1 The min-max theorem

Let q be a closed semi-bounded quadratic form on a dense domain $\mathcal{Q}$ in a Hilbert space $\mathcal{H}$. Let A be the self-adjoint operator corresponding to Q. The discrete points in the spectrum of A are eigenvalues of A. The *discrete spectrum* of A is the set of discrete points in the spectrum which are eigenvalues of finite multiplicity:

$$\sigma_{\rm d}(A) = \{\lambda \in \mathbb{R} \mid \lambda \text{ is a discrete eigenvalue of finite multiplicity}\}.$$

The discrete spectrum is a relatively open set in the spectrum and hence its complement *the essential spectrum*

$$\sigma_{\rm ess}(A) = \sigma(A) \setminus \sigma_{\rm d}(A)$$

is a closed set. We define the bottom of the essential spectrum

$$\Sigma(A) = \inf \sigma_{\rm ess}(A).$$

If $\sigma_{\rm ess}(A) = \emptyset$ we understand this as $\Sigma(A) = \infty$. Note that $\Sigma(A) > -\infty$ since A is a semibounded operator and hence the spectrum of A is contained in a half line. An operator for which $\Sigma(A) = \infty$ is said to have *compact resolvent*. We want to characterize the part of the spectrum below $\Sigma(A)$. It will consist entirely of eigenvalues and they may be calculated as what we call the min-max values of q.

Definition 1 (The min-max values)**.** The min-max values of q are

$$\mu_n = \inf\left\{ \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} \;\middle|\; M \subseteq \mathcal{Q} \text{ subspace},\ \dim(M) = n\right\}.$$

Note that the max is really a max and not just a sup, since we are taking the max of a continuous function q over the unit ball in a finite dimensional normed vector space M, where the unit ball is compact.

Lemma 1. *The min-max values form a non-decreasing (finite if $\mathcal{H}$ is finite dimensional) sequence, i.e.,*

$$-\infty < \mu_1 \le \mu_2 \le \cdots .$$

If $\mathcal{D} \subseteq \mathcal{Q}$ is a subspace which is dense in $\mathcal{Q}$ in the q-norm $\|\phi\|_q = (q(\phi) + (1-\gamma)\|\phi\|^2)^{1/2}$ (where γ is the lower bound on q) then

$$\mu_n = \inf\left\{ \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} \;\middle|\; M \subseteq \mathcal{D} \textit{ subspace},\ \textit{dim}(M) = n\right\}.$$

Proof. Since q is bounded below it is clear that $-\mu_1 > -\infty$. Assume that $\mathcal{H}$ has dimension at least $n+1$ (otherwise there are at most n min-max values). We will show that $\mu_n \leq \mu_{n+1}$. Given $\varepsilon > 0$ it is enough to show that $\mu_n \leq \mu_{n+1} + \varepsilon$. From the definition of the min-max values we may find a space $M \subseteq \mathcal{Q}$ of dimension $n+1$ such that

$$\mu_{n+1} \geq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} - \varepsilon.$$

Let M' be a subspace of M of dimension n. Hence

$$\begin{aligned} \mu_{n+1} + \varepsilon &\geq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} \\ &\geq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M'\} \geq \mu_n. \end{aligned}$$

To prove the second half of the lemma let us denote

$$\mu_n' = \inf\left\{ \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} \;\middle|\; M \subseteq \mathcal{D} \text{ subspace},\ \dim(M) = n \right\}.$$

Since a subspace of $\mathcal{D}$ is also a subspace of $\mathcal{Q}$ we immediately see that $\mu_n \leq \mu_n'$. To prove the opposite inequality given $0 < \varepsilon < 1$ choose an n-dimensional $M \subseteq \mathcal{Q}$ such that

$$\mu_n \geq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} - \varepsilon.$$

Let $\phi_1, \ldots \phi_n$ be an orthonormal basis for M. Then we can find $\phi_1', \ldots, \phi_n' \in \mathcal{D}$ such that $\|\phi_j - \phi_j'\| \leq \varepsilon$ for $j = 1, 2, \ldots, n$. Then since

$$\langle \phi, \psi \rangle_q = q(\phi, \psi) + (1-\gamma)\langle \phi, \psi \rangle$$

and $\|\phi\| \leq \|\phi\|_q$ for all $\phi, \psi \in \mathcal{Q}$ we obtain

$$\begin{aligned} |q(\phi_i', \phi_j') - q(\phi_i, \phi_j)| &\leq |\langle \phi_j', \phi_i' \rangle_q - \langle \phi_j, \phi_i \rangle_q| + |1-\gamma||\langle \phi_j', \phi_i' \rangle - \langle \phi_j, \phi_i \rangle| \\ &\leq \|\phi_j' - \phi_j\|_q \|\phi_i'\|_q + \|\phi_i' - \phi_i\|_q \|\phi_j\|_q \\ &\quad + |1-\gamma|(\|\phi_j' - \phi_j\|\|\phi_i'\| + \|\phi_i' - \phi_i\|\|\phi_j\|) \\ &\leq 2(1 + |1-\gamma|)(1+\varepsilon)\varepsilon \leq C\varepsilon \end{aligned}$$

for $C = 4(1 + |1-\gamma|)$. Hence

$$|q(\sum_{j=1}^n \alpha_j \phi_j') - q(\sum_{j=1}^n \alpha_j \phi_j)| \leq C \sum_{j=1}^n \sum_{i=1}^n |\alpha_i \alpha_j| \varepsilon \leq Cn\varepsilon \sum_{j=1}^n |\alpha_i|^2,$$

by the Cauchy-Schwarz inequality. It follows from this that

$$\begin{aligned} \mu_n' &\leq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M'\} \\ &\leq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} + Cn\varepsilon \leq \mu_n + (Cn+1)\varepsilon. \end{aligned}$$

Since this is true for all $\varepsilon > 0$ we conclude that $\mu_n' \leq \mu_n$. □

In particular, if A_0 is a semi-bounded symmetric operator on the domain $\mathcal{D}(A_0)$ then the form corresponding to A_0 may be closed to a quadratic form q and its min-max values may be calculated directly from the information about A_0

$$\mu_n = \inf\left\{ \max\{\langle\phi, A_0\phi\rangle \mid \|\phi\| = 1,\ \phi \in M\} \;\middle|\; M \subseteq \mathcal{D}(A_0) \text{ subspace}, \ \dim(M) = n \right\}.$$

We will therefore also refer to the min-max values as the min-max values of A_0. This holds in particular if A is the self-adjoint operator connected to a closed quadratic form. We shall use this in the proof of the min-max Theorem below. In particular, we point out that the min-max values of a semi-bounded symmetric operator are the same as the min-max values of its Friedrichs extension.

Theorem 1 (The min-max Theorem)**.** *Let A be the self-adjoint operator corresponding to a closed semi-bounded quadratic form. The min-max values of q satisfy $\mu_n \leq \Sigma(A)$. If $\mu_n < \Sigma(A)$ then μ_n is the n-th eigenvalue of A counted with multiplicity from the lowest value. If $\mu_n = \Sigma(A)$ then A has at most $n-1$ eigenvalues counted with multiplicity below $\Sigma(A)$.*

Proof. If $\Sigma(A) < \infty$ and $\varepsilon > 0$ then by the Spectral Theorem the spectral subspace

$$\mathbf{1}_{[\Sigma(A),\Sigma(A)+\varepsilon)}(\mathcal{H})$$

is infinite dimensional (otherwise $\Sigma(A)$ would be a discrete eigenvalue of finite multiplicity). We can thus find a subspace M of this space of arbitrarily high dimension. For all normalized $\phi \in M$ we have (again by the Spectral Theorem)

$$q(\phi) = \langle\phi, A\phi\rangle \leq \Sigma(A) + \varepsilon.$$

It follows that for all n, $\mu_n \leq \Sigma(A) + \varepsilon$. Since this is true for all $\varepsilon > 0$ we conclude that $\mu_n \leq \Sigma(A)$.

Assume that A has at least n eigenvalues counted with multiplicity below $\Sigma(A)$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \Sigma(A)$ denote the n lowest eigenvalues and let the corresponding orthonormal family of eigenvectors be $\phi_1, \ldots, \phi_n \in \mathcal{D}(A) \subseteq \mathcal{Q}$. Consider the n-dimensional space $M = \mathrm{span}\{\phi_1, \ldots, \phi_n\}$. Then, if $\phi \in M$ is normalized we have

$$q(\phi) = \langle \phi, A\phi \rangle = \sum_{j=1}^{n} \lambda_j |\langle \phi_j, \phi \rangle|^2 \leq \lambda_n.$$

We of course have equality above if $\phi = \phi_n$. Hence

$$\max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} = \lambda_n.$$

Thus $\mu_n \leq \lambda_n$. It follows that if $\mu_n \geq \Sigma(A)$ then A cannot have n eigenvalues below $\Sigma(A)$.

If on the other hand $\mu_n < \Sigma(A)$ we will now show that $\mu_n \geq \lambda_n$. Given $\varepsilon > 0$ it is enough to show that $\mu_n \geq \lambda_n - \varepsilon$. Choose $M \subseteq \mathcal{D}(A)$ with $\dim M = n$ such that

$$\mu_n \geq \max\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} - \varepsilon$$

The projection of $\mathrm{span}\{\phi_1, \ldots, \phi_{n-1}\}$ onto M is an at most $n-1$-dimensional subspace. Hence we can find a normalized $\phi \in M$ such that

$$\phi \perp \mathrm{span}\{\phi_1, \ldots, \phi_{n-1}\}.$$

It then follows by the Spectral Theorem that

$$q(\phi) = \langle \phi, A\phi \rangle \geq \lambda_n.$$

We conclude that $\mu_n \geq \lambda_n - \varepsilon$. □

If K is a bounded operator we can also define the *max-min* values

$$\nu_n = \sup\left\{ \min\{q(\phi) \mid \|\phi\| = 1,\ \phi \in M\} \,\middle|\, M \subseteq \mathcal{Q} \text{ subspace},\ \dim(M) = n \right\}.$$

We of course have a corresponding max-min Theorem. We leave it to the reader to formulate this.

Definition 2 (Compact, trace class and Hilbert Schmidt operators)**.** If K is a bounded operator such that $\lim_{n\to\infty} \nu_n(K^*K) = 0$ then K is said to be compact. We have the following two subclasses of the compact operators.

- If $\sum_{n=0}^{\infty} \nu_n(K^*K)^{1/2} < \infty$ then K is said to be trace class.
- If $\sum_{n=0}^{\infty} \nu_n(K^*K) < \infty$ then K is said to be Hilbert-Schmidt.

One can prove that the sets of compact, trace class, and Hilbert-Schmidt operators are subspaces.