The Matter of Instability\textsuperscript{a}

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\textsuperscript{a}Joint work with Elliott H. Lieb
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The charged gas in Quantum Mechanics

The Hamiltonian of a gas of charged particles:

\[ H_N = \sum_{i=1}^{N} -\frac{1}{2} \Delta_i + \sum_{1 \leq i < j \leq N} \frac{e_ie_j}{|x_i - x_j|} \]

We consider (for simplicity) the charges \( e_i = \pm 1, \) \( i = 1, \ldots, N \) as variables. Thus the Hilbert space is \( \mathcal{H} = L^2 \left( (\mathbb{R}^3 \times \{-1, 1\})^N \right) \). If

\[ \mathcal{H}_B = \bigotimes_{\text{sym}} L^2 \left( \mathbb{R}^3 \times \{-1, 1\} \right), \]

then

\[ E(N) := \inf \text{spec}_{\mathcal{H}} H_N = \inf \text{spec}_{\mathcal{H}_B} H_N \]

Stability of Matter (i.e., that \( H_N \) obeys a lower bound linear in \( N \)) holds on the subspace of \( \mathcal{H} \), where either the positively or negatively charged particles (or both) are fermions.
The Instability of the charged Bose Gas

THEOREM 1 (Instability of the charged (Bose) gas. Dyson ‘67). There is a constant $C_+ > 0$ such that

$$E(N) \leq -C_+ N^{7/5}.$$ 

INSTABILITY: $7/5 > 1$.

The trial state: The Dyson trial state is a complicated Bogolubov pair function. Stability cannot be proved with simple product state:

$$\Psi(x_1, e_1, \ldots, x_N, e_N) = \prod_{i=1}^{N} \phi(x_i) \ (\text{and} \ = 0 \ \text{if} \ \sum_{i=1}^{N} e_i \neq 0) \ (N \ \text{even}):$$

$$\langle \Psi, H_N \Psi \rangle = \underbrace{CNR^{-2}}_{\text{kinetic energy}} - \underbrace{CNR^{-1}}_{\text{potential= self-energy}} = -CN$$

where $R$ is the extent of the support of $\phi$. 


The $N^{7/5}$ and $N^{5/3}$ laws for Bosons

**THEOREM 2** (The $N^{7/5}$ law. Conlon-Lieb-Yau '88). There is a constant $C_- > 0$ such that $E(N) \geq -C_- N^{7/5}$.

**THEOREM 3** (The $N^{5/3}$ law. Dyson '67, Lieb '78). If the positive or negative bosons are infinitely heavy there are constant $C_\pm > 0$ such that $-C_- N^{5/3} \leq E(N) \leq -C_+ N^{5/3}$.

Proof of lower bound.

**Electrostatic inequality:**

$$\sum_{i<j} \frac{e_i e_j}{|x_i - x_j|} \geq \frac{1}{\max_{j \neq i} |x_i - x_j|^{-1}} \sum_{i=1}^{N}$$

**Sobolev’s inequality:**

$$-\Delta - \max_j |x - x_j|^{-1} \geq \sup_R \left\{ -NR^{1/2} - R^{-1} \right\} = N^{2/3}.$$

Stability of matter can be proved similarly except that one should use the *Lieb-Thirring inequality* instead of the Sobolev inequality.
Foldy’s law and Dyson’s conjecture

THEOREM 4 (Foldy’s law. Lieb-Solovej ‘01). The thermodynamic energy per particle $e(\rho)$ of positively charged bosons in a constant negative background of density $\rho$ satisfies

$$\lim_{\rho \to \infty} \frac{e(\rho)}{\rho^{1/4}} = J, \quad J = (2/\pi)^{3/4} \int_{0}^{\infty} 1 + x^4 - x^2 (x^4 + 2)^{1/2} \, dx.$$  

Foldy calculated this in ‘61 using the method of Bogolubov. This is what motivated Dyson in constructing his trial function for the upper bound in the two-component gas.

DYSON’S CONJECTURE (‘67): For the two component gas we have

$$\lim_{N \to \infty} \frac{E(N)}{N^{7/5}} = \inf \left\{ \frac{1}{2} \int |\nabla \phi|^2 - J \int \phi^{5/2} : \phi \geq 0, \int \phi^2 = 1 \right\}$$

THEOREM 5 (Dyson’s conjecture. Lieb-Solovej in prep.). Dyson’s conjecture is correct as a lower bound.
The Foldy-Bogolubov method (in a box)

$a_{p\pm}^*$: creation operators of momentum $p$ states charge $\pm 1$.

$\nu_{\pm} =$ number of particles of charge $\pm 1$. $\nu = \nu_+ + \nu_-$. 

$b_{p\pm}^* = (\nu_{\pm})^{-1/2} a_{p\pm}^* a_{0\pm}$. 

$d_{p}^* = (\nu_+ + \nu_-)^{-1/2} (\nu_+^{1/2} b_{p+}^* - \nu_-^{1/2} b_{p-}^*)$.

Condensation: most particles have momentum 0: $\nu_{\pm} \approx a_{0\pm}^* a_{0\pm}$.

Bogolubov approximation: The important part of the Hamiltonian can be written in terms of $b_{p\pm}^*$ or rather $d_{p}^*$: The Foldy-Bogolubov Hamiltonian:

$$\sum_{p \neq 0} \frac{1}{2} |p|^2 (d_{p}^* d_{p} + d_{-p}^* d_{-p}) + \frac{\nu}{|p|^2} (d_{p}^* d_{p} + d_{-p}^* d_{-p} + d_{p}^* d_{-p} + d_{p} d_{p})$$

$$= \sum_{p \neq 0} D(d_{p}^* + \alpha d_{-p})(d_{p}^* + \alpha d_{-p})^* + D(d_{-p}^* + \alpha d_{p})(d_{-p}^* + \alpha d_{p})^*$$

$$- D\alpha^2 ([d_{p}, d_{p}^*] + [d_{-p}, d_{-p}^*]), \quad \text{(Note: } [d_{p}, d_{p}^*] \leq 1)$$

For specific $D$ and $\alpha$. In particular,

$$2(2\pi)^{-3} \int D\alpha^2 dp = J \nu (\nu/\text{vol})^{1/4}.$$
Length scales

• Size $R$ of gas: $N(N/R^3)^{1/4} = NR^{-2} \Rightarrow R = N^{-1/5}$.

• Energy: $N(N/R^3)^{1/4} = NR^{-2} = N^{7/5}$.

• Momentum scale of the excited pairs:

\[ p^2 = (N/R^3)|p|^{-2} \Rightarrow |p| = (N/R^3)^{1/4} = N^{2/5} \]

• Separation of scales: $|p| = N^{2/5} \gg R^{-1} = N^{1/5}$. 
Steps in the rigorous proof

- Dirichlet localize gas into region of size $R = N^{-1/5}$.
- Neumann localize into boxes of size $|p|^{-1} = N^{-2/5}$.
- Electrostatic energy between regions is controlled by the method of sliding using the positivity of the Coulomb kernel.
- Control all terms in the Hamiltonian except the Foldy-Bogolubov part.
- Control condensation
- **DIFFICULTY** with kinetic energy localization: A pure Neumann localization is too crude. It ignores variation on scale $N^{1/5}$. One must use Neumann only for high momentum ($N^{2/5}$) and keep full energy for low momentum ($N^{1/5}$).
- **DIFFICULTY** with controlling condensation: It is not enough to know the *expectation* value of the condensation.
Kinetic energy bound

THEOREM 6 (A many body kinetic energy bound).
\( \chi_z = \text{“smooth characteristic” function of unit cube centered at } z \in \mathbb{R}^3. \ P_z = \text{projection orthogonal to constants in unit cube.} \)
\( \Omega \subset \mathbb{R}^3. \ e_1, e_2, e_3 \text{ standard basis.} \)
For all \( 0 < s < t < 1 \)
\[
(1 + \varepsilon(\chi, s)) \sum_{i=1}^{N} -\Delta_i \geq \int_{\Omega} \left[ \sum_{i=1}^{N} P_z^{(i)} \chi_z^{(i)} \frac{(-\Delta_i)^2}{-\Delta_i + s^{-2}} \chi_z^{(i)} P_z^{(i)} 
+ \sum_{j=1}^{3} \left( \sqrt{a_0^*(z + e_j) a_0(z + e_j)} + 1/2 - \sqrt{a_0^*(z) a_0(z) + 1/2} \right)^2 \right] \, dz 
- 3 \text{vol}(\Omega).
\]
THEOREM 7 (Localizing large matrices). Suppose that $A$ is an $N \times N$ Hermitean matrix and let $A^k$, with $k = 0, 1, ..., N - 1$, denote the matrix consisting of the $k^{th}$ supra- and infra-diagonal of $A$. Let $\psi \in \mathbb{C}^N$ be a normalized vector and set $d_k = (\psi, A^k \psi)$ and $\lambda = (\psi, A \psi) = \sum_{k=0}^{N-1} d_k$. (\psi need not be an eigenvector of $A$.)

Choose some positive integer $M \leq N$. Then, with $M$ fixed, there is some $n \in [0, N - M]$ and some normalized vector $\phi \in \mathbb{C}^N$ with the property that $\phi_j = 0$ unless $n + 1 \leq j \leq n + M$ (i.e., $\phi$ has length $M$) and such that

$$ (\phi, A \phi) \leq \lambda + \frac{C}{M^2} \sum_{k=1}^{M-1} k^2 |d_k| + C \sum_{k=M}^{N-1} |d_k|, \quad (1) $$

where $C > 0$ is a universal constant. (Note that the first sum starts with $k = 1$.)