MATTER UNDER THE INFLUENCE OF EXTREMELY STRONG MAGNETIC FIELDS

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There are huge magnetic fields at the surface of a neutron star - as large as $10^{13}$ Gauss, as measured spectroscopically. The atoms there are iron with nuclear charge $Z = 26$. The natural unit of magnetic field is

$$B^* = m^2 c^3 / \hbar^3 = 2.35 \times 10^9 \text{ Gauss},$$

so we are talking about large fields.

The cyclotron radius $= a_0 (B^* / B)^{1/2}$, $a_0 =$ Bohr radius.

These large fields are trapped by collapsing current loops when the neutron star is born from a collapsing star.
HAMiLToNian (non-relativistic)

\[ H_N = \sum_{i=1}^{N} (H_A^{(i)} - Z|x(i)|^{-1}) + \sum_{1 \leq i < j \leq N} |x(i) - x(j)|^{-1} \]

\[ H_A = \left( (p - A(x)) \cdot \sigma \right)^2 = (p - A(x))^2 - \sigma \cdot B \]

\[ B = (0, 0, B) = \text{constant}, \quad A = \frac{1}{2} B \times x \]

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

We want the ground state energy, \( E_N \) of \( H_N \) for \( N \) (e.g. \( N = Z \)) fermions with spin, i.e. \( \psi \in \bigwedge \mathcal{L}^2(\mathbb{R}^3; \mathbb{C}^2) \).

THOMAS-FERMI THEORY

As usual, we hope that we can replace the \( N \)-body problem by a functional of the electron density \( \rho \).

\[ E_{TF}^N = \inf \{ \mathcal{E}_{MTF}(\rho) \mid \rho : \mathbb{R}^3 \to \mathbb{R}^+, \int \rho = N \} \]

\[ \mathcal{E}_{MTF}(\rho) = \int \tau_B(\rho)dx - \int \frac{Z}{|x|} \rho(x)dx + \frac{1}{2} \int \rho(x)|x-y|^{-1} \rho(y)dxdy \]
What is $\tau_B(\rho)$? ($=\rho^{5/3}$ when $B = 0$). We first study the one-body problem with Hamiltonian

$$H = H_A - V(x).$$

Generalized Lieb-Thirring inequality:

**Theorem 1.** There exist universal constants $L_1, L_2 > 0$ such that if we let $e_j(B, V)$, $j = 1, 2, \ldots$ denote the negative eigenvalues of $H_A - V$ with $0 \leq V \in L^{3/2}(\mathbb{R}^3) \cap L^{5/2}(\mathbb{R}^3)$ then

$$\sum_j |e_j(B, V)| \leq L_1 B \int V(x)^{3/2} dx + L_2 \int V(x)^{5/2} dx.$$

We can choose $L_1$ as close to $2/3\pi$ as we please, compensating with $L_2$ large.

Note: $\sigma \cdot B$ is a constant. What we are really estimating is the sum of the eigenvalues of $(p - A)^2 - V(x)$ below $+B$ (the bottom of the continuous spectrum).
SCALING AND SEMI-CLASSICAL LIMIT

\[ H = [(\hbar p - b a(x)) \cdot \sigma]^2 - v(x), \]

\[ a(x) = \frac{1}{2}(0, 0, 1) \times x \text{ and } v \geq 0. \]

**Theorem 2.** Let \( e_j(h, b, v), j = 1, 2, \ldots, \) denote the negative eigenvalues of \( H \), with \( 0 \leq v \in L^{3/2}(\mathbb{R}^3) \cap L^{5/2}(\mathbb{R}^3) \). Then

\[ \lim_{h \to 0} \sum_j |e_j(h, b, v)| \Bigg/ E_{\text{scl}}(h, b, v) = 1 \]

uniformly in \( b \), where

\[ E_{\text{scl}}(h, b, v) = \frac{1}{3\pi^2} h^{-2} b \int \left( v(x)^{3/2} + 2 \sum_{\nu=1}^{\infty} [v(x) - 2\nu bh]_+^{3/2} \right) dx. \]

The effective parameter is \( bh \). For \( bh \ll 1 \), the right side reduces to the standard semiclassical formula

\[ \frac{2}{15\pi^2} h^{-3} \int v(x)^{5/2} dx. \]
We take $\tau_B(\rho)$ to be the Legendre transform of the semiclassical function
\[
V \leftrightarrow \frac{1}{3\pi^2} B \left( V^{3/2} + 2 \sum_{\nu=1}^{\infty} [V - 2\nu B]^{3/2} \right). \quad (*)
\]
Thus, $\tau_B(\rho) = \text{energy/unit volume of free particles in box with magnetic field } B$.

The many-body Hamiltonian $H_N$ can be reduced to a one-body operator with the mean field potential:
\[
V = Z |x|^{-1} - |x|^{-1} \ast \rho^{\text{MTF}}.
\]
From the scaling of the minimizer $\rho^{\text{MTF}}$ of $\mathcal{E}^{\text{MTF}}$ we find that the effective parameters are
\[
h = (B/Z^3)^{1/5} \quad \text{and} \quad b = (B^2/Z)^{1/5}.
\]
Thus when $B \ll Z^3$ the semiclassical approach is appropriate and our analysis is consistent.
1) $B \ll Z^{4/3}$, (i.e., $hb \ll 1, h$ small):
The effect of the magnetic field is negligible. We get standard Thomas-Fermi theory with $\tau_B(\rho) = \rho^{5/3}$.

2) $B \sim Z^{4/3}$ (i.e., $hb \sim 1, h$ small):
The magnetic field becomes important. The function $\tau_B$ is complicated because we have a finite number of terms in ($\ast$). The density is still spherical.

3) $Z^{4/3} \ll B \ll Z^3$ (i.e., $hb \gg 1, h$ small):
The magnetic field is increasingly important. (Most electrons will, in a certain sense, be confined to the lowest Landau band.) The function $\tau_B$ is simple since the sum is not present in ($\ast$) and therefore $\tau_B(\rho) \sim \rho^3/B^2$. The density is spherical and furthermore the atom is getting smaller. The atomic radius behaves as $Z^{1/5}B^{-2/5}$.

4) $B \sim Z^3$ (i.e., $h \gtrsim 1$):
In this regime one can no longer use semiclassics. The functional $\mathcal{E}^{\text{MTF}}$ is not a good approximation to the energy for any $\tau_B(\rho)$. The atom is no longer spherical. A density matrix functional works, however.

5) $B \gg Z^3$, (i.e., $h \gg 1$):
This is the hyper-strong regime. Atoms are highly cylindrical, almost one-dimensional.
THE NON-SEMICLASSICAL CASE $B \geq Z^3$

The following result is important in the study of the non-seciall case

**Theorem 3.** Let $B/Z^{4/3} \rightarrow \infty$ as $Z \rightarrow \infty$. Let $\Pi_0$ denote the projection in $\bigwedge^N L^2(\mathbb{R}^3; C^2)$ onto states in which all $N$ electrons are in the lowest Landau band. Consider

$$\Pi_0 H_N \Pi_0 = \Pi_0 \left( \sum_i p_3(i)^2 - Z|x(i)|^{-1} + \sum_{i<j} |x(i) - x(j)|^{-1} \right) \Pi_0,$$

(using $\Pi_0 H_\Lambda \Pi_0 = \Pi_0((p - A(x)) \cdot \sigma)^2 \Pi_0 = \Pi_0 p_3^2 \Pi_0$) and let $E_N^{(0)}$ be the corresponding ground state energy. Then, as $Z \rightarrow \infty$,

$$E_N^{(0)}/E_N \rightarrow 1.$$

We again reduce to a mean-field Hamiltonian:

$$\Pi_0 \sum_i \left( p_3(i)^2 - Z|x(i)|^{-1} + \rho^*|x(i)|^{-1} \right) \Pi_0 = \Pi_0 \sum_i h_i(x_\perp(i)) \Pi_0,$$

where $h_i(x_\perp(i))$ is a one-dimensional Schrödinger operator depending on the two-dimensional parameter $x_\perp(i) = (x_1(i), x_2(i))$. 

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Our problem is to find the infimum
\[ \inf \langle \Psi, \sum_i h_i(x_\perp(i))\Psi \rangle \]
over all \( \Psi \) satisfying
\[ \Psi \in \bigwedge^N L^2(\mathbb{R}^3; C^2), \quad \| \Psi \| = 1, \quad \Pi_0 \Psi = \Psi. \quad (**) \]
Define
\[
\gamma_{x_\perp}(x_3, y_3) = \\
N \int \Psi^*(x_\perp, y_3, x(2), \ldots) \Psi(x_\perp, x_3, x(2), \ldots) dx(2) \ldots dx(N).
\]
We can consider \( \gamma_{x_\perp} \) as a trace class operator on \( L^2(\mathbb{R}) \). Then because of the density of states in the lowest Landau band
\[ \| \Psi \| = 1 \implies \int \text{Tr}_{L^2(\mathbb{R})}[\gamma_{x_\perp}] dx_\perp = N \]
\[ \Psi \in \bigwedge^N L^2(\mathbb{R}^3; C^2), \quad \Pi_0 \Psi = \Psi \implies 0 \leq \gamma_{x_\perp} \leq \frac{B}{2\pi} 1. \quad (***) \]
We avoid \( \Pi_0 \) by relaxing (**) to the right side of (***)

The operator \( h(x_\perp) \) depends on the unknown density \( \rho(x) \). We get around this problem again by defining a functional:
\[ E^{DM}(\gamma) = \int \text{Tr}_{L^2(\mathbb{R})}[(p_3^2 - Z|x|^{-1})\gamma_{x_\perp}] dx_\perp + \frac{1}{2} \int \int \rho_\gamma(x)\rho_\gamma(y)|x-y|^{-1} dx dy \]
where \( \rho_\gamma(x) = \gamma_{x_\perp}(x_3, x_3). \)
We define the energy

\[ E^\text{DM}(N, Z, B) = \inf \{|\mathcal{E}^\text{DM}(\gamma)| \int \text{Tr}_{L^2(\mathbb{R})}[\gamma_x]dx = N, \; 0 \leq \gamma_x \leq \frac{B}{2\pi}\}. \]

The scaling is

\[ E^\text{DM}(N, Z, B) = Z^3 E^\text{DM}(\frac{N}{Z}, 1, \frac{B}{Z^3}). \]

This is the second time that we see the ratio \( B/Z^3 \) as a non-trivial parameter in the theory, it also played the role of an effective Planck’s constant.

**Theorem 1. (Energy)** Let \( E^Q(N, Z, B) \) denote the quantum energy. If \( N/Z \) is fixed and \( B/Z^{4/3} \to \infty \) (to ensure confinement in the lowest Landau band) as \( Z \to \infty \) then

\[ E^Q(N, Z, B)/E^\text{DM}(N, Z, B) \to 1 \]

**Theorem 2. (Regions)**

**Region 4:** If \( B/Z^3 \) is fixed then \( \gamma_x \) has finite rank (depending on \( B/Z^3 \)) for almost all \( x \).

**Region 3:** As \( B/Z^3 \to 0 \) the rank of \( \gamma_x \) tends to infinity allowing for the semiclassical treatment.
Region 5: There is a critical \( \eta_c \) such that for \( B/Z^3 \geq \eta_c \) the rank of \( \gamma_{x_\perp} \) is one. Then

\[
\gamma_{x_\perp}(x_3, y_3) = \sqrt{\rho_{\gamma}(x_\perp, x_3)} \sqrt{\rho_{\gamma}(x_\perp, y_3)}.
\]

Thus the energy is in this case again a functional only of the density

\[
E_{DM}(\gamma) = E_{SS}(\rho_{\gamma}).
\]

In the limit \( B/Z^3 \to \infty \), the functional \( E_{SS} \) reduces after an appropriate rescaling to a functional of a one-dimensional density which can be minimized in closed form. Examples of the conclusions we can draw from this explicit minimization:

As \( Z \to \infty \) and \( B/Z^3 \to \infty \) we get

- Maximal number of electrons in atom:

\[
\lim \inf N_c(Z)/Z \geq 2 \quad (\text{non-neutrality})
\]

- Energy:

\[
E^Q(N, Z, B) \approx \left( -\frac{1}{48} \left( \frac{N}{Z} \right)^3 + \frac{1}{8} \left( \frac{N}{Z} \right)^2 - \frac{1}{4} \left( \frac{N}{Z} \right) \right) Z^3 \left[ \ln(B/Z^3) \right]^2
\]

- Binding energy of neutral diatomic molecule \((Z + Z)\):
  (a) Energy of molecule \( \approx -\frac{7}{6} Z^3 \left[ \ln(B/Z^3) \right]^2 \)
  (b) Energy of two atoms \( \approx -\frac{7}{24} Z^3 \left[ \ln(B/Z^3) \right]^2 \)
  (b) Binding energy \( \approx \frac{7}{8} Z^3 \left[ \ln(B/Z^3) \right]^2 \)