ON THE K-THEORY OF HIGHER RANK GRAPH C^*-ALGEBRAS.

D. GWION EVANS

Abstract. Given a row-finite k-graph Λ with no sources we investigate the K-theory of the higher rank graph C^*-algebra, $C^*(\Lambda)$. When $k = 2$ we are able to give explicit formulae to calculate the K-groups of $C^*(\Lambda)$. The K-groups of $C^*(\Lambda)$ for $k > 2$ can be calculated under certain circumstances. We state that for all k, the torsion-free rank of $K_0(C^*(\Lambda))$ and $K_1(C^*(\Lambda))$ are equal when $C^*(\Lambda)$ is unital, and we determine the position of the class of the unit of $C^*(\Lambda)$ in $K_0(C^*(\Lambda))$.

1. Introduction

In [23], Spielberg realised that a crossed product algebra $C(\Omega) \rtimes \Gamma$, where Ω is the boundary of a certain tree and Γ is a free group, is isomorphic to a Cuntz-Krieger algebra [5, 4]. Noticing that such a tree may be regarded as an affine building of type $\tilde{A}_1$, Robertson and Steger studied the situation when a group Γ acts simply transitively on the verices of an affine building of type $\tilde{A}_2$ with boundary Ω [18]. They found that the corresponding crossed product algebra $C(\Omega) \rtimes \Gamma$ is generated by two Cuntz-Krieger algebras. This led them to define a C^*-algebra $\mathcal{A}$ via a finite sequence of finite 0–1 matrices (i.e. matrices with entries in $\{0, 1\}$) $M_1, \ldots, M_r$ satisfying certain conditions (H0)-(H3), such that $\mathcal{A}$ is generated by r Cuntz-Krieger algebras, one for each $M_1, \ldots, M_r$. Accordingly they named their algebras higher rank Cuntz-Krieger algebras, the rank being r.

Kumjian and Pask [11] noticed that Robertson and Steger had constructed their algebras from a set, W of *(higher rank) words* in a finite *alphabet* A - the common index set of the 0–1 matrices - and realised that W could be thought of as a special case of a generalised directed graph - a higher rank graph. Subsequently, Kumjian and Pask associated a C^*-algebra, $C^*(\Lambda)$ to the higher rank graph Λ and showed that $\mathcal{A} \cong C^*(W)$ [11, Corollary 3.5 (ii)]. Moreover, they derived a number of results elucidating the structure of higher rank graph C^*-algebras. They show in [11, Theorem 5.5] that a simple, purely infinite k-graph C^*-algebra $C^*(\Lambda)$ may be classified by its K-theory. This is a consequence of $C^*(\Lambda)$ satsifying the hypotheses of the Kirchberg-Phillips classificaiton theorem ([10, 15]). Moreover, criteria on the underlying k-graph Λ were found that decided when $C^*(\Lambda)$ was simple and purely infinite (see [11, Proposition 4.8, Proposition 4.9]). Thus a step towards the classification of k-graph C^*-algebras is the computation of their K-groups.

Date: 22 June 2004.

1991 *Mathematics Subject Classification.* Primary 46L80; Secondary 46L35.

Key words and phrases. K-theory, C^*-algebra, k-graphs, graph algebra.

Research supported by the European Union Research Training Network in Quantum Spaces - Noncommutative Geometry.

In [20, Proposition 4.1] Robertson and Steger proved that the K-groups of a rank 2 Cuntz-Krieger algebra is given in terms of the homology of a certain chain complex, whoose differentials are defined in terms of $M_1, \ldots, M_r$. Their proof relied on the fact that a rank 2 Cuntz-Krieger algebra is stably isomorphic to a crossed product of an AF-algebra by $\mathbb{Z}^2$. We will generalise their method to provide explicit formulae for the K-groups of 2-graph C^*-algebras and to gain information on the K-groups of k-graph C^*-algebras for $k > 2$.

The rest of this paper is organised as follows. We begin in §2 by recalling the fundamental definitions we will need from [11]. In §3 we show that the C^*-algebra of a row-finite k-graph Λ with no sources is stably isomorphic to a crossed product of an AF algebra B by $\mathbb{Z}^k$. This enables us to apply a theorem of Kasparov [9, 6.10 Theorem] to deduce that there is a homological spectral sequence ([25, Chapter 5]) converging to $K_*(C^*(\Lambda))$ with initial term $E^2_{p,q} \cong H_p(\mathbb{Z}^k, K_q(B))$ (see [3] for the definition of the homology of a group G with coefficients in a left G-module M, denoted by $H_*(G, M)$). We will see that it suffices to compute $H_*(\mathbb{Z}^k, K_0(B))$. It transpires that $H_*(\mathbb{Z}^k, K_0(B))$ is given by the so called vertex matrices of Λ. These are matrices over the non-negative integers that encode the structure of the category Λ. Next we assemble the results of §3 and state them in our main theorem, Theorem 1. We then specialise to the cases $k = 2$ and $k = 3$. For $k = 2$ a complete description of the K-groups in terms of the vertex matrices can be given. For $k = 3$ we illustrate how Theorem 1 can be used to give a description of the K-groups of 3-graph C^*-algebras under stronger hypotheses.

In section §4 we consider the K-theory of unital k-graph C^*-algebras. We show that the torsion-free rank of $K_0(C^*(\Lambda))$ is equal to that of $K_1(C^*(\Lambda))$ when $C^*(\Lambda)$ is unital and give formulae for the torsion-free rank and torsion parts of the K-groups of 2-graph C^*-algebras. Finally, we determine the position of the class of the unit of $C^*(\Lambda)$ in $K_0(C^*(\Lambda))$ to facilitate the application of the Kirchberg-Phillips classification theorem.

This paper was written while the author was an European Union Network in Quantum Spaces – Non-Commutative Geometry funded post-doc at the University of Copenhagen. The paper is based on a part of the author's PhD thesis, which was written under the supervision of David E. Evans at Cardiff University. I would like to take this opportunity to thank David for his guidance and support, and Johannes Kellendonk and Ryszard Nest for useful discussions on homological algebra. Thanks are due to the operator algebra groups in both universities also, for creating stimulating environments for research.

2. Preliminaries

The following notation will be used throughout this paper. We let $\mathbb{N}$ denote the abelian monoid of non-negative integers and we let $\mathbb{Z}$ be the group of integers. Note that a monoid N (hence a group) can be considered as a category with one object and morphism set equal to N, with composition given by multiplication in the monoid. For a positive integer k, we let $\mathbb{N}^k$ be the product monoid viewed as a category. Similarly, we let $\mathbb{Z}^k$ be the product group viewed, where appropriate, as a category. Let $\{e_i\}_{i=1}^k$ be the canonical generators of $\mathbb{N}^k$ as a monoid. Moreover, we choose to endow $\mathbb{N}^k$ and $\mathbb{Z}^k$ with the coordinatewise order induced by the usual order on $\mathbb{N}$ and $\mathbb{Z}$, i.e. for all $m, n \in \mathbb{Z}^k$ $m \leq n \iff m - n \in \mathbb{N}^k$. By slight abuse of notation we shall let the set of morphisms of a small category $\mathcal{C}$ be denoted by $\mathcal{C}$ and identify an object of $\mathcal{C}$ with its corresponding identity morphism.

The concept of a *higher rank graph* or k-graph ($k = 1, 2, \ldots$ being the rank) was introduced by A. Kumjian and D. Pask in [11]. We recall their definition of a k-graph.

Definition 1 ([11, Definitions 1.1])**.** A k**-graph** (rank k graph or higher rank graph) (Λ, d) consists of a countable small category Λ (with range and source maps r and s respectively) together with a functor $d : \Lambda \longrightarrow \mathbb{N}^k$ satisfying the **factorisation property:** for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m+n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$. For $n \in \mathbb{N}^k$ and $v \in \Lambda^0$ we write $\Lambda^n := d^{-1}(n)$, $\Lambda(v) := r^{-1}(v)$ and $\Lambda^n(v) := \{\lambda \in \Lambda^n \mid r(\lambda) = v\}$.

Definition 2 ([11, Definitions 1.4])**.** A k-graph Λ is **row-finite** if for each $n \in \mathbb{N}^k$ and $v \in \Lambda^0$ the set $\Lambda^n(v)$ is finite. We say that Λ has **no sources** if $\Lambda^n(v) \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

We refer to [12] as an appropriate reference on category theory. There is no need for a detailed knowledge of category theory as we will be interested in the combinatorial graph-like nature of higher rank graphs. As the name suggests a higher rank graph can be thought of as a higher rank analogue of a directed graph. Indeed, every 1-graph is isomorphic (in the natural sense) to the category of finite paths of a directed graph ([11, Example 1.3]). For examples of k-graphs see [11]. By [11, Remarks 1.2] Λ^0 is the set of identity morphisms of Λ. Indeed it is fruitful to view Λ^0 as a set of vertices and Λ as a set of (coloured) paths with composition in Λ being concatention of paths. This viewpoint is discussed further in [6, 17]. We will let (Λ, d) (or more succinctly Λ with the understanding that the degree functor will be denoted by d) stand for a generic row-finite k-graph with no sources.

Given a countable group G, a k-graph Λ and a functor $c : \Lambda \longrightarrow G$, Kumjian and Pask defined the *skew-product* k-graph $G \times_c \Lambda$ as follows [11, Definition 5.1] the objects are identified with $G \times \Lambda^0$ and the morphisms are identified with $G \times \Lambda$ with the following structure maps: $s(g, \lambda) = (gc(\lambda), s(\lambda))$ and $r(g, \lambda) = (g, r(\lambda))$. If $s(\lambda) = r(\mu)$ then (g, λ) and $(gc(\lambda), \mu)$ are composable in $G \times_c \Lambda$ and $(g, \lambda)(gc(\lambda), \mu) = (g, \lambda\mu)$. The degree map is given by $d(g, \lambda) = d(\lambda)$.

3. The K-groups of k-graph C^*-algebras

We begin by noticing that $C^*(\Lambda)$ is stably isomorphic to a crossed product of an AF-algebra by $\mathbb{Z}^k$. As observed by Kumjian and Pask [11, §5], given a k-graph Λ with degree map d, we may form the skew-product graph $\mathbb{Z}^k \times_d \Lambda$ by considering d as a functor from Λ into $\mathbb{Z}^k$. Kumjian and Pask showed that $C^*(\mathbb{Z}^k \times_d \Lambda)$ is an AF-algebra as follows.

Lemma 1 ([11, Lemma 5.4])**.** *Let Λ be a k-graph and suppose there is a map $b : \Lambda^0 \longrightarrow \mathbb{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$, then $C^*(\Lambda)$ is AF.*

In the proof of [11, Theorem 5.5] Kumjian and Pask noticed that the map $b : \mathbb{Z}^k \times_d \Lambda : (n, \lambda) \mapsto n$ satisfies the hypothesis of Lemma 1, and they deduced that $C^*(\mathbb{Z}^k \times_d \Lambda)$ is AF. Henceforth we denote $C^*(\mathbb{Z}^k \times_d \Lambda)$ by B.

Moreover, by Remark 5.6 and its preceeding paragraph in [11], $\mathbb{Z}^k$ acts freely on $\mathbb{Z}^k \times_d \Lambda$ by $n(m, \lambda) = (m + n, \lambda)$ for all $m, n \in \mathbb{Z}^k$ and $\lambda \in \Lambda$. Thus there is an action β on B induced by the above action of $\mathbb{Z}^k$ on $\mathbb{Z}^k \times_d \Lambda$ such that

$\beta_n(s_{(m,\lambda)}) = s_{(m+n,\lambda)}$ for all $m, n \in \mathbb{Z}^k$ and $\lambda \in \Lambda$. Furthermore, by [11, Theorem 5.7], we can see now that $C^*(\Lambda)$ is stably isomorphic to a crossed product of an AF-algebra by $\mathbb{Z}^k$, namely

$$C^*(\Lambda) \otimes \mathbb{K} \cong B \rtimes_\beta \mathbb{Z}^k.$$

Hence, as in the proof of [20, Proposition 4.1], by [9, 6.10 Theorem], the K-groups of $C^*(\Lambda)$ are given by a homological spectral sequence with E^2 term given by $E^2_{p,q} \cong H_p(\mathbb{Z}^k, K_q(B))$. For future reference, we state this in the following Lemma.

Lemma 2 (c.f. [20, Proposition 4.1]). *There exists a spectral sequence (E^r, d^r) converging to $K_*(C^*(\Lambda)) := \bigoplus_{n\in\mathbb{Z}} H_n$ where*

$$H_n := \begin{cases} K_0(C^*(\Lambda)) & \text{if } n \text{ is even,} \\ K_1(C^*(\Lambda)) & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, for $p, q \in \mathbb{Z}$,

$$E^2_{p,q} \cong \begin{cases} H_p(\mathbb{Z}^k, K_0(B)) & \text{if } p \in \{0, 1, \ldots, k\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

$E^\infty \cong E^{k+1}$ and $E^{k+1}_{p,q} = 0$ if $p \in \mathbb{Z}\backslash\{0, 1, \ldots, k\}$ or q is odd.

Proof. The first assertion follows from [9, 6.10 Theorem] applied to $B \rtimes_\beta \mathbb{Z}^k \cong C^*(\Lambda)$ after noting that $K_*(B \rtimes_\beta \mathbb{Z}^k)$ coincides with its "γ-part" since the Baum-Connes Conjecture with coefficients in an arbitrary C^*-algebra is true for the amenable group $\mathbb{Z}^k$ for all $k \geq 1$.

By the proof of [9, 6.10 Theorem], $K_*(B \rtimes_\beta \mathbb{Z}^k) \cong K_*(D)$ for some C^*-algebra D which has a finite filtration by ideals: $0 \subset D_0 \subset D_1 \subset \cdots \subset D_k = D$ since the dimension of the universal covering space of the classifying space of $\mathbb{Z}^k$ is k.

The spectral sequence we are considering is the spectral sequence $\{(E^r, d^r)\}$ in homology K_* associated with the finite filtration $0 \subset D_0 \subset D_1 \subset \cdots \subset D_k = D$ of D ([21, §6]) which has $E^1_{p,q} = K_{(p+q \mod 2)}(D_p/D_{p-1})$ where $D_n = 0$ for $n < 0$ and $D_n = D$ for $n \geq k$. It follows easily that $E^r_{p,q} = 0$ for $p \in \mathbb{Z}\backslash\{0, 1, \ldots, k\}$, for all $q \in \mathbb{Z}$ and for all $r \geq 1$ and $E^\infty \cong E^{k+1}$ (see also [21, Theorem 2.1]). This combined with Kasparov's calculation in the proof of [9, 6.10 Theorem], giving $E^2_{p,q} \cong H_p(\mathbb{Z}^k, K_q(B))$, along with the observation that $K_q(B) = 0$ for odd q, since B is an AF-algebra, proves the second assertion. □

Now we will compute $H_*(\mathbb{Z}^k, K_0(B))$. First, let us examine the structure of B in a little more detail. As noted earlier, the map $b : (\mathbb{Z}^k \times_d \Lambda)^0 \longrightarrow \mathbb{Z}^k : (n, \lambda) \mapsto n$ satisfies $b(s(n, \lambda)) - b(r(n, \lambda)) = d(n, \lambda)$ for all $(n, \lambda) \in \mathbb{Z}^k \times_d \Lambda$. Also note that for all $n \in \mathbb{Z}^k$, $b^{-1}(n) = \{n\} \times \Lambda^0$ and we may identify $s^{-1}(n, v)$ with $s^{-1}(v)$ via $(n - d(\lambda), \lambda) \mapsto \lambda$ for all $\lambda \in s^{-1}(v)$, $v \in \Lambda^0$. Thus, by the proof of [11, Lemma 5.4], $B = \overline{\bigcup_{n\in\mathbb{Z}^k} B_n}$ where $B_n = \bigoplus_{v\in\Lambda^0} B_n(v)$ and $B_n(v) := \overline{\text{span}}\{s_\lambda s^*_\mu \mid \lambda, \mu \in \mathbb{Z}^k \times_d \Lambda,\ s(\lambda) = s(\mu) = (n, v)\} \cong \mathbb{K}(\ell^2(s^{-1}(v)))$ for all $v \in \Lambda^0$ and $n \in \mathbb{Z}^k$.

Definition 3. Let $\mathbb{Z}\Lambda^0$ be the group of all maps from Λ^0 into $\mathbb{Z}$ that have finite support under pointwise addition.

Definition 4 (c.f [11, §6]). Define the vertex matrices of Λ, M_i, by the following. For $u, v \in \Lambda^0$ and $i = 1, 2, \ldots, k$, $M_i(u, v) := |\{\lambda \in \Lambda^{e_i} \mid r(\lambda) = u, s(\lambda) = v\}|$. Also, let $(M^t_1, \ldots, M^t_k)^n := \prod_{j=1}^k (M^t_j)^{n_j}$ for all $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, where S^t denotes the transpose of a matix S.

Lemma 3 (c.f. [20] Lemma 4.5 and [14] Proposition 4.1.2)**.** *For all $n, m \in \mathbb{Z}^k$ such that $m \le n$, the groups $A_m := \mathbb{Z}\Lambda^0$ and homomorphisms $\jmath_{nm} : A_m \longrightarrow A_n$ defined by $\jmath_{nm} := (M_1^t, \dots, M_k^t)^{(n-m)}$ form a direct system. Moreover the groups $K_0(B)$ and $\lim_{\rightarrow}(A_m; \jmath_{nm})$ are isomorphic.*

Proof. It is clear that $(A_m; \jmath_{nm})$ is a direct system. Using the above notation we see that $K_0(B) \cong \varinjlim(K_0(B_n); K_0(\iota_{n,m}))$ where for $m, n \in \mathbb{Z}^k$ with $m \le n$, $\iota_{n,m} : B_m \longrightarrow B_n$ are the inclusion maps [24, Proposition 6.2.9].

For $v \in \Lambda^0$, $K_0(B_n(v))$ is isomorphic to $\mathbb{Z}$ and is generated by the class of any minimal projection in $B_n(v)$. Therefore to get a set of generators for $K_0(B_n) \cong \bigoplus_{v\in\Lambda^0} K_0(B_n(v))$ it suffices to choose a minimal projection from each $B_n(v)$. We will choose $\{[p_{(n,v)}]_n \mid v \in \Lambda^0\}$ to be our generators of $K_0(B_n)$, where $[\,\cdot\,]_n$ denotes the equivalence classes of $K_0(B_n)$ for all $n \in \mathbb{Z}^k$. Thus the map $\theta_n : A_n \longrightarrow K_0(B_n)$ given by $f \mapsto \sum_{u\in\Lambda^0} f(u)[p_{(n,u)}]_n$ is a group isomorphism for all $n \in \mathbb{Z}^k$.

The embedding $\iota_{n,m} : B_m \longrightarrow B_n$ is given by

$$\iota_{n,m}(s_{(m-d(\lambda),\lambda)} s^*_{(m-d(\mu),\mu)}) = \sum_{\alpha\in\Lambda^{n-m}(s(\lambda))} s_{(m-d(\lambda),\lambda\alpha)} s^*_{(m-d(\mu),\mu\alpha)}$$

for all $\lambda, \mu \in \Lambda$. Therefore,

$$\begin{aligned} K_0(\iota_{n+e_i,n})\left([p_{(n,v)}]_n\right) &= \left[\iota_{n+e_i,n}\left(p_{(n,v)}\right)\right]_{n+e_i} = \left[\sum_{\alpha\in\Lambda^{e_i}(v)} p_{(n,\alpha)}\right]_{n+e_i} \\ &= \sum_{u\in\Lambda^0} M_i(v,u)[p_{(n+e_i,u)}]_{n+e_i} \end{aligned}$$

and

$$\begin{aligned} K_0(\iota_{n+e_i,n})\left(\sum_{v\in\Lambda^0} f(v)[p_{(n,v)}]_n\right) &= \sum_{u\in\Lambda^0}\left(\sum_{v\in\Lambda^0} M_i(v,u)f(v)\right)[p_{(n+e_i,u)}]_{n+e_i} \\ &= \sum_{u\in\Lambda^0} (M_i^t f)(u)[p_{(n+e_i,u)}]_{n+e_i}. \end{aligned}$$

Thus the following squares commute for all $i \in \{1, 2, \dots, k\}$.

$$\begin{array}{ccc} K_0(B_n) & \xrightarrow{K_0(\iota_{n+e_i,n})} & K_0(B_{n+e_i}) \\ \theta_n \uparrow & & \uparrow \theta_{n+e_i} \\ A_n & \xrightarrow{\jmath_{n+e_i,n}} & A_{n+e_i} \end{array}$$

The result follows. □

Henceforth, we shall follow the notation in Lemma 3.

Lemma 4 (c.f. [20] Lemma 4.10)**.** *Fix $i \in \{1, \dots, k\}$ and let $\phi_{i,n} := M_i^t$ for all $n \in \mathbb{Z}^k$. Let the homomorphism induced by the system of homomorphisms $\{\phi_{i,n} : A_n \longrightarrow A_n \mid n \in \mathbb{Z}^k\}$ be denoted by $\phi_i : \lim_{\rightarrow}(A_m; \jmath_{nm}) \longrightarrow \lim_{\rightarrow}(A_m; \jmath_{nm})$. Then $\psi\phi_i = K_0(\beta_{e_i})\psi$, where $\psi : \lim_{\rightarrow}(A_m; \jmath_{nm}) \longrightarrow K_0(B)$ is the isomorphism constructed in Lemma 3.*

Proof. Fix $i \in \{1, \ldots, k\}$ and let $A := \lim_{\to}(A_m; \jmath_{nm})$. First we should check that ϕ_i is well-defined by showing that

$$\begin{array}{ccc} A_m & \xrightarrow{(M_1,\ldots,M_k)^{n-m}} & A_n \\ {\scriptstyle M_i^t}\downarrow & & \downarrow{\scriptstyle M_i^t} \\ A_m & \xrightarrow{(M_1,\ldots,M_k)^{n-m}} & A_n \end{array}$$

is commutative for all $i \in \{1, \ldots, k\}$ and $n, m \in \mathbb{Z}^k$ such that $n \geq m$. However, this is clear since composition of the maps involved is merely matrix multiplication and the vertex matrices of Λ commute [11, §6].

Next, we let $\tilde{\psi} : A \longrightarrow \lim_{\to}(K_0(B_m); K_0(\iota_{n,m}))$ be the unique isomorphism such that $K_0(\iota_n) \circ \theta_n = \tilde{\psi} \circ \jmath_n$ for all $n \in \mathbb{Z}^k$ where $\theta_n : A_n \longrightarrow K_0(B_n) : f \mapsto \sum_{u \in \Lambda^0} f(u)[p_{(n,u)}]_n$ (c.f. proof of Lemma 3). Then $\psi : A \longrightarrow K_0(B)$ is the composition of $\tilde{\psi}$ with the canonical isomorphism of $\lim_{\to}(K_0(B_n); K_0(\iota_{n,m}))$ onto $K_0(B)$.

Finally we will prove that

$$\begin{array}{ccc} K_0(B_n) & \xrightarrow{K_0(\iota_n)} & K_0(B) \\ {\scriptstyle \tilde{\phi}_{i,n}}\downarrow & & \downarrow{\scriptstyle K_0(\beta_{e_i})} \\ K_0(B_n) & \xrightarrow{K_0(\iota_n)} & K_0(B) \end{array}$$

commutes for all $i = 1, 2, \ldots, k$ and $n \in \mathbb{Z}^k$ where $\iota_n : B_n \longrightarrow B$ is the inclusion map for all $n \in \mathbb{Z}^k$ and $\tilde{\phi}_{i,n} = \theta_n \circ \phi_{i,n} \circ \theta_n^{-1}$. For then the Lemma follows from the universal properties of direct limits.

Fix $i \in \{1, \ldots, k\}$ and $n \in \mathbb{Z}^k$. Let $[\,\cdot\,], [\,\cdot\,]_n$ denote the equivalence classes in $K_0(B), K_0(B_n)$ respectively. Now $K_0(B_n)$ is generated by $\{[p_{(n,v)}]_n \mid v \in \Lambda^0\}$, therefore it is enough to show that $K_0(\beta_{e_i}) \circ K_0(\iota_n)([p_{(n,v)}]_n) = K_0(\iota_n) \circ \tilde{\phi}_{i,n}([p_{(n,v)}]_n)$ for all $v \in \Lambda^0$. To see that this holds let $v \in \Lambda^0$, then

$$K_0(\beta_{e_i}) \circ K_0(\iota_n)([p_{(n,v)}]_n) = K_0(\beta_{e_i})([p_{(n,v)}]) = [p_{(n+e_i,v)}].$$

While

$$K_0(\iota_n) \circ \tilde{\phi}_{i,n}([p_{(n,v)}]_n) = \sum_{u \in \Lambda^0} M_i(v,u)[p_{(n,u)}] = \sum_{\alpha \in \Lambda^{e_i}(v)} [p_{(n+e_i,\alpha)}] = [p_{(n+e_i,v)}].$$

□

We are now in a position to commence the computation of $H_p(\mathbb{Z}^k, K_0(B))$. It will be convenient to use multiplicative notation for the free abelian group $\mathbb{Z}^k$, generated by k generators. Thus we set $G := \langle s_i \mid s_i s_j = s_j s_i \text{ for all } i, j \in \{1, \ldots, k\}\rangle$ and $R := \mathbb{Z}G$ the group ring of G [3]. An efficient way of computing the homology groups $H_j(\mathbb{Z}^k, K_q(B))$ is by means of a Koszul resolution [25, Corollary 4.5.5].

Definition 5. For any non-negative integer n, let $\mathcal{E}^n(R^k)$ denote the n^{th} term of the exterior algebra [2] of the free R-module $R^k := \bigoplus_{i=1}^k R_i$, over R, where $R_i = R$ for $i = 1, \ldots, k$. Moreover, for any negative integer n, let $\mathcal{E}^n(R^k) = \{0\}$.

For $l \in \mathbb{Z}$ let $N_l := \begin{cases} \{(\mu_1, \ldots, \mu_l) \in \{1, \ldots, k\}^l \mid \mu_1 < \cdots < \mu_l\} & \text{if } l \in \{1, \ldots, k\}, \\ \{\star\} & \text{if } l = 0, \\ \emptyset & \text{otherwise.} \end{cases}$

For $l \in \{1, \dots, k\}$ and $\mu = (\mu_1, \dots, \mu_l) \in N_l, i = 1, \dots, l$ we let

$$\mu^i := \begin{cases} (\mu_1, \mu_2 \dots, \mu_{i-1}, \mu_{i+1}, \mu_{i+2} \dots, \mu_l) \in N_{l-1} & \text{if } l \neq 1, \\ \star & \text{if } l = 1. \end{cases}$$

For $n = 1, 2, \dots$ and $r \in \mathbb{Z}$, let

$$\binom{n}{r} := \begin{cases} \frac{n!}{(n-r)!r!} & \text{if } 0 \leq r \leq n, \\ 0 & \text{if } r < 0 \text{ or } r > n. \end{cases}$$

Thus, for $l \in \mathbb{Z}$, $\mathcal{E}^l(R^k)$ is generated as a free R-module by the set N_l and has rank $\binom{k}{l}$.

One checks that for any basis $\{t_1, \dots, t_k\}$ of G, the subset $\boldsymbol{x} = \{x_i\}_{i=1}^k$, where $x_i := 1 - t_i$ for $i = 1, \dots, k$, is a regular sequence on R (for the definition of a regular sequence see [8, §3.1]). Let $K(\boldsymbol{x})$ be the chain complex

$$0 \longleftarrow \mathcal{E}^0(R^k) \longleftarrow \mathcal{E}^1(R^k) \longleftarrow \cdots \longleftarrow \mathcal{E}^k(R^k) \longleftarrow 0$$

where the differentials $\mathcal{E}^l(R^k) \longrightarrow \mathcal{E}^{l-1}(R^k)$ are given by sending

$$\mu \mapsto \sum_{j=1}^{l} (-1)^{j+1} x_{\mu_j} \mu^j \qquad \text{for all } \mu = (\mu_1, \dots, \mu_l) \in N_l,$$

if $l \in \{1, \dots, k\}$ and the zero map otherwise. By [25, Corollary 4.5.5] $K(\boldsymbol{x})$ is a free resolution of R/I over R where I is the ideal of R generated by $\boldsymbol{x}$. It is well known (see e.g. [25, Chapter 6],[3, §I.2]) that $I = \ker \epsilon$ where $\epsilon : R \longrightarrow \mathbb{Z} : g \mapsto 1$ is the augmentation map of the group ring $R = \mathbb{Z}G$. Thus we have a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, which we may use to compute $H_*(G, K_0(B))$.

If we choose $t_i = s_i^{-1}$ for $i = 1, \dots, k$, it follows that $H_*(G, K_0(B))$ is isomorphic to the homology of the chain complex

$$\mathcal{C} : 0 \longleftarrow K_0(B) \longleftarrow \cdots \longleftarrow \bigoplus_{N_l} K_0(B) \longleftarrow \cdots \longleftarrow K_0(B) \longleftarrow 0, \tag{1}$$

where the differentials $\bigoplus_{N_l} K_0(B) \longrightarrow \bigoplus_{N_{l-1}} K_0(B)$ $(l \in \{1, \dots, k\})$ are defined by

$$\bigoplus_{\mu \in N_l} m_\mu \mapsto \bigoplus_{\lambda \in N_{l-1}} \sum_{\mu \in N_l} \sum_{i=1}^{l} (-1)^{i+1} \delta_{\lambda, \mu^i} (m_\mu - s_{\mu_i} \cdot m_\mu).$$

Recall that the G-action on $K_0(B)$ is given by $s_i \cdot m = K_0(\beta_{e_i})(m)$ for all $m \in K_0(B)$, $i = 1, \dots, k$.

For $m, n \in \mathbb{Z}^k$ with $m \leq n$, let $\mathcal{C}^{(n)}$ be the chain complex

$$0 \longleftarrow A_n \longleftarrow \cdots \longleftarrow \bigoplus_{N_l} A_n \longleftarrow \cdots \longleftarrow A_n \longleftarrow 0,$$

with $A_n = \mathbb{Z}\Lambda^0$ and differentials, $\partial_l^{(n)} : \bigoplus_{N_l} A_n \longrightarrow \bigoplus_{N_{l-1}} A_n$ $(l \in \{1, \dots, k\})$, defined by

$$\bigoplus_{\mu \in N_l} m_\mu \mapsto \bigoplus_{\lambda \in N_{l-1}} \sum_{\mu \in N_l} \sum_{i=1}^{l} (-1)^{i+1} \delta_{\lambda, \mu^i} (1 - M_{\mu_i}^t) m_\mu.$$

Furthermore, let $\tau_m^n = \{(\tau_m^n)_p \mid p \in \mathbb{Z}\} : \mathcal{C}^{(m)} \longrightarrow \mathcal{C}^{(n)}$ be the chain map defined by $(\tau_m^n)_p(\bigoplus_{\mu \in N_p} m_\mu) = \bigoplus_{\mu \in N_p} (M_1^t, \dots, M_k^t)^{n-m} m_\mu$ for all $p \in \mathbb{Z}$ (c.f. Lemma 3). By Lemma 3 and Lemma 4, $(\mathcal{C}^{(n)}; \tau_m^n)$ is a direct system of chain complexes

([22, Chapter 4, §1]) isomorphic to $\mathcal{C}$. Note that the chain complexes $\mathcal{C}^{(n)}$ do not actually depend on $n \in \mathbb{Z}^k$, thus we let $\mathcal{D}$ denote this common chain complex with differentials $\partial_p := \partial_p^{(n)}$ for all $p \in \mathbb{Z}$.

Proposition 1. *Using the above notation we have $H_*(G, K_0(B)) \cong \mathrm{Hom}(\mathcal{D})$, where* Hom *is the homology functor.*

Proof. The homology functor commutes with direct limits ([22, Chapter 4, §1, Theorem 7]), therefore it follows that $H_*(G, K_0(B)) \cong \lim_{\rightarrow}(\mathrm{Hom}(\mathcal{C}^{(n)}), \mathrm{Hom}(\tau_m^n))$. Thus, it suffices to prove that $\mathrm{Hom}(\tau_m^{m+e_j})_p$ is the identity map for all $p \in \mathbb{Z}$, $m \in \mathbb{Z}^k$, $j \in \{1, \ldots, k\}$. To see that this is indeed true we need to show that $\bigoplus_{\mu \in N_p}(1 - M_j^t)(y) \in \mathrm{im}\, \partial_{p+1}$ for all $y \in \ker \partial_p$, $p \in \mathbb{Z}$, $j \in \{1, \ldots, k\}$. Indeed, we claim that given $y = \bigoplus_{\mu \in N_p} y_\mu \in \ker \partial_p$ we have

$$\bigoplus_{\mu \in N_p}(1 - M_j^t)y_\mu = \partial_{p+1}\left(\bigoplus_{\lambda \in N_{p+1}} z_\lambda\right)$$

where $z_\lambda = \sum_{i=1}^{p+1}(-1)^{i+1}\delta_{\lambda_i, j}y_{\lambda^i}$ for all $\lambda \in N_{p+1}$.

Fix $j \in \{1, \ldots, k\}$. For any $p \in \{1, \ldots, k\}$ and $\mu = (\mu_1, \ldots, \mu_p) \in N_p$ let $j(\mu) := \sum_{i=1}^{p} \delta_{i,\mu_i} i$, i.e. if j is a component of μ, then $j(\mu)$ denotes the unique $i \in \{1, \ldots, k\}$ such that $\mu_i = j$, otherwise $j(\mu) = 0$. Now fix a $p \in \{1, \ldots, k\}$ and a $\mu' = (\mu'_1, \ldots, \mu'_p) \in N_p$ and let $y = \bigoplus_{\mu \in N_p} y_\mu$ be in $\ker \partial_p$.

First, suppose that $j(\mu') > 0$ and let $\eta = (\mu')^{j(\mu')}$. Then

$$\begin{aligned}
0 &= \partial_p(y)_\eta = \sum_{\mu \in N_p}\sum_{i=1}^{p}(-1)^{i+1}\delta_{\eta,\mu^i}(1 - M_{\mu_i}^t)y_\mu \\
&= \sum_{i=1}^{p}(-1)^{i+1}\delta_{\eta,(\mu')^i}(1 - M_{\mu_i}^t)x_{\mu'} \\
&\quad + \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}}\sum_{i=1}^{p}(-1)^{i+1}\delta_{\eta,\mu^i}(1 - M_{\mu_i}^t)x_\mu \\
&= (-1)^{j(\mu')+1}(1 - M_j^t)x_{\mu'} + \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}}\sum_{i=1}^{p}(-1)^{i+1}\delta_{\eta,\mu^i}(1 - M_{\mu_i}^t)x_\mu,
\end{aligned}$$

so that

$$(1 - M_j^t)x_{\mu'} = \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}}\sum_{i=1}^{p}(-1)^{i+j(\mu')+1}\delta_{\eta,\mu^i}(1 - M_{\mu_i}^t)x_\mu.$$

Now

$$\begin{aligned}
\partial_{p+1}\left(\bigoplus_{\lambda\in N_{p+1}} z_\lambda\right)_{\mu'} &= \sum_{\lambda\in N_{p+1}}\sum_{i,r=1}^{p+1}(-1)^{i+r+2}\delta_{\mu',\lambda^i}\delta_{j,\lambda_r}(1-M^t_{\lambda_i})y_{\lambda^r} \\
&= \sum_{\substack{\lambda\in N_{p+1}\\ j(\lambda)>0}}\sum_{i=1}^{p+1}(-1)^{i+j(\lambda)}\delta_{\mu',\lambda^i}(1-M^t_{\lambda_i})y_{\lambda^{j(\lambda)}} \\
&= \sum_{\substack{\lambda\in N_{p+1}\\ j(\lambda)>0}}\left\{\sum_{i=1}^{j(\lambda)-1}(-1)^{i+j(\mu')+1}\delta_{\mu',\lambda^i}(1-M^t_{\lambda_i})y_{\lambda^{j(\lambda)}}\right. \\
&+ \left.\sum_{i=j(\lambda)+1}^{p+1}(-1)^{i+j(\mu')}\delta_{\mu',\lambda^i}(1-M^t_{\lambda_i})y_{\lambda^{j(\lambda)}}\right\} \\
&= \sum_{\substack{\mu\in N_p\\ j(\mu)=0}}\sum_{i=1}^{p}(-1)^{i+j(\mu')+1}\delta_{\eta,\mu^i}(1-M^t_{\mu_i})y_\mu
\end{aligned}$$

since for every $\lambda \in N_{p+1}$ such that $j(\lambda) > 0$ and for every $i \in \{1,\dots,p+1\}\setminus\{j(\lambda)\}$ we have

(1) $\delta_{\mu',\lambda^{j(\lambda)}} = 0$,

(2) $j(\lambda) = \begin{cases} j(\mu')+1 & \text{if } \mu' = \lambda^i \text{ with } i < j(\lambda), \\ j(\mu') & \text{if } \mu' = \lambda^i \text{ with } j(\lambda) < i, \end{cases}$

(3) $\mu' = \lambda^i \iff \eta = \begin{cases} (\lambda^{j(\lambda)})^i & \text{if } i < j(\lambda), \\ (\lambda^{j(\lambda)})^{i-1} & \text{if } j(\lambda) < i, \end{cases}$

and if $\mu \in N_p$ then $j(\mu) = 0$, $\eta = \mu^i$ for some $i \in \{1,\dots,k\} \iff \mu \neq \mu'$, $\eta = \mu^i$ for some $i \in \{1,\dots,k\}$.

Hence,

$$\begin{aligned}
\partial_{p+1}\left(\bigoplus_{\lambda\in N_{p+1}} z_\lambda\right)_{\mu'} &= \sum_{\substack{\mu\in N_p\\ \mu\neq\mu'}}\sum_{i=1}^{p}(-1)^{i+j(\mu')+1}\delta_{\eta,\mu^i}(1-M^t_{\mu_i})y_\mu \\
&= (1-M^t_j)y_{\mu'}.
\end{aligned}$$

Now suppose that $j(\mu') = 0$. Then

$$\begin{aligned}
\partial_{p+1}\left(\bigoplus_{\lambda\in N_{p+1}} z_\lambda\right)_{\mu'} &= \sum_{\lambda\in N_{p+1}}\sum_{i,r=1}^{p+1}(-1)^{i+r+2}\delta_{\mu',\lambda^i}\delta_{j,\lambda_r}(1-M^t_{\lambda_i})y_{\lambda^r} \\
&= (-1)^{j(\xi)+j(\xi)+2}(1-M^t_{\xi_{j(\xi)}})y_{\xi^{j(\xi)}} \\
&= (1-M^t_j)y_{\mu'},
\end{aligned}$$

where ξ is the unique element of N_{p+1} satisfying $j(\xi) > 0$ and $\xi^{j(\xi)} = \mu'$. □

Combining the results of this section we get the following theorem.

Theorem 1. *Let Λ be a row-finite k-graph with no sources. Then there exists a spectral sequence $\{(E^r, d^r)\}$ converging to $K_*(C^*(\Lambda))$ with $E^\infty_{p,q} \cong E^{k+1}_{p,q}$ and*

$$E^2_{p,q} \cong \begin{cases} \operatorname{Hom}_p \mathcal{D} & \text{if } p \in \{0,1,\dots,k\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{D}$ is the chain complex with $\mathcal{D}_p$ trivial for $p \in \mathbb{Z}\backslash\{0,1,\dots,k\}$, $\mathcal{D}_p := \bigoplus_{\mu \in N_p} \mathbb{Z}\Lambda^0$ for $p \in \{0,1,\dots,k\}$ and differentials

$$\partial_p : \mathcal{D}_p \longrightarrow \mathcal{D}_{p-1} : \bigoplus_{\mu \in N_p} m_\mu \mapsto \bigoplus_{\lambda \in N_{p-1}} \sum_{\mu \in N_p} \sum_{i=1}^{p} (-1)^{i+1} \delta_{\lambda,\mu^i} (1 - M^t_{\mu_i}) m_\mu$$

for $p \in \{1,\dots,k\}$.

Specialising Theorem 1 to the case when $k = 2$ gives us explicit formulae to compute the K-groups of 2-graph C^*-algebras under mild assumptions.

Proposition 2. *Given a row-finite 2-graph Λ with no sources, the K-groups of $C^*(\Lambda)$ are given by*

$$\begin{aligned} K_0(C^*(\Lambda)) &\cong \operatorname{coker}(1 - M^t_1, 1 - M^t_2) \oplus \ker \begin{pmatrix} M^t_2 - 1 \\ 1 - M^t_1 \end{pmatrix} \\ K_1(C^*(\Lambda)) &\cong \ker(1 - M^t_1, 1 - M^t_2) / \operatorname{im} \begin{pmatrix} M^t_2 - 1 \\ 1 - M^t_1 \end{pmatrix} \end{aligned}$$

where we regard $(1 - M^t_1, 1 - M^t_2) : \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 \longrightarrow \mathbb{Z}\Lambda^0$ and $\begin{pmatrix} M^t_2 - 1 \\ 1 - M^t_1 \end{pmatrix} : \mathbb{Z}\Lambda^0 \longrightarrow \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0$ as group homorphisms defined in the natural way.

Proof. The Kasparov spectral sequence converging to $K_*(C^*(\Lambda))$ of Proposition 1 has $E^\infty_{p,q} \cong E^3_{p,q}$ for all $p, q \in \mathbb{Z}$. However, it follows from $E^2_{p,q} = 0$ for odd q that the differential d^2 is the zero map and $E^3_{p,q} \cong E^2_{p,q} \cong \operatorname{Hom}_p(\mathcal{D})$ for all $p \in \{0,1,\dots,k\}$ and even q, where $\mathcal{D}$ is the chain complex

$$0 \longleftarrow \mathbb{Z}\Lambda^0 \stackrel{\partial_1}{\longleftarrow} \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 \stackrel{\partial_2}{\longleftarrow} \mathbb{Z}\Lambda^0 \longleftarrow 0$$

with $\partial_1 = (1 - M^t_1, 1 - M^t_2)$ and $\partial_2 = \begin{pmatrix} M^t_2 - 1 \\ 1 - M^t_1 \end{pmatrix}$ for a suitable choice of bases.

Convergence of the spectral sequence to $K_*(C^*(\Lambda))$ means that (c.f. proof of Proposition 4)

$$K_1(C^*(\Lambda)) \cong E^2_{1,0}$$

and there exists the following short exact sequence of groups

$$0 \longrightarrow E^2_{0,0} \longrightarrow K_0(C^*(\Lambda)) \longrightarrow E^2_{2,0} \longrightarrow 0.$$

However, $E^2_{2,0} \cong \ker \partial_2$ is a free abelian group, thus the short exact sequence splits and the result follows. ☐

Evidently complications arise when $k > 2$, however it is worth noting that under some extra assumptions on the vertex matrices it is possible to say a fair amount about the K-groups of higher rank graph C^*-algebras. For example, the case $k = 3$ is considered below.

Suppose that Λ is a row-finite 3-graph with no sources. By Theorem 1, there exist short exact sequences

$$0 \longrightarrow E^4_{0,0} \longrightarrow K_0(C^*(\Lambda)) \longrightarrow E^4_{2,0},$$
$$0 \longrightarrow E^4_{1,0} \longrightarrow K_1(C^*(\Lambda)) \longrightarrow E^4_{3,0}.$$

However, since $E^4_{p,q} = 0$ if $p \in \mathbb{Z}\backslash\{0,1,2,3\}$ the only non-zero component of the differential d^3 is $d^3_{3,0} : E^3_{3,0} \longrightarrow E^3_{0,2}$, thus we have $E^4_{1,0} \cong E^2_{1,0} \cong \mathrm{Hom}_1\,\mathcal{D}$ and $E^4_{2,0} \cong E^2_{2,0} \cong \mathrm{Hom}_2\,\mathcal{D}$. Moreover, as in the proof of Proposition 2, the differential d^2 is the zero map, thus $E^3_{3,0} \cong E^2_{3,0}$ and $E^3_{0,2} \cong E^2_{0,0}$. Also note that $E^4_{3,0} \cong \ker d^3_{3,0}$ is a free group, thus the exact sequence for $K_1(C^*(\Lambda))$ splits. Therefore there are two obvious cases for immediate consideration, namely when $\ker\partial_3 = 0$, and when $\operatorname{im}\partial_1 = \mathbb{Z}\Lambda^0$. Thus, writing $\partial_p,\ p = 1,2,3$ in matrix form yields the following proposition.

Proposition 3. *Let Λ be a row-finite 3-graph with no sources. Consider the following group homomorphisms defined by block matrices:*

$$\partial_1 = (1-M_1^t \ \ 1-M_2^t \ \ 1-M_3^t) : \bigoplus_{i=1}^{3} \mathbb{Z}\Lambda^0 \longrightarrow \mathbb{Z}\Lambda^0,$$
$$\partial_2 = \begin{pmatrix} M_2^t-1 & M_3^t-1 & 0 \\ 1-M_1^t & 0 & M_3^t-1 \\ 0 & 1-M_1^t & 1-M_2^t \end{pmatrix} : \bigoplus_{i=1}^{3} \mathbb{Z}\Lambda^0 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z}\Lambda^0,$$
$$\partial_3 = \begin{pmatrix} 1-M_3^t \\ M_2^t-1 \\ 1-M_1^t \end{pmatrix} : \mathbb{Z}\Lambda^0 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z}\Lambda^0.$$

If ∂_1 is surjective then

$$K_0(C^*(\Lambda)) \cong \ker\partial_2/\operatorname{im}\partial_3,$$
$$K_1(C^*(\Lambda)) \cong \ker\partial_1/\operatorname{im}\partial_2 \oplus \ker\partial_3.$$

If $\cap_{i=1}^3 \ker(1-M_i^t) = 0$ then

$$K_1(C^*(\Lambda)) = \ker\partial_1/\operatorname{im}\partial_2$$

and there exists a short exact sequence

$$0 \longrightarrow \operatorname{coker}\partial_1 \longrightarrow K_0(C^*(\Lambda)) \longrightarrow \ker\partial_2/\operatorname{im}\partial_3 \longrightarrow 0.$$

Remarks 1. (i) One may recover [13, Theorem 3.1] from Theorem 1 by setting k equal to 1.

(ii) By [11, Corollary 3.5 (ii)] a rank k Cuntz-Krieger algebra ([19, 20]) is isomorphic to a k-graph C^*-algebra. Thus, Propsition 2 generalises [20, Proposition 4.1], the proof of which inspired the methods used throughout this paper.

(iii) By showing that the C^*-algebra of a row-finite 2-graph, Λ, with no sources and finite vertex set, satisfying some further conditions, is isomorphic to a rank 2 Cuntz-Krieger algebra, Allen, Pask and Sims used Robertson and Steger's [20, Proposition 4.1] result to calculate the K-groups of $C^*(\Lambda)$ [1, Theorem 4.1]. Moreover, in [1, Remark 4.7. (1)] they note that their formulae for the K-groups holds for more general 2-graph C^*-algebras, namely the C^*-algebras of row-finite 2-graphs, Λ, with no sinks (i.e. $s^{-1}(v)\cap\Lambda^n \neq \emptyset$ for all $n \in \mathbb{N}^k,\ v \in \Lambda^0$) nor sources and finite vertex set.

4. The K-groups of unital k-graph C^*-algebras

Recall that if Λ is a row-finite higher rank graph with no sources then Λ^0 finite is equivalent to $C^*(\Lambda)$ being unital ([11, Remarks 1.6 (v)]). Thus in this section we specialise in the case where the vertex set of our higher rank graph, hence each vertex matrix, is finite. We will continue to denote the Kasparov spectral sequence converging to $K_*(C^*(\Lambda))$ of the previous section by $\{(E^r, d^r)\}$ and we shall denote the torsion-free rank of an abelian group G by $r_0(G)$ (see e.g. [7]).

Proposition 4. *If Λ is a row-finite higher rank graph with no sources and Λ^0 finite then $K_0(C^*(\Lambda))$ and $K_1(C^*(\Lambda))$ have equal torsion-free rank.*

Proof. Let the rank of the given higher rank graph Λ be k and let $|\Lambda^0| = n$.

Since, $E^{\infty}_{p,q} \cong E^{k+1}_{p,q}$ for all $p, q \in \mathbb{Z}$ and $E^{k+1}_{p,q} = 0$ if $p \in \mathbb{Z}\backslash\{0, 1, \dots, k\}$ or q odd by Lemma 2, it follows from the definition of convergence of $\{(E^r, d^r)\}$ ([25, 5.2.5]) that there exists finite filtrations,

$$0 = F_{-1}(H_0) \subseteq E^{k+1}_{0,0} \cong F_0(H_0) \subseteq F_1(H_0) \subseteq \cdots \subseteq F_{k-1}(H_0) \subseteq F_k(H_0) = H_0,$$

and

$$0 = F_0(H_1) \subseteq E^{k+1}_{1,0} \cong F_1(H_1) \subseteq F_2(H_1) \subseteq \cdots \subseteq F_{k-1}(H_1) \subseteq F_k(H_1) = H_1$$

of $H_0 = K_0(C^*(\Lambda))$ and $H_1 = K_1(C^*(\Lambda))$ respectively, such that

$$E^{k+1}_{p,q} \cong F_p(H_{p+q})/F_{p-1}(H_{p+q}).$$

Thus,

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) &= r_0(F_k(H_0)) = r_0(F_{k-1}(H_0)) + r_0(E^{k+1}_{k,-k}) = \cdots \\ &= r_0(F_0(H_0)) + \sum_{s\geq 1} r_0(E^{k+1}_{s,-s}) = \sum_{s\in\mathbb{Z}} r_0(E^{k+1}_{s,-s}), \end{aligned}$$

and

$$\begin{aligned} r_0(K_1(C^*(\Lambda))) &= r_1(F_k(H_1)) = r_0(F_{k-1}(H_0)) + r_0(E^{k+1}_{k,-k+1}) = \cdots \\ &= r_0(F_1(H_1)) + \sum_{s\geq 2} r_0(E^{k+1}_{s,-s+1}) = \sum_{s\in\mathbb{Z}} r_0(E^{k+1}_{s,-s+1}). \end{aligned}$$

Now we claim that

$$\sum_{s\in\mathbb{Z}} r_0(E^{k+1}_{s,-s}) - r_0(E^{k+1}_{s,-s+1}) = \sum_{s\in\mathbb{Z}} r_0(E^{2}_{s,-s}) - r_0(E^{2}_{s,-s+1}).$$

To see that this holds it is sufficient to prove that for all $r \geq 2$ we have

$$\sum_{s\in\mathbb{Z}} r_0(E^{r+1}_{s,-s}) - r_0(E^{r+1}_{s,-s+1}) = \sum_{s\in\mathbb{Z}} r_0(E^{r}_{s,-s}) - r_0(E^{r}_{s,-s+1}).$$

Recall that for all $r \geq 1,\ p, q \in \mathbb{Z},\ E^{r+1}_{p,q} \cong Z(E^r)_{p,q}/B(E^r)_{p,q}$ where $Z(E^r)_{p,q} = \ker d^r_{p,q}$ and $B(E^r)_{p,q} = \operatorname{im} d^r_{p+r,q-r+1}$. Thus

$$\begin{aligned} r_0(E^{r+1}_{p,q}) &= r_0(Z(E^r)_{p,q}) + r_0(B(E^r)_{p,q}) \\ &= r_0(Z(E^r)_{p,q}) + r_0(E^r_{p+r,q-r+1}) - r_0(Z(E^r)_{p+r,q-r+1}) \end{aligned}$$

for all $r \geq 1$, $p, q \in \mathbb{Z}$. Moreover, it follows from the definition of the Kasparov spectral sequence that given any $r \geq 1$ and $p, q, q' \in \mathbb{Z}$ with $q = q' \mod 2$ there exist isomorphisms $\rho : E^r_{p,q} \longrightarrow E^r_{p,q'}$, $\sigma : E^r_{p-r,q+r-1} \longrightarrow E^r_{p-r,q'+r-1}$ such that $d^r_{p,q'} \circ \rho = \sigma \circ d^r_{p,q}$. Therefore,

$$\begin{aligned}\sum_{s\in\mathbb{Z}} r_0(E^{r+1}_{s,-s}) - r_0(E^{r+1}_{s,-s+1}) &= \sum_{s\in\mathbb{Z}} \{r_0(Z(E^r)_{s,-s}) - r_0(Z(E^r)_{s+r,-s-r+2}) \\ &- \ r_0(Z(E^r)_{s,-s+1}) + r_0(Z(E^r)_{s+r,-s-r+1}) \\ &+ \ r_0(E^r_{s+r,-s-r+2}) - r_0(E^r_{s+r,-s-r+1})\} \\ &= \sum_{s\in\mathbb{Z}} r_0(E^r_{s,-s}) - r_0(E^r_{s,-s+1})\end{aligned}$$

for all $r \geq 1$. Combining the above gives

$$r_0(K_0(C^*(\Lambda))) - r_0(K_1(C^*(\Lambda))) = \sum_{s\in\mathbb{Z}} r_0(E^2_{s,-s}) - r_0(E^2_{s,-s+1}).$$

Now, recall that for all $p \in \mathbb{Z}$ and $q \in 2\mathbb{Z}$, $E^2_{p,q} \cong H_p(\mathbb{Z}^k, K_0(B)) \cong \ker \partial_p / \operatorname{im} \partial_{p+1}$ by Proposition 1. Therefore,

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) - r_0(K_1(C^*(\Lambda))) &= \sum_{s\in\mathbb{Z}} r_0(E^2_{2s,-2s}) - r_0(E^2_{2s+1,-2s}) \\ = \ & \sum_{s\in\mathbb{Z}} r_0(\ker \partial_{2s}) - r_0(\operatorname{im} \partial_{2s+1}) - r_0(\ker \partial_{2s+1}) + r_0(\operatorname{im} \partial_{2s+2}) \\ = \ & \sum_{s\in\mathbb{Z}} r_0(\ker \partial_{2s}) - \binom{k}{2s+1} n + r_0(\ker \partial_{2s+1}) - r_0(\ker \partial_{2s+1}) \\ + \ & \binom{k}{2s+2} n - r_0(\ker \partial_{2s+2}) \\ = \ & \sum_{s\in\mathbb{Z}} \left\{ \binom{k}{2s} - \binom{k}{2s-1} \right\} n \\ = \ & \sum_{s\in\mathbb{Z}} \left\{ \binom{k-1}{2s} + \binom{k-1}{2s-1} - \binom{k-1}{2s-1} - \binom{k-1}{2s-2} \right\} n = 0. \end{aligned}$$

□

Corollary 1. *If Λ is a row-finite higher rank graph with no sources and Λ^0 is finite then there exists a non-negative integer r such that for $i = 1, 2$,*

$$K_i(C^*(\Lambda)) \cong \mathbb{Z}^r \oplus T_i$$

for some finite group T_i, where $\mathbb{Z}^0 := \{0\}$.

Proof. It is well-known that if B is a finitely generated subgroup of an abelian group A such that A/B is also finitely generated then A must be finitely generated too [7]. Now, for all $p, q \in \mathbb{Z}$, $E^{k+1}_{p,q}$ is isomorphic to a sub-quotient of the finitely generated abelian group $E^2_{p,q} \cong H_p(\mathbb{Z}^k, K_q(B))$, therefore $E^{k+1}_{p,-p}$ is also finitely generated. Moreover, $E^{k+1}_{0,i} \cong F_0(K_i(C^*(\Lambda)))$ and for $p \in \{1, 2, \ldots, k\}$, $E^{k+1}_{p,-p+i} \cong F_p(K_i(C^*(\Lambda)))/F_{p-1}(K_i(C^*(\Lambda)))$, which implies that $K_i(C^*(\Lambda)) = F_k(K_i(C^*(\Lambda)))$ is finitely generated. The result follows from Proposition 4 by noting that every finitely generated abelian group A is isomorphic to the direct sum of a finite group with $\mathbb{Z}^r$, where $r = r_0(A)$ (see e.g. [7, Theorem 15.5]). □

Under mild assumptions, formulae for the torsion-free rank and torsion parts of the K-groups of unital 2-graph C^*-algebras can be given in terms of the vertex matrices (c.f. [20, Proposition 4.13]). This we do in Proposition 5 below. Similar formulae may be given for higher rank graph C^*-algebras if we impose extra conditions on the vertex matrices, (c.f. Proposition 3). However, since at present we must compute these on a case by case basis we leave the details to the interested reader.

Proposition 5 (c.f. [20, Proposition 4.13]). *Let Λ be a row-finite 2-graph with no sources and finite vertex set. Then*

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) &= r_0(K_1(C^*(\Lambda)) \\ &= r_0(\operatorname{coker}(1 - M_1^t, 1 - M_2^t)) + r_0(\operatorname{coker}(1 - M_1, 1 - M_2)), \\ \operatorname{tor}(K_0(C^*(\Lambda))) &\cong \operatorname{tor}(\operatorname{coker}(1 - M_1^t, 1 - M_2^t)), \\ \operatorname{tor}(K_1(C^*(\Lambda))) &\cong \operatorname{tor}(\operatorname{coker}(1 - M_1, 1 - M_2)). \end{aligned}$$

Proof. We have already seen in Proposition 4 that the torsion-free rank of the K_0-group and K_1-group of a k-graph are equal so it is sufficient to calculate the torsion-free rank of $K_0(C^*(\Lambda))$. Let $n := |\Lambda^0|$. By Proposition 2 we have

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) &= r_0(\operatorname{coker}(1 - M_1^t, 1 - M_2^t)) + r_0\left(\ker\begin{pmatrix} 1 - M_1^t \\ 1 - M_2^t \end{pmatrix}\right) \\ &= r_0(\operatorname{coker}(1 - M_1^t, 1 - M_2^t)) + n - r_0(\operatorname{im}(1 - M_1, 1 - M_2)) \\ &= r_0(\operatorname{coker}(1 - M_1^t, 1 - M_2^t)) + r_0(\operatorname{coker}(1 - M_1, 1 - M_2)). \end{aligned}$$

Furthermore, the assertion about the torsion part of $K_0(C^*(\Lambda))$ is obvious. The torsion part of $K_1(C^*(\Lambda))$ is given by $\operatorname{tor}(K_1(C^*(\Lambda))) \cong \operatorname{tor}(\ker(1 - M_1^t, 1 - M_2^t)/\operatorname{im}\binom{M_2^t - 1}{1 - M_1^t}$, which is clearly isomorphic to $\operatorname{tor}(\operatorname{coker}\binom{M_2^t - 1}{1 - M_1^t})$. However, by reduction to Smith normal forms, $\operatorname{coker}\binom{M_2^t - 1}{1 - M_1^t}$ is isomorphic to $\operatorname{coker}(1 - M_1, 1 - M_2)$. □

As already stated earlier, by [11, Theorem 5.7] $C^*(\Lambda) \otimes \mathbb{K}$ is isomorphic to $B \rtimes_\beta \mathbb{Z}^k$. The proof of [11, Theorem 5.7] relies on groupoid techniques and is rather convoluted. We note that in our special case an isomorphism can be derived fairly easily using the universal property of k-graph C^*-algbebras as follows.

First note that given any k_1-graph, (Λ_1, d_1), and any k_2-graph, (Λ_2, d_2), such that both are row-finite and have no sources a $(k_1 + k_2)$-graph may be formed as in [11, Proposition 1.8], which is denoted by $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$. The structure of $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is given by saying that $\Lambda_1 \times \Lambda_2$ is the product category and $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \longrightarrow \mathbb{N}^{k_1 + k_2}$ is given by $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ for all $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Furthermore, by [11, Corollary 3.5 (iv)], $C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$.

Lemma 5. *Let $\Delta := \{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k \mid m \le n\}$ be the k-graph with structure maps defined by $r(m, n) = m$, $s(m, n) = n$, composition defined by $(m, l)(l, n) = (m, n)$ and degree functor $d_\Delta : \Delta \longrightarrow \mathbb{N}^k$ defined by $d_\Delta(m, n) = n - m$. Then*

$$C^*(\mathbb{Z}^k \times_d \Lambda) \rtimes_\beta \mathbb{Z}^k \cong C^*(\Lambda \times \Delta) \cong C^*(\Lambda) \otimes \mathbb{K}.$$

Proof. Let $(B \rtimes_\beta \mathbb{Z}^k,\, i_B,\, i_{\mathbb{Z}^k})$ be a crossed product for the dynamical system $(B,\, \mathbb{Z}^k,\, \beta)$ in the sense of [16]. One checks that $\{t_{(\lambda,(m,n))} \mid (\lambda, (m, n)) \in \Lambda \times \Delta\}$

is a *-representation of $\Lambda \times \Delta$, where for $(\lambda, (m, n)) \in \Lambda \times \Delta$ we let $t_{(\lambda,(m,n))} := i_B(s_{(m,\lambda)})i_{\mathbb{Z}^k}(m + d(\lambda) - n)$. Moreover $C^*(t_\xi \mid \xi \in \Lambda \times \Delta) = B \rtimes_\beta \mathbb{Z}^k$. Thus by the universal property of $C^*(\Lambda \times \Delta)$, there exists a *-homomorphism $\pi : C^*(\Lambda \times \Delta) \longrightarrow B \rtimes_\beta \mathbb{Z}^k$ such that $\pi(s_\xi) = t_\xi$ for all $\xi \in \Lambda \times \Delta$. Let $\alpha : \mathbb{T}^k \longrightarrow \mathrm{Aut}(B)$ denote the canonical gauge action on B and let $\hat{\beta} : \mathbb{T}^k \longrightarrow \mathrm{Aut}(B \rtimes_\beta \mathbb{Z}^k)$ denote the dual action of β. There exists an action $\tilde{\alpha}$ of $\mathbb{T}^k$ on $B \rtimes_\beta \mathbb{Z}^k$ such that $i_B\alpha_z = \tilde{\alpha}_z i_B$ for all $z \in \mathbb{T}^k$. It is clear that setting $\gamma_{(z_1,z_2)} := \tilde{\alpha}_{z_1 z_2}\hat{\beta}_{z_2^{-1}}$ for all $(z_1, z_2) \in \mathbb{T}^k \times \mathbb{T}^k$ defines an action γ of $\mathbb{T}^{2k}$ on $B \rtimes_\beta \mathbb{Z}^k$. Moreover, it satisfies $\pi\alpha^\times_z = \gamma_z\pi$ for all $z \in \mathbb{T}^{2k}$ where $\alpha^\times$ is the canonical gauge action on $\Lambda \times \Delta$. Clearly $\pi(p_v) = 0$ for all $v \in \Lambda \times \Delta$, hence by the gauge-invariant uniquness theorem [11, Theorem 3.4] we see that $C^*(\Lambda \times \Delta) \cong B \rtimes_\beta \mathbb{Z}^k$.

As discussed above $C^*(\Lambda \times \Delta) \cong C^*(\Lambda) \otimes C^*(\Delta)$ therefore it suffices to show that $C^*(\Delta) \cong \mathbb{K}$. To see that this holds note that $\{e_{m,n} \mid m, n \in \mathbb{Z}^k\}$ is a complete system of matrix units if $e_{m,n} := s_{(m,q)}s^*_{(n,q)}$ where $q := \sup\{m, n\}$ (c.f. [11, Examples 1.7 (ii)]). □

Consider the image of the class of the unit of $C^*(\Lambda)$ in $K_0(C^*(\Lambda))$ under the isomorphism induced by the above isomorphism.

For the identity element $0 \in \mathbb{Z}^k$, we see that $p_{(0,0)} = s_{(0,0)}$ is a minimal projection in $C^*(\Delta)$. Therefore the homomorphism $x \mapsto x \otimes p_{(0,0)}$ induces an isomorphism between $K_0(C^*(\Lambda))$ and $K_0(C^*(\Lambda) \otimes C^*(\Delta))(\cong K_0(C^*(\Lambda) \otimes \mathbb{K}))$ which in turn is isomorphic to $K_0(C^*(\Lambda) \times \Delta)$ and $K_0(B \rtimes_\beta \mathbb{Z}^k)$. The action of the above isomorphism,

$$K_0(C^*(\Lambda)) \longrightarrow K_0(C^*(\Lambda) \otimes C^*(\Delta)) \longrightarrow K_0(C^*(\Lambda \times \Delta)) \longrightarrow K_0(B \rtimes \mathbb{Z}^k),$$

on the class of the unit of $C^*(\Lambda)$ in $K_0(C^*(\Lambda))$ is as follows:

$$[1] \quad = \quad \sum_{v\in\Lambda^0} [p_v] \mapsto \sum_{\Lambda^0} [p_v \otimes p_{(0,0)}] \mapsto \sum_{\Lambda^0} [p_{(v,(0,0))}] \mapsto \sum_{v\in\Lambda^0} [i_B(p_{(0,v)})].$$

Proposition 6. *Let Λ be a row-finite k-graph with no sources and finite vertex set. Then there exists a group homomorphism $\Phi : \operatorname{coker}\partial_1 \longrightarrow K_0(C^*(\Lambda))$. When $k = 2$, Φ coincides with the embedding $\mathrm{Hom}_0(\mathcal{D}) \longrightarrow K_0(C^*(\Lambda))$ of Proposition 2. Moreover, if the K_0-class of the unit of $C^*(\Lambda)$ is denoted by* [1], *then $\Phi(e+\operatorname{im}\partial_0) =$ [1] where $e(v) = 1$ for all $v \in \Lambda^0$.*

Proof. By Proposition 1 $\operatorname{coker}\partial_0 \cong K_0(B)/\operatorname{im}(1 - \sigma_1, \dots, 1 - \sigma_k)$ where $\sigma_i = K_0(\beta_{e_i})$ for all $i \in \{1, \dots, k\}$. The isomorphism, Φ_1 say, sends $e + \operatorname{im}\partial_0$ to $\sum_{v\in\Lambda^0}[p_{(0,v)}] + \operatorname{im}(1 - \sigma_1, \dots, 1 - \sigma_k)$.

For $j \in \{1, \dots, k\}$ let i_j be the natural embedding of B into $B \rtimes_{\beta_{e_1}} \mathbb{Z} \rtimes_{\tilde{\beta}_{e_2}} \cdots \rtimes_{\tilde{\beta}_{e_j}} \mathbb{Z}$ where for $j \in \{2, \dots, k\}$, $\tilde{\beta}_{e_j}$ is the automorphism on $B \rtimes_{\beta_{e_1}} \mathbb{Z} \rtimes_{\tilde{\beta}_{e_2}} \cdots \rtimes_{\tilde{\beta}_{e_{j-1}}} \mathbb{Z}$ satisfying $i_{j-1}\beta_{e_j} = \tilde{\beta}_{e_j} i_{j-1}$. Moreover, let $\tilde{\sigma}_j := K_0(\tilde{\beta}_{e_j})$ for all $j \in \{1, \dots, k\}$. Note that $K_0(i_{j-1})(1 - \sigma_j) = (1 - \tilde{\sigma}_j)K_0(i_{j-1})$.

For $j \in \{2, \dots, k\}$ let f_j be the natural embedding of $B \rtimes_{\beta_{e_1}} \mathbb{Z} \rtimes_{\tilde{\beta}_{e_2}} \cdots \rtimes_{\tilde{\beta}_{e_{j-1}}} \mathbb{Z}$ into $B \rtimes_{\beta_{e_1}} \mathbb{Z} \rtimes_{\tilde{\beta}_{e_2}} \cdots \rtimes_{\tilde{\beta}_{e_j}} \mathbb{Z}$ and let $f_1 := i_1$. Then $i_j = f_j i_{j-1}$ for all $j \in \{2, \dots, k\}$.

Applying the Pimsner-Voiculescu exact sequence in succession we see that

$$\ker K_0(f_j) = \operatorname{im}(1 - \tilde{\sigma}_j) \text{ for all } j \in \{1, \dots, k\}.$$

We claim that $\mathrm{im}(1-\sigma_1,\dots,1-\sigma_k) \subseteq \ker K_0(i_k)$. Let $g \in \mathrm{im}(1-\sigma_1,\dots,1-\sigma_k)$ then

$$\begin{aligned} g &= \sum_{j=1}^{k}(1-\sigma_j)x_j \quad \text{for some } x_j \in K_0(B) \\ \Longrightarrow K_0(i_k)(g) &= K_0(i_1)(1-\sigma_1)x_1 + \sum_{j=2}^{k-1} K_0(f_k\cdots f_j)K_0(i_{j-1})(1-\sigma_j)x_j \\ &+ K_0(f_k)K_0(i_{k-1})(1-\sigma_k)x_k \\ &= \sum_{j=2}^{k-1} K_0(f_k\cdots f_{j+1})K_0(f_j)(1-\tilde{\sigma}_j)K_0(i_{j-1})x_j \\ &+ K_0(f_k)(1-\tilde{\sigma}_k)K_0(i_{k-1})x_k = 0. \end{aligned}$$

Therefore, $g \in \ker K_0(i_k)$.

Thus we may define $\Phi_2 : K_0(B)/\mathrm{im}(1-\sigma_1,\dots,1-\sigma_k) \longrightarrow K_0(B)/\ker K_0(i_k)$ to be the natural homomorphism i.e. $\Phi_2(g+\mathrm{im}(1-\sigma_1,\dots,1-\sigma_k)) = g+\ker K_0(i_k)$.

Let $i_B : B \longrightarrow B \rtimes_\beta \mathbb{Z}^k$ be the natural embedding. Clearly $\ker K_0(i_B) = \ker K_0(i_k)$. Let $\pi : K_0(B) \longrightarrow K_0(B)/\ker K_0(i_B)$ be the natural homomorphism and define $\Phi_3 : K_0(B)/\ker K_0(i_k) \longrightarrow K_0(B \rtimes_\beta \mathbb{Z}^k)$ by $\Phi_3\pi = K_0(i_B)$. Finally let $\Phi_4 : K_0(B \rtimes_\beta \mathbb{Z}^k) \longrightarrow K_0(C^*(\Lambda))$ be the inverse of the composition of the isomorphisms stated earlier. Then $\Phi := \Phi_4\Phi_3\Phi_2\Phi_1$ is the required homomorphism.

Moreover, when $k=2$ we claim that $\mathrm{im}(1-\sigma_1, 1-\sigma_2) = \ker K_0(i_2)$. To see that this holds, first note that $K_0(i_1)$ is surjective since $K_1(B) = 0$. Now let $g \in \ker K_0(i_2)$, then

$$\begin{aligned} K_0(f_2)K_0(i_1)(g) &= 0 \\ \Longrightarrow K_0(i_1)(g) &= (1-\tilde{\sigma}_2)K_0(i_1)(x) \quad \text{for some } y \in K_0(B) \\ \Longrightarrow K_0(i_1)(g) &= K_0(i_1)(1-\sigma_2)y \\ \Longrightarrow g &= (1-\sigma_1)x + (1-\sigma_2)y \quad \text{for some } x, y \in K_0(B). \end{aligned}$$

Therefore $g \in \mathrm{im}(1-\sigma_1, 1-\sigma_2)$. It follows that Φ coincides with the embedding $\mathrm{Hom}_0(\mathcal{D}) \longrightarrow K_0(C^*(\Lambda))$ of Proposition 2 in this case (c.f. [20, Remark 4.3]). □

Remarks 2. Recall that there exists a pair of 2-graphs Λ_1, Λ_2 both sharing the same vertex matrices but having very different C^*-algebras (e.g. [11, Example 6.1] in which the Cuntz algebra, $\mathcal{O}_2$, and $C(\mathbb{T}) \otimes \mathcal{O}_2$ are seen to be 2-graph C^*-algebras with their underlying 2-graphs sharing common vertex matrices). However, by Proposition 2 and Proposition 6 they share isomorphic K-groups with the isomorphism sending the class of the unit (if any) in $K_0(C^*(\Lambda_1))$ onto that of $K_0(C^*(\Lambda_2))$. Moreover, if $C^*(\Lambda_1)$ and $C^*(\Lambda_2)$ are simple and purely infinite (e.g. if Λ_1 and Λ_2 satisfy the hypotheses of [11, Proposition 4.8, Proposition 4.9]) then by [11, Theorem 5.5] and the Kirchberg-Phillips classification theorem ([10, 15]) we have $C^*(\Lambda_1) \cong C^*(\Lambda_2)$.

References

[1] S. Allen, D. Pask, and A. Sims, *A dual graph construction for higher-rank graphs, and K-theory for finite 2-graphs*, Preprint, arXiv:math.OA/0402126.

[2] M. Auslander and D.A. Buchsbaum, *Groups, Rings, Modules*, Harper & Row Publishers, New York, 1974, Harper's Series in Modern Mathematics, MR0366959 (51 #3205), Zbl 0325.13001.

[3] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, MR1324339 (96a:20072), Zbl 0584.20036.

[4] J. Cuntz, *A class of C^*-algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^*-algebras*, Invent. Math. **63** (1981), no. 1, 25–40, MR0608527 (82f:46073b), Zbl 0461.46047.

[5] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268, MR0561974 (82f:46073a), Zbl 0434.46045.

[6] D. G. Evans, *On higher rank graph C^*-algebras*, PhD Thesis, Univ. Wales, 2002.

[7] L. Fuchs, *Infinite Abelian Groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, 1970, MR0255673 (41 #333), Zbl 0209.05503.

[8] I. Kaplansky, *Commutative Rings*, revised ed., The University of Chicago Press, Chicago, Ill.-London, 1974, MR0345945 (49 #10674), Zbl 0296.13001.

[9] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201, MR0918241 (88j:58123), Zbl 0647.46053.

[10] E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, in, Fields Inst. Commun, Amer. Math. Soc. Providence, to appear.

[11] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20 (electronic), MR1745529 (2001b:46102), Zbl 0946.46044.

[12] S. Mac Lane, *Categories for the Working Mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998, MR1712872 (2001j:18001), Zbl 0906.18001.

[13] D. Pask, *Cuntz-Krieger algebras associated to directed graphs*, Operator algebras and quantum field theory (Rome, 1996), Internat. Press, Cambridge, MA, 1997, pp. 85–92, MR1491109 (98m:46072), Zbl 0921.46050.

[14] D. Pask and I. Raeburn, *On the K-theory of Cuntz-Krieger algebras*, Publ. Res. Inst. Math. Sci. **32** (1996), no. 3, 415–443, MR1409796 (97m:46111), Zbl 0862.46043.

[15] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114 (electronic), MR1745197 (2001d:46086b), Zbl 0943.46037.

[16] I. Raeburn, *On crossed products and Takai duality*, Proc. Edinburgh Math. Soc. (2) **31** (1988), no. 2, 321–330, MR0989764 (90d:46093), Zbl 0674.46038.

[17] I. Raeburn, A. Sims, and T. Yeend, *Higher-rank graphs and their C^*-algebras*, Proc. Edinb. Math. Soc. (2) **46** (2003), no. 1, 99–115, MR1961175 (2004f:46068), Zbl pre01925883.

[18] G. Robertson and T. Steger, *C^*-algebras arising from group actions on the boundary of a triangle building*, Proc. London Math. Soc. (3) **72** (1996), no. 3, 613–637, MR1376771 (98b:46088), Zbl 0869.46035.

[19] ———, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. Reine Angew. Math. **513** (1999), 115–144, MR1713322 (2000j:46109), Zbl pre01333057.

[20] ———, *Asymptotic K-theory for groups acting on $\tilde{A}_2$ buildings*, Canad. J. Math. **53** (2001), no. 4, 809–833, MR1848508 (2002f:46141), Zbl 0993.46039.

[21] C. Schochet, *Topological methods for C^*-algebras. I. Spectral sequences*, Pacific J. Math. **96** (1981), no. 1, 193–211, MR0634772 (84g:46105a), Zbl 0426.46057.

[22] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York, 1966, MR0210112 (35 #1007), Zbl 0145.43303.

[23] J. Spielberg, *Free-product groups, Cuntz-Krieger algebras, and covariant maps*, Internat. J. Math. **2** (1991), no. 4, 457–476, MR1113572 (92j:46120), Zbl 0769.46044.

[24] N. E. Wegge-Olsen, *K-theory and C^*-algebras: A Friendly Approach*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1993, MR1222415 (95c:46116), Zbl 0780.46038.

[25] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994, MR1269324 (95f:18001), Zbl 0797.18001.

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

E-mail address: smadge@math.ku.dk