LECTURE 2: Martingales and the Itô formula

Properties of the Itô integral:
1) \[ \int_0^T \left( af(s) + bg(s) \right) dB_s = a \int_0^T f(s) dB_s + b \int_0^T g(s) dB_s, \]
   when \( f, g \in \mathcal{H}(0,T), a, b \) are constants.

2) \[ E \left[ \int_0^T f(s) dB_s \right] = 0 \quad \text{if} \quad f \in \mathcal{H}(0,T). \]

3) (The Itô isometry) \[ E \left[ (\int_0^T g(s) dB_s)^2 \right] = \int_0^T g(s)^2 ds \quad \text{if} \quad f \in \mathcal{H}(0,T). \]

4) The process \( I(t) := \int_0^t f(s) dB_s \) is a martingale with respect to \( \{ \mathcal{F}_t \}_{t \geq 0} \) (and \( \mathbb{P} \)). (For \( f \in \mathcal{H}(0,T) \)).

A filtration is an increasing family of \( \sigma \)-algebras \( \{ \mathcal{F}_t \}_{t \geq 0} \) (if \( s < t \) then \( \mathcal{F}_s \subseteq \mathcal{F}_t \)).

**Definition.** A stochastic process \( \{ T(t) \} \) is a martingale with respect to a given filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) (and \( \mathbb{P} \)) if the following holds:

1) \( \{ T(t) \}_{t \geq 0} \) is adapted to \( \{ \mathcal{F}_t \}_{t \geq 0} \), i.e., for each \( t \), \( T(t) \) is \( \mathcal{F}_t \)-measurable.
2) \( E[|Y(t)|] < \infty \) for all \( t \)

3) \( E[Y(s) | N_t] = Y(t) \) for all \( s > t \).

RECALL THE BASICS OF CONDITIONAL EXPECTATION:

Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \( X \)
be a r.v, \( E[|X|] < \infty \). Let \( \mathcal{G} \subset \mathcal{F} \) be another
\( \sigma \)-algebra. Then the conditional expectation of \( X \)
with respect to \( \mathcal{G} \), \( E[X | \mathcal{G}] \), is defined by
the following properties:

(i) \( E[X | \mathcal{G}] \) is \( \mathcal{G} \)-measurable

(ii) \( \int_H E[X | \mathcal{G}] dP = \int_H X dP \) for all \( H \in \mathcal{G} \).

Some properties:

1) \( E[aX + bY | \mathcal{G}] = a E[X | \mathcal{G}] + b E[Y | \mathcal{G}] \)

2) \( E[X | \mathcal{G}] = X \) if \( X \) is \( \mathcal{G} \)-measurable

3) \( E[X Y | \mathcal{G}] = E[X | \mathcal{G}] E[Y | \mathcal{G}] \) if \( X \) is \( \mathcal{G} \)-measurable

4) \( E[Y | \mathcal{G}] = E[Y] \) if \( Y \) is independent of \( \mathcal{G} \)

5) The tower property: If \( \mathcal{G} \subset \mathcal{H} \subset \mathcal{F} \) are
\( \sigma \)-algebras, then

\( E[X | \mathcal{G}] = E[E[X | \mathcal{H}] | \mathcal{G}] \)
**Example**  \( Y(t) := B(t) \) is an \( \mathcal{F}_t \)-martingale.

_Proof._ We must show that
\[
E \left[ B(s) \mid \mathcal{F}_t \right] = B(t) \quad \forall \ s > t.
\]

1. \( E \left[ B(s) \mid \mathcal{F}_t \right] = E \left[ B(s) - B(t) + B(t) \mid \mathcal{F}_t \right] \)
2. \( = E \left[ B(s) - B(t) \mid \mathcal{F}_t \right] + E \left[ B(t) \mid \mathcal{F}_t \right] \)
3. \( = E \left[ B(s) - B(t) \right] + B(t) = B(t). \quad \text{OK} \)

**Extension of the Ito integral**

Let \( \mathcal{F}_t \) be a filtration such that \( \mathcal{F}_t \subseteq \mathcal{F}_t \) for all \( t \). Let \( W(0,T) \) be the set of measurable processes \( f(t,w) \) such that

1. \( f(t,w) \) is \( \mathcal{F}_t \)-adapted
2. \( \int_0^T f^2(s)ds < \infty \) a.s.
3. \( E \left[ \int_0^T f^2(s)ds \right] < \infty \)

Then if \( B(t) \) is a martingale w.r.t. \( \mathcal{F}_t \), we can define the _Itô integral_
\[
\int_0^T f(s)dB(s) ; \ f \in W(0,T)
\]
in a similar way as we did for \( W(0,T) \), replacing convergence in \( L^2 \) by convergence in probability.
This extension is important for the Ito's formula.

**THE ITO'S FORMULA (1-DIMENSIONAL CASE)**

An Ito process (1-dimensional) is a process $X_t$ of the form

$$X_t = X_0 + \int_0^t U(s)ds + \int_0^t V(s)dB(s); \quad \sigma \leq t \leq T$$

where $U(s)$ and $V(s)$ are $\mathcal{F}_t$-adapted, for some $\mathcal{F}_t$ s.t. $B$ is a martingale w.r.t. $\mathcal{F}_t$, and

$$\int_0^T |U(s)| ds < \infty, \quad \int_0^T V^2(s) ds < \infty \quad a.s.$$

In short hand notation,

$$dX_t = U_t dt + V_t dB_t; \quad X_0 = x \in \mathbb{R}$$

**THEOREM (The Ito's formula)**

Let $X_t$ be an Ito process as above.

Let $g \in C^{1,2}([\sigma, \infty) \times \mathbb{R})$, i.e. $g$ is c寨t, diff. w.r.t. the first variable and twice c寨t, diff. w.r.t. the second variable. Define

$$Y_t = g(t, X_t)$$
Then $Y(t)$ is again an Itô process, and in differential form $Y(t)$ is given by

$$dY(t) = \frac{∂g}{∂t}(t, Y(t)) dt + \frac{∂g}{∂x}(t, Y(t)) dX(t) + \frac{1}{2} \frac{∂^2g}{∂x^2}(t, Y(t)) (dX(t))^2$$

where $(dX(t))^2 = (u(t) dt + \nu(t) dB(t))^2$

$$= (u(t) dt)^2 + 2 u(t) \nu(t) dt dB(t) + (\nu(t) dB(t))^2$$

where multiplication of the differentials is carried out according to the following multiplication table

<table>
<thead>
<tr>
<th></th>
<th>dt</th>
<th>dB(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dt</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>dB(t)</td>
<td>0</td>
<td>dt</td>
</tr>
</tbody>
</table>

Recall that $E \left[ (B_{t+\Delta t} - B_t)^2 \right] = \Delta t$

**Remark** Note that $\int_0^t \nu(s) dB(s)$ is not necessarily a martingale if we only assume $f \in \mathcal{W}(0, T)$. (But it is a local martingale.)

**Example** Consider $Y(t) = \frac{1}{2} B_t^2$

$$dY(t) = ?$$

Choose $g(t, x) = \frac{1}{2} x^2$, $X(t) = B(t)$
Then by the Ito formula, \( (\frac{\partial g}{\partial x} = \gamma, \frac{\partial g}{\partial x^2} = 1) \)

\[
dY(t) = \mathbb{E}(X(t)) dB(t) + \frac{1}{2} \gamma(t) (dB(t))^2
\]

So,

\[
dY(t) = \frac{1}{2} dt + B(t) dB(t)
\]

i.e.,

\[
Y(t) = Y(0) + \frac{t}{2} \gamma + \int_{0}^{t} B(s) dB(s)
\]

If \( B(0) = 0 \), this gives

\[
\frac{1}{2} B^2(t) = \frac{1}{2} t + \int_{0}^{t} B(s) dB(s)
\]

This implies that

\[
\int_{0}^{t} B(s) dB(s) = \frac{1}{2} B^2(t) - \frac{1}{2} t
\]

**EXAMPLE**

\[\gamma(t) = \exp (\alpha t + \beta B(t))\]

\(\alpha, \beta\) constants.

\[dY(t) = ?\]

Choose \( X(t) = \alpha t + \beta B(t) \), \( g(x) = e^x \)

Then

\[
dY(t) = Y(t) \left( \alpha dt + \beta dB(t) \right) + \frac{1}{2} Y(t) \beta^2 dt
\]

\[
= Y(t) \left[ (\alpha + \frac{1}{2} \beta^2) dt + \beta dB(t) \right]
\]
Alternatively, we could have chosen
\[
X(t) = B(t), \quad g(x, t) = \exp(\alpha t + \beta x)
\]
Then we get the same final result, but with different computation.

THE MULTI-DIMENSIONAL CASE

Suppose we have \( n \) Itô processes
\[
\begin{align*}
\frac{dX_1(t)}{dt} &= u_1(t) dt + v_{11}(t) dB_1(t) + \cdots + v_{1n}(t) dB_n(t) \\
\frac{dX_2(t)}{dt} &= u_2(t) dt + v_{21}(t) dB_1(t) + \cdots + v_{2n}(t) dB_n(t) \\
&\vdots \\
\frac{dX_n(t)}{dt} &= u_n(t) dt + v_{n1}(t) dB_1(t) + \cdots + v_{nn}(t) dB_n(t)
\end{align*}
\]
Introduce
\[
\begin{align*}
\mathbf{dX}(t) &= \begin{bmatrix} dX_1(t) \\ \vdots \\ dX_n(t) \end{bmatrix}, \\
\mathbf{u}(t) &= \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \\
\mathbf{dB}(t) &= \begin{bmatrix} dB_1(t) \\ \vdots \\ dB_n(t) \end{bmatrix}
\end{align*}
\]
\[
\mathbf{V}(t) = \begin{bmatrix} v_{11}(t) & \cdots & v_{1n}(t) \\ \vdots & \ddots & \vdots \\ v_{n1}(t) & \cdots & v_{nn}(t) \end{bmatrix} \in \mathbb{R}^{n \times n}
\]
Then the system gets the form
\[
\mathbf{dX}(t) = \mathbf{u}(t) dt + \mathbf{V}(t) \mathbf{dB}(t)
\]
THE MULTI-DIMENSIONAL ITÔ FORMULA

Let \( \mathbf{X}(t) \) be as above and let

\[ g = g(t, x) = g(t, x_1, \ldots, x_n) \in C^{1,2}([0, \infty) \times \mathbb{R}^n) \]

Define \( \gamma(t) = g(t, \mathbf{X}(t)) \).

Then

\[
d\gamma(t) = \frac{\partial g}{\partial t}(t, \mathbf{X}(t)) \, dt + \sum_{i=1}^n \frac{\partial g}{\partial x_i}(t, \mathbf{X}(t)) \, d\mathbf{X}_i(t) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j}(t, \mathbf{X}(t)) \, d\mathbf{X}_i(t) \, d\mathbf{X}_j(t),
\]

where \( d\mathbf{X}_i(t) \) is computed by using the multiplication rules:

\[
dt \cdot dt = dt \cdot dB_i(t) = 0 \]

\[
dB_i(t) \cdot dB_j(t) = \begin{cases} 0 & \text{if } \hat{i} \neq \hat{j} \\ dt & \text{if } \hat{i} = \hat{j} \end{cases}
\]

EXAMPLE (Integration by parts)

Let

\[
d\mathbf{X}_\hat{i}(t) = u_{\hat{i}}(t) \, dt + \sum_{j=1}^m v_{\hat{i}j}(t) \, dB_j(t), \quad \hat{i} = 1, 2.
\]

Then

\[
d(\mathbf{X}_1(t) \mathbf{X}_2(t)) = X_1(t) \, d\mathbf{X}_2(t) + X_2(t) \, d\mathbf{X}_1(t) + d\mathbf{X}_1(t) \, d\mathbf{X}_2(t)
\]
Proof: Apply the Itô formula with

\[ g(x_1, x_2) = x_1 x_2. \]

THE MARTINGALE REPRESENTATION THEOREM

We have seen that if \( f \in \mathcal{U}(0, T) \) then

\[ \mathbb{I}(t) := \int_0^t f(s) dB(s); \quad 0 \leq t \leq T \]

is an \( \mathcal{F}_t \)-martingale (and continuous).

The martingale representation theorem states that the converse is also true:

THEOREM. Let \( M(t) \) be a continuous \( \mathcal{F}_t \)-martingale with \( \mathbb{E}[M^2(t)] < \infty \) for all \( t \in [0, T] \). Then there exists \( f \in \mathcal{U}(0, T) \) such that

\[ M(t) = M(0) + \int_0^t f(s) dB(s). \]

A proof of this can be given by using the following related result:

THEOREM (The Itô representation theorem)
Let $F \in L^2(\mathbb{P})$ be $\mathcal{F}_t$-measurable.

Then there exists $f \in \mathcal{D}(0,\infty)$ such that

$$F = \mathbb{E}[F] + \int_0^T f(s) dB(s).$$

The classical proof gives the existence and uniqueness of $f$, but no information about how to find it. But using Malliavin calculus one can show that $f$ can be given the representation

$$f(s) = \mathbb{E}\left[D_s F \mid \mathcal{F}_s \right] \quad \text{(The Clark-Obremski formula)}$$

where $D_s F$ is the Malliavin derivative of $F$ at $s$. 
