsilon^{-1}(s) \sigma(X(s)).$$

□

LEMMA 2 Let  $a(t)$  be a continuous, determ. function such that  $\int_0^T a(t) dt = 1$ .

Then

$$Y(T) = \int_0^T D_s X(T) a(s) \sigma^{-1}(X(s)) Y(s) ds.$$

Proof. Choose  $t = T$  in Lemma 1, multiply by  $a(s)$  and integrate.  $\square$

Proof of theorem:

We may assume that  $\varphi$  is smooth.

Then

$$\frac{d}{dx} E[\varphi(X^x(T))] = E[\varphi'(X^x(T)) Y(T)]$$

Lemma 2

$$= E\left[\varphi'(X^x(T)) \int_0^T D_s X^x(T) a(s) \sigma^{-1}(X^x(s)) Y(s) ds\right]$$

$$= E\left[\int_0^T \underbrace{\varphi'(X^x(T)) D_s X^x(T)}_{D_s(\varphi(X^x(T)))} a(s) \sigma^{-1}(X^x(s)) Y(s) ds\right]$$

duality

$$= E\left[\varphi(X^x(T)) \int_0^T \underbrace{a(s) \sigma^{-1}(X^x(s)) Y(s)}_{\Lambda} d\mathcal{B}(s)\right].$$