THE IMPACT OF BIRMAN'S WORK

on the

THEORIES OF STABILITY OF MATTER

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The Slides can be seen at

`http://www.math.ku.dk/~solovej/slides.html`

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- Stability of relativistic matter interacting with classical electromagnetic fields
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Stability of Matter

N electrons and K nuclei. Hamiltonian:

$$H_{N,K} = \sum_{i=1}^{N} t_i + V_C.$$

The electron kinetic energy operator

non-relativistic case: $t = -\frac{1}{2}\Delta_{x_i}$. We shall consider other possible choices for t.

The Coulomb potential V_C is the function

$$\begin{aligned} V_C &= -\sum_{k=1}^{K}\sum_{i=1}^{N} Z_k |x_i - R_k|^{-1} + \sum_{i<j} |x_i - x_j|^{-1} \\ &\quad + \sum_{k<\ell} Z_k Z_\ell |R_k - R_\ell|^{-1} \end{aligned}$$

$R_1, \ldots, R_K \in \mathbb{R}^3$ are the nuclear coordinates.

Units: $m_e = \hbar = e = 1$.

Physical Hilbert space $\mathcal{H} = \bigwedge_i^N L^2(\mathbb{R}^3; \mathbb{C}^2)$.

$$
\begin{aligned}
H_{N,K} &= \sum_{i=1}^{N} t_i + V_C \qquad \mathcal{H} = \bigwedge_i^N L^2(\mathbb{R}^3;\mathbb{C}^2) \\
t &= -\frac{1}{2}\Delta_{x_i}. \\
V_C &= -\sum_{k=1}^{K}\sum_{i=1}^{N} Z_k|x_i - R_k|^{-1} + \sum_{i<j}|x_i - x_j|^{-1} \\
&\quad + \sum_{k<\ell} Z_k Z_\ell |R_k - R_\ell|^{-1}
\end{aligned}
$$

1. THEOREM. (Stability of Matter)

If $\psi \in \mathcal{H}$ normalized then uniformly in the nuclear positions

$$(H_{N,K}\psi,\psi) \geq -C(\max\{Z_k\})(K+N).$$

Dyson-Lenard, Federbush, Lieb-Thirring,...

Reducing problem to spectral estimates

Lieb-Yau inequality:

$$V_c \geq \sum_{i=1}^{N} W(x_i) + c \sum_{k=1}^{K} D_k^{-1} - cN, \; D_k = \min_{j \neq k} |R_j - R_k|$$

Here $W(x) \sim -Z_k |x - R_k|^{-1}$ when x near R_k.

Thus $(H_{N,K}\psi, \psi) \geq -\mathrm{Tr}_{L^2(\mathbb{R}^3;\mathbb{C}^2)} [t + W]_- - cN$.

Stability follows from $d = 3$, $s = 1$ case of

2. THEOREM.

$$\mathrm{Tr}_{L^2(\mathbb{R}^d)} [-\Delta + W]_-^s \leq L_s \int [W]_-^{s+(d/2)}.$$

Lieb-Thirring: $(d \geq 2,\ s > 0)$, $(d = 1,\ s > 1/2)$

Cwikel-Lieb-Rozenblum: $(d \geq 3,\ s = 0)$

Weidl: $(d = 1,\ s = 1/2)$

The Birman-Schwinger Principle:

(Birman 59, Schwinger 61)

If $W \leq 0$

$$N_e\left(-(-\Delta + W)\right) = N_1\left(\sqrt{[W]_-}(-\Delta + e)^{-1}\sqrt{[W]_-}\right)$$

$$N_e(A) := \text{ CARD}\left(\text{ spec}(A) \cap [e, -\infty)\right).$$

Lieb-Thirring's proof ($d = 3$, $s = 1$):

$$\begin{aligned}
\text{Tr}_{L^2(\mathbb{R}^d)}[-\Delta + W]_- &= \int_0^\infty N_{\frac{e}{2}}\left(-(-\Delta + W + \frac{e}{2})\right) de \\
\leq & \int_0^\infty N_1\left(\sqrt{[W + \frac{e}{2}]_-}(-\Delta + \frac{e}{2})^{-1}\sqrt{[W + \frac{e}{2}]_-}\right) de \\
\leq & \int_0^\infty \text{Tr}\left(\sqrt{[W + \frac{e}{2}]_-}(-\Delta + \frac{e}{2})^{-1}\sqrt{[W + \frac{e}{2}]_-}\right)^2 de \\
\leq & \int [W(x) + \frac{e}{2}]_- \frac{e^{(-2(\frac{e}{2})^{1/2}|x-y|)}}{|x-y|^2}[W(y) + \frac{e}{2}]_- dx dy de \\
\leq & \; L_1 \int [W(x)]_-^{1+(d/2)} dx
\end{aligned}$$

Using Cauchy-Schwarz in the last step.

Stability of Relativistic Matter

Now $t = \sqrt{-c^2\Delta + c^4}$ or simply $t = c\sqrt{-\Delta}$. Here the speed of light $c = \alpha^{-1} \approx 137$. Using the equivalent of Lieb-Thirring type inequalities for $\sqrt{-\Delta}$ due to Daubechies, Lieb and Yau prove:

3. THEOREM (Relativistic Stability). *If $\alpha \max\{Z_k\} \leq 2/\pi$ and α is small enough then*

$$\sum_{i=1}^{N} c\sqrt{-\Delta_i} + V_c \geq 0.$$

The condition $\alpha \max\{Z_k\} \leq 2/\pi$ is not only sufficient it is also necessary.

Matter interacting with classical electromagnetic fields

New dynamic variable $\mathbf{A}$.

Magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. t=Pauli=Dirac^2

$$t = [\boldsymbol{\sigma} \cdot (-i\nabla - \mathbf{A})]^2 = (-i\nabla - \mathbf{A})^2 - \mathbf{B} \cdot \boldsymbol{\sigma}.$$

Stability:

$$(H_{N,K}\psi, \psi) + \frac{1}{8\pi\alpha^2} \int |\mathbf{B}|^2 \geq -C(\max\{Z_k\})(K+N)$$

uniformly in nuclear coordinates and in $\mathbf{A}$.

4. THEOREM (Fefferman, Lieb-Loss-Solovej).

If $Z\alpha^2$ *small enough and* α *small enough then non-relativistic matter interacting with classical fields is stable.*

Relativistic matter interacting

with classical electromagnetic fields

$t =$ relativistic Dirac $= c\boldsymbol{\alpha} \cdot (-i\nabla - \mathbf{A}) + \beta c^2$

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-particle Hilbert space is $P(t > 0)L^2(\mathbb{R}^3; \mathbb{C}^4)$.

Stability:

$$(H_{N,K}\psi, \psi) + \frac{1}{8\pi\alpha^2} \int |\mathbf{B}|^2 \geq -C(\max\{Z_k\})(K+N)$$

for all $\psi \in \bigwedge^N P(t > 0)L^2(\mathbb{R}^3; \mathbb{C}^4)$ *uniformly in the nucelar coordinates and in* $\mathbf{A}$.

5. THEOREM (Lieb-Siedentop-Solovej).

If $Z\alpha$ *small enough and* α *small enough then relativistic matter interacting with classical fields is stable.*

Birman-Koplienko-Solomyak Inequality:

6. THEOREM (BKS '75). *If A and B are positive semi-definite operators then*

$$\mathrm{Tr}\left(\sqrt{A}-\sqrt{B}\right)_- \leq \mathrm{Tr}\sqrt{(A-B)_-}$$

Proof of Relativistic Stability using BKS:

Lieb-Yau+diamagnetism: $V_c \geq -\sum_{j=1}^{N} \kappa \left|-i\nabla_j - \mathbf{A}\right|$.

$$H_{N,K} \geq \sum_{j=1}^{N} \left[|t_j| - \kappa \left|-i\nabla_j - \mathbf{A}\right|\right]$$

$$(\psi, H_{N,K}\psi) \geq -\mathrm{Tr}\left[|t| - \kappa\left|-i\nabla - \mathbf{A}\right|\right]_-$$

$$\geq -\mathrm{Tr}\left[t^2 - \kappa^2(-i\nabla - \mathbf{A})^2\right]_-^{1/2}$$

$$\geq -\mathrm{Tr}\left[(c^2-\kappa^2)(-i\nabla - \mathbf{A})^2 - |\mathbf{B}|\right]_-^{1/2}$$

$$\geq -\mathrm{const}\int |\mathbf{B}|^2, \text{ By LT with } d=3, s=1/2.$$

For non-relativistic case use $t^2 \geq 2|t| - 1$.