Dyson’s conjecture for the energy of a charged Bose gas\textsuperscript{a}

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\textsuperscript{a}Joint work with E.H. Lieb
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The oldest problem in Quantum Mechanics

The energy of a charged gas of \( N \) particles, with charges \( e_i \in \{1, -1\} \):

\[
H_N = \sum_{i=1}^{N} -\frac{1}{2} \Delta_i + \sum_{1 \leq i < j \leq N} \frac{e_i e_j}{|x_i - x_j|}, \quad \text{on } \bigotimes L^2(\mathbb{R}^3) = L^2(\mathbb{R}^{3N})
\]

\[
E(N) = \min \{ \inf \text{spec} H_N : e_i = \pm 1, \ i = 1, 2, \ldots, N \} = \inf \text{spec}_{L^2((\mathbb{R}^3 \times \{1, -1\})^N)} H_N
\]

On \( L^2((\mathbb{R}^3 \times \{1, -1\})^N) = \bigotimes^N L^2(\mathbb{R}^3 \times \{1, -1\}) \) we may restrict to fully symmetric functions. Therefore this is a \textit{charged Bose gas}.

Note that \( E(1) = 0, \ E(2) = -1/4 \) (Schrödinger 1926-27 (hydrogen)).

\textbf{Dyson’s conjecture (1967):}

\[
\lim_{N \to \infty} \frac{E(N)}{N^{7/5}} = \inf \left\{ \frac{1}{2} \int (\nabla \Phi)^2 - J \int \Phi^{5/2} : \int \Phi^2 = 1, \ \Phi \geq 0 \right\}
\]

\[
J = (2/\pi)^{3/4} \int 1 + x^4 - x^2(x^4 + 2)^{1/2} dx.
\]
History (after 1927)

THEOREM 1 (Dyson 1967 Instability of charged bose gas).

\[ E(N) \leq -CN^{7/5} \]

THEOREM 2 (Dyson-Lenard 1967). \( E(N) \geq -CN^{5/3} \)

THEOREM 3 (Conlon-Lieb-Yau 1988). \( E(N) \geq -CN^{7/5} \)

THEOREM 4 (Lieb-Solovej 2003).

\[ \liminf_{N \to \infty} \frac{E(N)}{N^{7/5}} \geq \inf \left\{ \int (\nabla \Phi)^2 - J \int \Phi^{5/2} : \int \Phi^2 = 1, \Phi \geq 0 \right\} \]

Remark on proof of Thm. 1: If we use a product trial function \( \prod_{i=1}^{N} \phi(x_i) \), with \( e_i = (-1)^i \) we only get \( E(N) \leq -CN \) (not instability).

Dyson uses a BCS type trial function (for \( 2N \) particles):

\[
\prod_{i=1}^{2N} \phi(x_i) \sum \text{Perm.} \prod_{\sigma j=1}^{N} \left[ 1 - e_{\sigma(2j)} e_{\sigma(2j-1)} \sum_{\alpha} \psi_{\alpha}(x_{\sigma(2j)}) \psi_{\alpha}(x_{\sigma(2j-1)}) \right]
\]
Sketching the proof of Dyson’s conjecture

Physics: Global length scale (of $\phi$) is $N^{-1/5}$, $\phi^2 \sim N^{8/5}$. Local length scales (of $\psi_\alpha$) is $N^{-2/5} \ll N^{-1/5}$.

Step 1: The local (short scale) energy: Consider gas confined to box of size $\ell$, with $N^{-2/5} \ll \ell \ll N^{-1/5}$ and with particle number $\nu \sim N^{8/5}\ell^3$. I.e., $\ell^{-1} \ll \nu \ll \ell^{-5}$. Use second quantization

$$H_{\text{Box}} = \sum_{p,e} \epsilon(p) a^*_p,e a_{p,e} + \frac{1}{2} \sum_{p,q,e,e'} \epsilon e' \omega_{pq;\mu\nu} a^*_p,e a^*_q,e' a_{\nu,e',\mu,e}$$

Relevant part is $\omega_{pq;00} = \omega_{00;pq} = \omega_{p0;0-q} = \omega_{0p;-q0} \sim g(p)\delta(p+q)$. Conclude energy $\sim -J\nu(\nu/\ell^3)^{1/4}$.

Step 2: The global energy:

$$\sum_{\text{n.n. boxes } i,j} \frac{1}{2} \ell^{-2} \left( \sqrt{\nu(i)} - \sqrt{\nu(j)} \right)^2 - \sum_{\text{boxes } i} J\nu(i) \left( \nu(i)/\ell^3 \right)^{1/4}.$$

A discrete approximation to $\frac{1}{2} \int (\nabla \Phi)^2 - J \int \Phi^{5/2}$, $\int \Phi^2 = N$. 

3
Local energy calculation

For $p \neq 0$ let $b_{p,e}^* = a_{p,e}^* a_{0,e} / \sqrt{\nu_e}$. Then

$$b_{p,e}^* b_{p,e} \leq a_{p,e}^* a_{p,e}, \quad [b_{p,e}^*, e', b_{p,e}^*] = 0, \quad [b_{p,e}, b_{p,e}^*] \leq 1 \quad (1)$$

and $H_{\text{Box}} \geq \sum_{p \neq 0} h_p$ where $h_p$ is

$$\frac{\epsilon(p)}{2} \sum_{e = \pm 1} (b_{p,e}^* b_{p,e} + b_{-p,e}^* b_{-p,e})$$

$$+ g(p) \sum_{e, e' = \pm 1} \sqrt{\nu_e \nu_{e'}} e e' (b_{p,e}^* b_{p,e'} + b_{-p,e}^* b_{-p,e'} + b_{*}^* b_{-p,e'} + b_{p,e} b_{-p,e'})$$

It follows from (1) (by completing squares) that

$$h_p \geq -(\epsilon(p)/2 + \nu g(p)) + \sqrt{(\epsilon(p)/2 + \nu g(p))^2 - (\nu g(p))^2}$$

After replacing sums by integrals this gives

$$H_{\text{Box}} \geq -J \nu (\nu/\ell^3)^{1/4}$$
The localization scheme

THEOREM 5 (The sliding method (Conlon-Lieb-Yau 1988)).

\( \chi_z = "\text{smooth characteristic" function of } \ell\text{-cube centered at } z \in \mathbb{R}^3. \) Then

\[
\sum_{1 \leq i < j \leq N} \frac{e_i e_j}{|x_i - x_j|} \geq \int \sum_{1 \leq i < j \leq N} \chi_z(x_i) \frac{e_i e_j}{|x_i - x_j|} \chi_z(x_j) dz - C \frac{N}{\ell}
\]

THEOREM 6 (A many body kinetic energy bound (\( \ell = 1 \))).

\( \chi_z = "\text{smooth characteristic" function of unit cube centered at } z \in \mathbb{R}^3. \) 
\( a^*(z) \) creation operator of constant in cube. \( P_z = \text{projection orthogonal to constants in cube. } \Omega \subset \mathbb{R}^3. \) \( e_1, e_2, e_3 \) standard basis. For all \( 0 < s < 1 \)

\[
(1 + \varepsilon(\chi, s)) \sum_{i=1}^{N} -\Delta_i \geq \int _{\Omega} \left[ \sum_{i=1}^{N} P_z^{(i)} \chi_z^{(i)} \frac{(-\Delta_i)^2}{-\Delta_i + s^{-2} \chi_z^{(i)} P_z^{(i)}} + \sum_{j=1}^{3} \left( \sqrt{a_0^*(z + e_j) a_0(z + e_j) + 1/2} - \sqrt{a_0^*(z) a_0(z) + 1/2} \right)^2 \right] dz
\]

\(-3\text{vol}(\Omega), \quad \varepsilon(\chi, s) \to 0 \) as \( s \to 0.\)