# 1 The Lieb-Thirring inequality

We consider the Schrödinger operator

$$H = -\Delta + V$$

defined on $\mathcal{D}(H) = C_0^2(\mathbb{R}^d)$, which is dense in $L^2(\mathbb{R}^d)$. Here $V \in L^2_{\text{loc}}(\mathbb{R}^d)$, such that the operator maps into $L^2(\mathbb{R}^d)$.

**Theorem 1** (Lieb-Thirring inequality)**.** *If $H$ is as above and the negative part $V_- = \min\{V, 0\}$ of $V$ satisfies $V_- \in L^{\frac{d}{2}+1}(\mathbb{R}^d)$ then $H$ is bounded below and its min-max values $\mu_n$ satisfy the Lieb-Thirring inequality*

$$\sum_{n=1}^{\infty} [\mu_n]_- \geq -C_d \int |V_-(x)|^{\frac{d}{2}+1} dx.$$

From the Lieb-Thirring inequality we see that the bottom of the essential spectrum of $H$ must satisfy $\Sigma(H) \geq 0$, because otherwise all $\mu_n \leq \Sigma(H) < 0$ and hence we would have $\sum_{n=1}^{\infty} [\mu_n]_- = -\infty$. It follows that all negative min-max values are eigenvalues and hence the Lieb-Thirring inequality is an estimate on the sum of negative eigenvalues. The proof is based on M. Rumin, *Balanced distribution-energy inequalities and related entropy bounds*, Duke Math. J., 160 (2011), pp. 567-597.

*Proof of the Lieb-Thirring Inequality.*
**Step 1:** It is enough to show that

$$\sum_{j=1}^{N} \langle \phi_j, H\phi_j \rangle \geq -C_d \int |V_-(x)|^{\frac{d}{2}+1} dx \tag{1}$$

for all finite orthonormal families $\{\phi_j\}_{j=1}^N$ in $C_0^2(\mathbb{R}^d)$.

Indeed, if (1) holds it is clear that $H$ is bounded below (the case $N = 1$). Moreover, it is clear from (1) that $\Sigma(H) \geq 0$. Otherwise, we could find a subspace $M \subseteq C_0^2(\mathbb{R}^d)$ of arbitrarily large dimension $N$ such that $\langle \phi, H\phi \rangle < \Sigma(H)/2 < 0$ for all normalized $\phi \in M$. Choosing any orthonormal basis in $M$ the sum on the left side of (1) would be less than $-N\Sigma(H)/2$ contradicting (1). Since $\Sigma(H) \geq 0$ all negative min-max values are eigenvalues and we can therefore choose an orthonormal family in $C_0^2(\mathbb{R}^d)$ approximating the corresponding eigenvectors such that the left side of (1) approximates arbirarily well the sum of the negative eigenvalues.

**Step 2:** We will use the following convention for the Fourier transform of $f \in L^1(\mathbb{R}^d)$

23.3.2011 Convention of Fourier transform changed

$$\widehat{f}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ipx} dx.$$

Then the Fourier transform extends to a unitary map on $L^2(\mathbb{R}^d)$.

For all $e > 0$ and $\phi \in L^2(\mathbb{R}^d)$ we write

$$\phi = \phi^{e,+} + \phi^{e,-}$$

where

$$\widehat{\phi}^{e,+}(p) = \begin{cases} \widehat{\phi}(p), & p^2 > e \\ 0, & p^2 \le e \end{cases}, \qquad \widehat{\phi}^{e,-}(p) = \begin{cases} 0, & p^2 > e \\ \widehat{\phi}(p), & p^2 \le e \end{cases}.$$

Then since the Fourier transform is unitary we obtain for $\phi \in C_0^1(\mathbb{R}^d)$

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} |\phi^{e,+}(x)|^2 dx de &= \int_{\mathbb{R}^d} \int_0^\infty |\widehat{\phi}^{e,+}(p)|^2 dp de \\ &= \int_{\mathbb{R}^d} \int_0^{p^2} |\widehat{\phi}(p)|^2 de dp \\ &= \int_{\mathbb{R}^d} p^2 |\widehat{\phi}(p)|^2 dx = \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 dx. \end{aligned} \tag{2}$$

**Step 3:** For any family $\{\phi_j\}_{j=1}^N$ in $L^2(\mathbb{R}^d)$ we have using the triangle inequality in $\mathbb{C}^N$

$$\left( \sum_{j=1}^N |\phi_j{}^{e,+}(x)|^2 \right)^{1/2} \ge \left[ \left( \sum_{j=1}^N |\phi_j(x)|^2 \right)^{1/2} - \left( \sum_{j=1}^N |\phi_j{}^{e,-}(x)|^2 \right)^{1/2} \right]_+ \tag{3}$$

for all $e > 0$. Here we have again used $t_+ = \max\{t, 0\}$.

**Step 4:** For any orthonormal family $\{\phi_j\}_{j=1}^N$ we have using Bessel's inequality that for all $x \in \mathbb{R}^d$

$$\begin{aligned} \sum_{j=1}^N |\phi_j^{e,-}(x)|^2 &= \sum_{j=1}^N \left| (2\pi)^{-d/2} \int e^{ipx} \mathbf{1}_{(0,e)}(p^2) \widehat{\phi}_j(p) dp \right|^2 \\ &\le (2\pi)^{-d} \int |e^{ipx} \mathbf{1}_{(0,e)}(p^2)|^2 dp = (2\pi)^{-d} \kappa_d e^{d/2}. \end{aligned} \tag{4}$$

**Step 5:** Combining (2–4) we obtain for any orthonormal family $\{\phi_j\}_{j=1}^N$

$$\begin{aligned}\sum_{j=1}^N \int_{\mathbb{R}^d} |\nabla\phi_j(x)|^2 dx &\geq \int\int_0^\infty \left[\left(\sum_{j=1}^N |\phi_j(x)|^2\right)^{1/2} - (2\pi)^{-d/2}\kappa_d^{1/2} e^{d/4}\right]_+^2 dedx \\ &\geq \frac{(2\pi)^2 d^2 \kappa_d^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \left(\sum_{j=1}^N |\phi_j(x)|^2\right)^{\frac{d+2}{d}} dx.\end{aligned}$$

**Step 6:** Using Step 5 and Hölder's inequality we arrive at the final result as follows

$$\begin{aligned}\sum_{j=1}^N \langle\phi_j, H\phi_j\rangle &= \sum_{[j=1}^N \int |\nabla\phi_j(x)|^2 + V(x)|\phi_j(x)|^2 dx \\ &\geq \sum_{j=1}^N \int |\nabla\phi_j(x)|^2 - |V_-(x)||\phi_j(x)|^2 dx \\ &\geq \frac{(2\pi)^2 d^2 \kappa_d^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \left(\sum_{j=1}^N |\phi_j(x)|^2\right)^{\frac{d+2}{d}} dx \\ &\quad - \left(\int |V_-(x)|^{\frac{d+2}{2}} dx\right)^{\frac{2}{d+2}} \left(\int_{\mathbb{R}^d} \left(\sum_{j=1}^N |\phi_j(x)|^2\right)^{\frac{d+2}{d}} dx\right)^{\frac{d}{d+2}} \\ &\geq -\frac{2(2\pi)^{-d}\kappa_d}{d+2}\left(\frac{d+4}{d}\right)^{d/2} \int |V_-(x)|^{\frac{d+2}{2}} dx,\end{aligned}$$

where we have used the fact that the function $\mathbb{R}_+ \ni t \mapsto At - Bt^{\frac{d}{d+2}}$ for $A, B > 0$ has the minimal value $-\frac{2}{d+2}\left(\frac{d}{d+2}\right)^{d/2} A^{-d/2} B^{\frac{d+2}{2}}$. □

The estimate in the Lieb-Thirring inequality should be compared to the classical phase space integral

$$(2\pi)^{-d}\iint [p^2 + V(x)]_- dpdx = -\frac{2(2\pi)^{-d}\kappa_d}{d+2}\int |V_-(x)|^{\frac{d+2}{2}} dx.$$

The celebrated Lieb-Thirring conjecture states that the inequality holds with the classical constant above, i.e.,

$$\sum_{n=1}^\infty [\mu_n]_- \geq (2\pi)^{-d}\int [p^2 + V(x)]_- dpdx.$$

As we saw the Lieb-Thirring inequality implies that $\Sigma(H) \geq 0$. If $V$ tends to zero at infinity we have $\Sigma(H) \leq 0$.

**Theorem 2.** *If $V \in L^2_{\rm loc}(\mathbb{R}^d)$ is such that $\{x \mid V(x) > \varepsilon\}$ is a bounded set (except for a set of measure zero) for all $\varepsilon > 0$ then $\Sigma(H) \leq 0$.*

*Proof.* Given $\varepsilon > 0$ we will show that there exists a subspace $M$ of $C_0^2(\mathbb{R}^d)$ of arbitrarily large dimension such that $\langle \phi, H\phi\rangle \leq \varepsilon\|\phi\|^2$ for all $\phi \in M$. In order to construct $M$ choose $g \in C_0^2(\mathbb{R}^d)$ with $\int |g|^2 = 1$. Choose $R$ so large that

$$R^{-2}\int |\nabla g|^2 < \varepsilon/2.$$

Given an integer $N$ we may define $\phi_j(x) = R^{-d/2} g((x-u_j)/R)$, $j = 1, \ldots, N$ where the $u_j \in \mathbb{R}^d$ are chosen in such a way that all $\phi_j$ have disjoint support and are supported away from the set $\{x \mid V(x) > \varepsilon/2\}$. Then the $\phi_j$ form an orthonormal family. If $\phi = \sum_{j=1}^N \alpha_j\phi_j$ then

$$\langle \phi, H\phi\rangle \leq \sum_{j=1}^N |\alpha_j|^2 \left(\int R^{-2}|\nabla g|^2 + \varepsilon/2\int |g|^2\right) \leq \varepsilon \sum_{j=1}^N |\alpha_j|^2.$$

Hence the space $M = \operatorname{span}\{\phi_1, \ldots, \phi_N\}$ has the desired property. □

# 2 The CLR bound

We will now prove a bound on the number of negative eigenvalues in dimension $d \geq 3$. The argument is due to Rupert Frank.

**STEP 1:**
We proceed as in the proof of the Lieb-Thirring inequality, but instead of assuming that $\{\phi_j\}_{j=1}^N$ is an orthonormal family, we assume that $\{\sqrt{-\Delta}\phi_j\}_{j=1}^N$ is an orthonormal family. The calculation of the kinetic energy is unchanged, but Bessel's inequality now yields (under the assumption $d \geq 3$),

$$\begin{aligned}\sum_{j=1}^N |\phi_j^{e,-}(x)|^2 &= \sum_{j=1}^N \left|(2\pi)^{-d/2}\int \frac{e^{ipx}\mathbf{1}_{(0,e)}(p^2)}{|p|}\big(|p|\widehat{\phi}_j(p)\big)\,dp\right|^2 \\ &\leq (2\pi)^{-d}\int \frac{|e^{ipx}\mathbf{1}_{(0,e)}(p^2)|^2}{|p|^2}\,dp = (2\pi)^{-d}\kappa_d e^{(d-2)/2}. \qquad (5)\end{aligned}$$

**STEP 2:**
Upon inserting (5) in the old **Step 5**, we get (with $\rho(x) = \sum_{j=1}^N |\phi_j(x)|^2$)

$$\sum_{j=1}^N \int_{\mathbb{R}^d} |\nabla\phi_j(x)|^2 dx \geq \int\int_0^\infty \Big[\rho(x)^{1/2} - (2\pi)^{-d/2}\kappa_d^{1/2} e^{(d-2)/4}\Big]_+^2 de dx$$
$$\geq C_d \int \rho(x)^{1+\frac{2}{d-2}}\, dx.$$

**STEP 3:**
Let now $\{\psi_1, \ldots, \psi_N\}$ be linearly independent eigenvectors of $H$ with eigenvalue below 0 (or more generally satisfy $1_{(-\infty,0]}(H)\psi_j = \psi_j$). We can now apply Gram-Schmidt to obtain a collection $\{\phi_1, \ldots, \phi_N\}$ with $\langle\sqrt{-\Delta}\phi_j, \sqrt{-\Delta}\phi_k\rangle = \delta_{j,k}$.

Notice that the number $N$ is unchanged because if $\sqrt{-\Delta}\psi = 0$, then $|p|\widehat{\psi}(p) = 0$ in $L^2$ which implies that $\widehat{\psi} = 0$ almost everywhere.

We now get that (using Hölder and STEP 2)

$$\begin{aligned} 0 \geq \sum_{j=1}^N \langle\phi_j, (-\Delta - V)\phi_j\rangle &= N - \int V(x)\rho(x)\, dx \\ &\geq N - (\int V^{d/2})^{2/d}(\int \rho^{d/(d-2)})^{(d-2)/d} \\ &\geq N - C(\int V^{d/2})^{2/d} N^{(d-2)/d}. \end{aligned} \tag{6}$$

A Hölder inequality now gives that

$$N \leq \tilde{C}\int V^{d/2}. \tag{7}$$