Rigorous Results on the energy and structure of ground states of large many-body systems IV. Stability and Instability of matter

Jan Philip Solovej
Department of Mathematics
University of Copenhagen

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The Theorem on Stability of Matter

My goal in this last lecture is to discuss stability of fermionic matter and instability of bosonic matter.

THEOREM 1 (Stability of Matter). *Matter consisting of nuclei and fermionic electrons satisfies stability of the 2nd kind*

$$E^{\mathrm{F}}_{N,K} > -C(N+K).$$

This was first proved by Dyson and Lenard. Shortly after Lieb and Lebowitz used this to prove that the **thermodynamic limit exists** for ordinary matter.

Lieb and Thirring later gave a simplified proof of stability of matter using the Lieb-Thirring inequality and **Thomas-Fermi theory**. In Thomas-Fermi theory one has the celebrated **No-binding theorem**.

I will sketch a somewhat different proof based on a **correlation inequality** of Baxter; still using the Lieb-Thirring inequality.

Correlation estimates

THEOREM 2 (Baxter's correlation estimate). *For all* $z_1, \ldots, z_M \in \mathbb{R}$ *and* $x_1, \ldots, x_M \in \mathbb{R}^3$.

$$\sum_{1 \leq i < j \leq M} z_i z_j |x_i - x_j|^{-1} \geq \sum_{i,\ z_i < 0} z_i V(x_i)$$

where

$$V(x) = (2 \max_k \{z_k\} + 1) \max_{j: z_j > 0} \{|x - x_j|^{-1}\}.$$

An improvement (and more analytic proof) of this was given by Lieb and Yau. A simpler version was proved already by Onsager in 1939:

$$\sum_{1 \leq i < j \leq M} z_i z_j |x_i - x_j|^{-1} \geq -\sum_{i=1}^{M} z_i^2 \max_{j: z_i z_j < 0} \{|x_i - x_j|^{-1}\}$$

A proof of stability of matter

For matter with electrons of charge -1, Baxter's correlation inequality gives

$$H_{N,K} \geq \sum_{i=1}^{N} -\tfrac{1}{2}\Delta_i - V(x_i)$$

where

$$V(x) = (2\max_k\{Z_k\} + 1) \max_{j:z_j>0}\{|x - x_j|^{-1}\}$$

By the Lieb Thirring inequality

$$E_{N,K} \geq -C_{\mathrm{LT}} \int_\Lambda V(x)^{5/2} - N \sup_{\mathbb{R}^3\setminus\Lambda} V.$$

With an appropriate choice $\Lambda \subset \mathbb{R}^3$ we find

$$E_{N+K} \geq -C(N+K).$$

Instability of bosonic matter

THEOREM 3 (Dyson's formula. Lieb-Sol. 2004 ($\geq$), Sol. 2004($\leq$)).

$$\lim_{N\to\infty} \frac{E_N^B}{N^{7/5}} = \inf\left\{ \tfrac{1}{2}\int |\nabla\phi|^2 - J\int \phi^{5/2} \mid \phi \geq 0,\ \int \phi^2 = 1 \right\}$$

History: Dyson 1967 proved $E_N^{\mathrm{B}} \leq -CN^{7/5}$. Implies no stability ($7/5 > 1$). No thermodynamics. Dyson conjectured above formula.

Conlon-Lieb-Yau 1988: $E_N \geq -CN^{7/5}$.

A Hartree trial state $\phi(x_1)\cdots\phi(x_N)$ would give

$$\begin{aligned} E^{\mathrm{B}} &\leq N\tfrac{1}{2}\int |\nabla\phi|^2 + \left(\sum_{1\leq i<j\leq N} z_i z_j\right)\int \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|}dxdy \\ &= \frac{N}{2}\left(\int |\nabla\phi|^2 - \int \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|}dxdy\right), \end{aligned}$$

i.e., linear in N, assuming that $\sum_i z_i = 0$ and $\sum_i z_i^2 = N$.

Heuristic derivation of Dyson's formula

Most particles are in condensed state $\widetilde{\phi}$ with $\int \widetilde{\phi}^2 = 1$.

The local density is $\rho = N\widetilde{\phi}^2$.

Local energy density is according to Foldy $-J\rho^{5/4} = -JN^{5/4}\widetilde{\phi}^{5/2}$. It is not quite that simple. One again has to do a Bogolubov approximation. In this special case the Bogolubov approximation is in fact an exact upper bound. The local energy density is however still an approximation.

The kinetic energy of the condensate is $\frac{1}{2}N\int |\nabla\widetilde{\phi}|^2$

The total energy is then

$$\frac{1}{2}N\int |\nabla\widetilde{\phi}|^2 - JN^{5/4}\int \widetilde{\phi}^{5/2}$$

If we set $\phi(x) = N^{-3/10}\widetilde{\phi}(xN^{-1/5})$ then

$$\frac{1}{2}N\int |\nabla\widetilde{\phi}|^2 - JN^{5/4}\int \widetilde{\phi}^{5/2} = N^{7/5}\left(\frac{1}{2}\int |\nabla\phi|^2 - J\int \phi^{5/2}\right).$$

Conclusion

I have in these lectures discussed non-relativistic many-body quantum mechanics and reviewed som rigorous results. Among the things I have not discussed one can mention

- Excited states and positive temperature
- Relativistic effects and corrections
- Coupling to quantum fields, such as the electromagnetic field or other gauge fields or coupling to the gravitational field.
- Perturbation theory