

# LECTURE 3: Stochastic differential equations.

Note Title

06/08/2008

Return to the population growth equation

$$dN_t = (r + \alpha \text{"noise"}) N_t dt$$

Formally, replace "noise" by " $\frac{dB}{dt}$ " and multiply by  $dt$ :

$$dN_t = r N_t dt + \alpha N_t dB_t$$

How do we solve this stochastic differential equation?

Define  $Y_t = \ln N_t$

Then by the Ito formula

$$dY_t = \frac{1}{N_t} dN_t + \frac{1}{2} \left( -\frac{1}{N_t^2} \right) (dN_t)^2$$

$$(dN_t)^2 = \alpha^2 N_t^2 dt, \quad \text{Hence}$$

$$dY_t = r dt + \alpha dB_t - \frac{1}{2} \alpha^2 dt$$

$$\text{So } Y_t = \ln N_t = Y_0 + \int_0^t (r - \frac{1}{2} \alpha^2) ds + \int_0^t \alpha dB_s$$

Therefore

$$\underline{N_t = N_0 \exp\left((r - \frac{1}{2} \alpha^2)t + \alpha B_t\right)}$$

Some consequences of the solution formula:

The law of the iterated logarithm for  $B_t$  states that

$$\overline{\lim}_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \ln(\ln t)}} = 1 \text{ a.s.}$$

Therefore

(i) If  $r - \frac{1}{2}\sigma^2 > 0$  then  $N_t \rightarrow \infty$  as  $t \rightarrow \infty$

(ii) If  $r - \frac{1}{2}\sigma^2 < 0$  then  $N_t \rightarrow 0$  as  $t \rightarrow \infty$

(iii) If  $r - \frac{1}{2}\sigma^2 = 0$  then  $N_t$  fluctuates between arbitrary large and arbitrary small values.

Observe that

$$\begin{aligned} E[N_t] &= N_0 E\left[\exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)\right] \\ &= N_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t\right) E\left[\exp(\sigma B_t)\right] = N_0 e^{rt}, \end{aligned}$$

which is the solution in case there is no noise.

Alternatively, one could model population growth in a random environment as follows:

$$\frac{dN_t}{dt} = rN_t + \alpha \text{"noise"}$$

i.e.

$$dN_t = rN_t dt + \alpha dB_t \quad (\text{Ornstein-Uhlenbeck})$$

To solve this we proceed as follows:

$$dN_t - rN_t dt = \alpha dB_t$$

Multiply by  $e^{-rt}$ :

$$e^{-rt} dN_t - r e^{-rt} N_t dt = \alpha e^{-rt} dB_t$$

$$d(e^{-rt} N_t) = \alpha e^{-rt} dB_t$$

$$e^{-rt} N_t = N_0 + \int_0^t \alpha e^{-rs} dB_s$$

Hence

$$N_t = N_0 e^{rt} + e^{rt} \int_0^t \alpha e^{-rs} dB_s$$

Note that  $E[N_t] = N_0 e^{rt}$ ,

as in the previous example.

Now consider a general stochastic differential equation of the form

$$\textcircled{*} \quad dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t; \quad 0 \leq t \leq T$$

$X_0$  independent of  $\{B_s\}$ .

Here  $b: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\sigma: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  are given functions.

THEOREM Suppose there exist constants  $C, D$  such that

$$(1) \text{ (Lipschitz)} \quad |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|$$

for all  $x, y \in \mathbb{R}^n, t \in [0, T]$

$$(2) \text{ (Linear growth)} \quad |b(t, x)| + |\sigma(t, x)| \leq D(1 + |x|)$$

for all  $x \in \mathbb{R}^n, t \in [0, T]$

Suppose  $E[|\mathcal{X}_0|^2] < \infty$ . Then there exists a unique adapted solution  $\mathcal{X}_t$  of the equation (\*) such that  $E[|\mathcal{X}_t|^2] < \infty$  for all  $t \in [0, T]$ .

REMARK Conditions (1) and (2) are sufficient to guarantee uniqueness and existence.

Example 1 Consider the equation

$$\frac{dX_t}{dt} = X_t^2; \quad X_0 = 1$$

Here the linear growth condition fails.

The solution of the equation is

$$X_t = \frac{1}{1-t}; \quad 0 \leq t < 1$$

which explodes at time  $t=1$ .

Example 2 Consider the equation

$$\frac{dX_t}{dt} = 3X_t^{2/3}; \quad X_0 = 0$$

For any  $a > 0$  the function

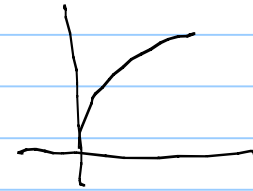
$$X_t = \begin{cases} 0 & \text{for } t \leq a \\ (t-a)^3 & \text{for } t > a \end{cases}$$

solves the equation. So the solution is not unique.

The function

$$b(x) = 3x^{2/3}$$

is not Lipschitz cont. at 0.



Nevertheless, there are many examples of equations with unique solutions where conditions (1) or (2) do not hold.

Example 1 A stochastic logistic equation for population growth could be

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t; X_0 = x > 0$$

The solution exists and is unique (even though we do not have linear growth as required in the theorem), and its given by

$$X_t = \frac{\exp\left((rK - \frac{1}{2}\beta^2)t + \beta B_t\right)}{x^{-1} + r \int_0^t \exp\left\{(rK - \frac{1}{2}\beta^2)s + \beta B_s\right\} ds} \quad (\text{Gard})$$

Example 2 A related logistic model is

$$dY_t = rY_t(K - Y_t)dt + \beta Y_t(K - Y_t)dB_t; Y_0 > 0$$

For  $r > 0, K > 0$  there is a unique solution of this equation (but no solution formula seems

to be known) (Lungu & Ø.)

Not all SDEs have solutions:

Example (the Tanaka equation)

$$\underline{dX_t = \text{sign}(X_t) dB_t} ; \quad \underline{X_0 = 0}$$

where

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

It can be proved that this equation has no (strong) solution. (Lipschitz continuity fails)

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An interesting example:

The Brownian bridge

Let  $a, b$  be given real numbers. Consider the equation

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t; \quad 0 \leq t < 1$$

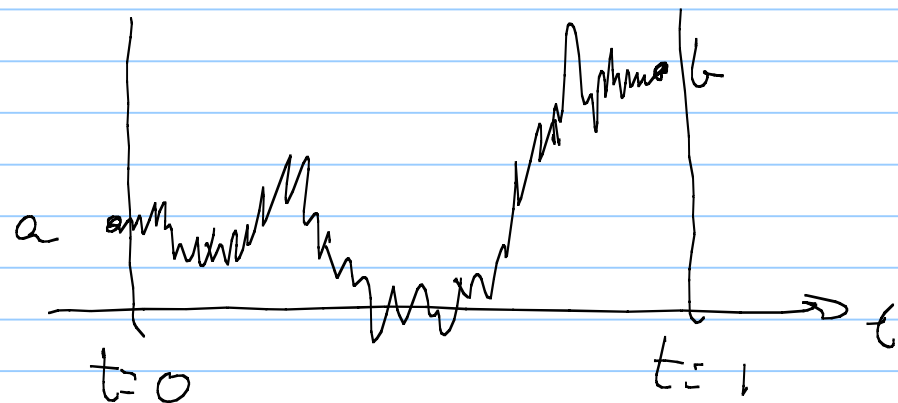
$$Y_0 = a$$

The solution is

$$Y_t = a(1-t) + bt + (1-t) \int_0^t \frac{dB_s}{1-s}; \quad 0 \leq t < 1$$

One can show that

$$\lim_{t \rightarrow 1^-} Y_t = b$$



# AN OPTIMAL HARVESTING PROBLEM

(N. Chr. Stenseth, K. A. Brodke, B. Ø.,) PNAS (2007)

Consider the following equation

$$dX(t) = X(t) [\mu(t)dt + \sigma(t)dB(t)] - c(t)dt ; t \geq 0$$
$$X(0) = x > 0$$

This models a population with size  $X(t)$  at time  $t$ , being harvested at the rate  $c(t)$ .

The coefficients  $\mu(t)$ ,  $\sigma(t)$  are  $\mathcal{F}_t$ -adapted processes.

Consider the following performance criterion.

$$J(c) = E \left[ \int_0^{\infty} e^{-\rho t} \log c(t) dt \right]$$

Interpretation: This is the expected total discounted logarithmic utility of the harvesting.  
( $\rho > 0$  is a given constant)

Introduce 
$$\lambda(t) = \frac{c(t)}{X(t)}$$

Then the system becomes

$$dX(t) = X(t) \left[ (\mu(t) - \lambda(t)) dt + \sigma(t) dB(t) \right]$$

By the Ito formula the solution is ( $X(0) = 1$ )

$$X(t) = \exp \left( \int_0^t \left\{ \mu(s) - \lambda(s) - \frac{1}{2} \sigma^2(s) \right\} ds + \int_0^t \sigma(s) dB(s) \right)$$

The performance criterion gets the form

$$\begin{aligned} J(\lambda) &= E \left[ \int_0^{\infty} e^{-\rho t} \log c(t) dt \right] \\ &= E \left[ \int_0^{\infty} e^{-\rho t} (\log \lambda(t) + \log \underline{X}(t)) dt \right] \\ &= E \left[ \int_0^{\infty} e^{-\rho t} (\log \lambda(t) + \int_0^t \{ \mu(s) - \lambda(s) - \frac{1}{2} \sigma^2(s) \} ds + \int_0^t \sigma(s) dB(s)) dt \right] \\ &= E \left[ \int_0^{\infty} e^{-\rho t} (\log \lambda(t) - \int_0^t \lambda(s) ds) dt \right] + C, \end{aligned}$$

where

$$C = E \left[ \int_0^{\infty} e^{-\rho t} \left( \int_0^t \{ \mu(s) - \frac{1}{2} \sigma^2(s) \} ds + \int_0^t \sigma(s) dB(s) \right) dt \right]$$

does not depend on  $\lambda$ .

So it suffices to maximize

$$\textcircled{1} J_0(\lambda) = E \left[ \int_0^{\infty} \underline{e^{-\rho t}} (\log \lambda(t) - \int_0^t \underline{\lambda(s) ds}) dt \right]$$

over all  $\lambda \geq 0$ .

Note that, by the Fubini theorem

$$\begin{aligned} \int_0^{\infty} e^{-\rho t} \left( \int_0^t \lambda(s) ds \right) dt &= \int_0^{\infty} \left( \int_s^{\infty} e^{-\rho t} dt \right) \lambda(s) ds \\ &= \int_0^{\infty} \frac{1}{\rho} e^{-\rho s} \lambda(s) ds \\ &= \int_0^{\infty} \frac{1}{\rho} e^{-\rho t} \lambda(t) dt \end{aligned}$$

Substituting this into  $\textcircled{1}$  we get

$$J_0(\lambda) = E \left[ \int_0^{\infty} e^{-\rho t} (\log \lambda(t) - \frac{1}{\rho} \lambda(t)) dt \right]$$

↳ We can maximize this pointwise, for each  $t$ :

The function  $h(\lambda) := \log \lambda - \frac{1}{\rho} \lambda$

is maximal when

$$\frac{1}{\lambda} - \frac{1}{\rho} = 0$$

i.e.  $\lambda = \lambda^*(t) = \rho$

We have proved:

THEOREM The optimal harvesting rate

for the performance  $J(c) = E \left[ \int_0^{\infty} e^{-\rho t} \log C(t) dt \right]$

is

$$\underline{c(t) = c^*(t) = \rho \mathbb{I}(t)}$$

POSSIBLE FURTHER STUDIES:

- 1) Stochastic control theory  
(dynamic programming, HJB equations)
- 2) Optimal stopping  
(Variational inequalities)
- 3) Filtering theory
  - a) Linear filter (Kalman filter)
  - b) Non-linear case ( $\rightarrow$  stochastic partial diff. eq.)

#### 4) Extensions to Lévy processes

Any Lévy process  $\eta(t)$  with  $E[\eta^2(\omega)] < \infty$

can be written

$$\eta(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz).$$