On Higher Rank Graph C^*-Algebras

by

Dafydd Gwion Evans

This thesis is submitted to the University of Wales in fulfillment of the requirements of candidature for the degree of Doctor of Philosophy.

December 2002

School of Mathematics,
Cardiff University.

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed .... (Dafydd Gwion Evans)

Date ....

Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by citations to an appended bibliography.

Signed .... (Dafydd Gwion Evans)

Date ....

Statement 2

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Signed .... (Dafydd Gwion Evans)

Date ....

Acknowledgements

My sincere gratitude goes to my supervisor Prof. David E. Evans for giving me the opportunity to study in the fascinating field of Operator Algebras, for his guidance and for his careful reading of the thesis. I have gained much insight from the active Operator Algebra group in Cardiff - the wide-ranging seminars and informal discussions all proved to be valuable. My thanks go to the people involved especially Paulo Pinto for his encouragement and interest in my work along with many interesting discussions.

I would like to take this opportunity to thank the staff at the School of Mathematics, Cardiff University whose support I have benefited from during my time at Cardiff.

Last, but most certainly not least, I thank my family, especially my wife Angharad, for their constant support and understanding.

Dedication

I'r merched - Angharad, Gwenllian ac Esyllt.

Abstract

The higher rank graphs and higher rank graph C^*-algebras introduced by Kumjian and Pask generalise row finite directed graphs and row finite directed graph C^*-algebras respectively. Higher rank graph C^*-algebras provide even more examples of Kirchberg algebras (i.e. separable, nuclear, purely infinite, unital C^*-algebras), which are classified by their K-theory. Under a mild condition we calculate the K-theory of rank 2 graph C^*-algebras in terms of the underlying rank 2 graph.

We also show that the representation of a Cuntz-Krieger-Toeplitz algebra on a subspace of the full Fock space has a higher rank analogue and that a higher rank graph C^*-algebra is isomorphic to a Cuntz-Pimsner algebra, which may prove useful in the analysis of more general Cuntz-Pimsner algebras.

A discussion on necessary and sufficient conditions on a higher rank graph to ensure that its C^*-algebra is approximately finite dimensional is included. Some necessary conditions are proved and a number of increasingly ambitious conjectures, with ideas of proofs, on sufficient conditions are proposed culminating in a conjecture about a single necessary and sufficient condition.

Contents

List of Figures

Chapter 1

Introduction

In this chapter we review the background and motivation of our study of higher rank graph C^*-algebras and state the organisation of the thesis.

1.1 Background

A class of C^*-algebras, $\mathcal{O}_A$, generated by a family of (non-zero) partial isometries $\{S_i \mid i \in \Sigma\}$ satisfying

$$S_i^* S_j = 0 \quad \text{if } i \neq j, \qquad S_i^* S_i = \sum_{j \in \Sigma} A(i,j) S_j S_j^* \tag{1.1}$$

where $A = (A(i,j))_{i,j \in \Sigma}$ is a matrix with no zero row nor column, Σ a finite set and $A(i,j) \in \{0,1\}$ for $i,j \in \Sigma$, was introduced by Cuntz and Krieger in [12, 11]. They generalised an earlier construction of Cuntz [10] and had a close relationship to topological Markov chains - a key theory in symbolic dynamics. These C^*-algebras immediately became the focus of considerable interest (see [19]) and were themselves, in time, generalised in numerous ways. Cuntz-Krieger algebras have played a part in diverse fields of mathematics. Representations of the Cuntz algebra have appeared in wavelet theory (see [6] and the references therein). Using Cuntz algebras, Izumi [30] introduced a new technique to construct subfactors with finite Jones-Watatani indices (see [20]). Another occurrence of the Cuntz algebras is in the large-N matrix models for open strings [40]. Such models include Quantum Chromodynamics and the M(atrix) theory of strings.

We refer the interested reader to Kumjian's survey [34], on the generalisations of Cuntz-Krieger algebras concentrating mainly on the generalisations relating to graphs. However, we state some relevant notions from Kumjian's survey and some of the many developments since its publication.

A directed graph $E = (E^0, E^1, r, s)$ consists of a countable *vertex* set E^0, a countable *edge* set E^1 and maps $r, s : E^1 \longrightarrow E^0$ that determine the range and source of an edge, respectively. Thus we may picture a directed graph as a collection of discrete points (the vertices) which may be connected to each other by one or more arrows (the directed edges) (see [1, 37] for more details). The graph C^*-algebras introduced in [37], built on earlier constructions [18, 44, 29, 38], were based on row finite directed graphs - those directected graphs whose vertices emitted a finite number of edges (i.e $s^{-1}(v)$ finite for all $v \in E^0$).

The class of directed graphs used to define C^*-algebras has been extended in recent years from row finite directed graphs to arbitrary directed graphs [25, 14]. Moreover, in [60], Tomforde defines the notion of an ultragraph, which is a generalisation of a directed graph, and associates a C^*-algebra to it. The class of ultragraph C^*-algebras contains not only directed graph C^*-algebras but it also contains Exel-Laca algebras [21, 22, 23], which defined by arbitrary infinite matrices with entries in $\{0, 1\}$ and with no zero row nor column, may be thought of as a more direct generalisation of Cuntz-Krieger algebras. Much of the analysis on row finite graph C^*-algebras carries over to ultragraph C^*-algebras.

Spielberg [58] has provided a nice functorial theory of graph C^*-algebras.

An even larger class of C^*-algebras generalising Cuntz-Krieger algebras, graph C^*-algebras and crossed products by $\mathbb{Z}$ [45, 48] is the class of Cuntz-Pimsner algbebras introduced by Pimsner in [47]. It was shown in [59] that the class of Cuntz-Pimsner algebras include Exel-Laca algebras.

In [57], Spielberg realised that a crossed product algebra $C(\Omega) \rtimes \Gamma$ where Ω is the boundary of a tree and Γ is a free group, is isomorphic to a Cuntz-Krieger algebra. Robertson and Steger generalised this observation in a series of papers [51, 52, 53]. Ultimately they defined a C^*-algebra $\mathcal{A}$ via a finite sequence of finite $\{0, 1\}$-matrices (i.e. a matrix with entries in $\{0, 1\}$) $M_1, \ldots, M_r$ made to satisfy certain conditions (H0)-(H3). The C^*-algebra $\mathcal{A}$ is in fact generated by r Cuntz-Krieger algebras, one for each $M_1, \ldots, M_r$. Accordingly they named their algebras higher rank Cuntz-Krieger algebras, the rank being r. Furthermore if $\mathcal{B}$ is an affine building of type $\tilde{A}_2$ [8, 54] and Γ is a group of type rotating automorphisms of $\mathcal{B}$ which acts freely on the vertex set with finitely many orbits then there is a natural action of Γ on the boundary Ω of $\mathcal{B}$ so that the crossed product algebra $C(\Omega) \rtimes \Gamma$ can be formed. Robertson and Steger show that $C(\Omega) \rtimes \Gamma \cong \mathcal{A}$ where $\mathcal{A}$ is a rank 2 Cuntz-Krieger algebra - M_1, M_2 being defined in terms of the building $\mathcal{B}$ (see [52, §7]).

Kumjian and Pask [35] noticed that Robertson and Steger had constructed their algebras from a set, W of *(higher rank) words* in a finite *alphabet* A - the common index set of the $\{0, 1\}$-matrices - and realised that W could be thought of as a generalised directed graph - a higher rank graph. Subsequently, Kumjian and Pask associated a C^*-algebra, $C^*(\Lambda)$ to the higher rank graph Λ and showed that $\mathcal{A} \cong C^*(W)$ [35, Corollary 3.5 (ii)]. They have also shown that the original results of Cuntz and Krieger on the relationship between Cuntz-Krieger algebras and topological Markov chains have a higher dimensional analogue. In [36] they construct Ruelle algebras R_s, R_u from a higher dimensional two-sided path space of a k rank graph Λ on which there is a $\mathbb{Z}^k$ action and stable, unstable equivalence relations. They show that R_s is strongly Morita equivalent to $C^*(\Lambda)$ with a similar result for R_u.

This brings us to our starting point. We shall continue Kumjian's and Pask's investigation of their higher rank graph C^*-algebras. The remainder of the thesis is organised as follows.

1.2 Layout

In Chapter 2 we introduce the various concepts used throughout the thesis. In Section 2.1 we recall the definition of a higher rank graph (or k-graph), a k-graph C^*-algebra and set some notation for the thesis. We also define the notion of the skeleton of a k-graph to be a k-coloured directed graph. The skeleton of a k-graph is useful as a visualisation of a k-graph and indeed in the construction of examples. In Section 2.2 we recall the minimum amount of information on Hilbert modules from [61, 50, 47]. Similarly in Section 2.3 we recall the bare essentials of homological algebra needed for Chapter 5.

In Chapter 3 we state two *-representations of a k-graph. Section 3.1 contains a *-representation reminiscent of the construction of the Cuntz-Krieger algebras from Fock space [18, 17]. More precisely, we define a concrete C^*-algebra $\mathcal{T}_\Lambda$ acting on a Hilbert space constructed from Λ and find that there is a *-representation of Λ on a quotient of this C^*-algebra. If Λ is a row finite 1-graph with no sinks nor sources and Λ^0 finite then the identity *-homomorphisms on $\mathcal{T}_\Lambda$ and $\mathcal{T}_{A_\Lambda}$, as defined in [18], where $A_\Lambda = (A(\lambda,\mu))_{\lambda,\mu\in\Lambda^1}$ is the edge matrix of Λ given by

$$A(\lambda,\mu) = \begin{cases} 1 & \text{if } s(\lambda) = r(\mu) \\ 0 & \text{otherwise} \end{cases},$$

are unitarily equivalent. In Section 3.2 we recall the definition of a Cuntz-Pimsner algebra from [47] and present a *-representation of a k-graph Λ on a Cuntz-Pimsner algebra, $\mathcal{O}_X$. This time the *-representation is always faithful in the sense that $C^*(\Lambda) \cong \mathcal{O}_X$.

In Chapter 4 we discuss the possibilities of when a k-graph C^*-algebra is an approximately finite dimensional (AF-)algebra [4, 15]. The AF-algebras are a very important class of C^*-algebras with uses in mathematical physics and C^*-algebra theory. Indeed the classification of C^*-algebras can be thought to have begun with Elliott's classification of AF-algebras via their ordered K-theory [16]. Some necessary conditions for a k-graph C^*-algebra to be AF are given and sufficient conditions are conjectured. The situation is not satisfactory.

In Chapter 5 we combine the methods of [36, 53] to calculate the K-theory of 2-graph C^*-algebras under a mild condition. Amongst other uses, the K-theory will be useful in deciding on the possibility of a 2-graph being AF. This is because it is often the case that the only obstruction towards being an AF algebra is that its K_1 group is trivial.

In Appendix A we give an alternative construction of a k-graph C^*-algebra as a Cuntz-Pimsner algebra that was inspired by [59].

In Appendix B we give an example of how to calculate the K-theory of a

2-graph using our results from Chapter 5. The example we have chosen is conjectured to be AF and we show that there is no obstruction from its K_1 group.

Finally in Appendix C we give a list of possible developments of the thesis.

Chapter 2

Preliminaries

We recall some notions that are used throughout the thesis. First some general conventions used throughout the thesis.

Unless otherwise stated, we let $\mathbb{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and $\mathbb{K}(\mathcal{H})$ the compact operators on $\mathcal{H}$. When a Hilbert space is not specified then $\mathbb{B}$ (respectively $\mathbb{K}$) denotes the bounded linear operators (respectively compact operators) on a separable infinite dimensional Hilbert space $\mathcal{H}$. We denote the multiplier algebra of a C^*-algebra A by $\mathcal{M}(A)$ [61, 50]. We let $\mathbb{N}$ be the set of non-negative integers and $\{e_i \,|\, i \in \{1, \dots, k\}\}$ be the canonical generators of $\mathbb{N}^k$ (i.e. $e_i = (0, \dots, 0, 1, 0 \dots, 0) \in \mathbb{N}^k$ where 1 is in the i^{th} component). Where appropriate we shall view $\mathbb{N}^k$ as a category with a single object $\{0\}$ and morphisms $n \in \mathbb{N}^k$ with $r(n) = s(n) = 0$ and composition defined by the pointwise addition of $\mathbb{N}^k$ or as an abelian monoid under addition with identity 0, which is the positive cone of $\mathbb{Z}^k$ under the usual coordinatewise partial order. Define a bilinear form $\langle \cdot, \cdot \rangle : \mathbb{N}^k \times \mathbb{N}^k \longrightarrow \mathbb{N}$ by $\langle n, m \rangle = \sum_{i=1}^k n_i m_i$ for all $n = (n_1, \dots, n_k), m = (m_1, \dots, m_k) \in \mathbb{N}^k$.

2.1 Higher rank graphs

The concept of a *higher rank graph* or k-graph ($k = 1, 2, \ldots$ being the rank) was introduced by A. Kumjian and D. Pask in [35]. We recall their definition of a k-graph.

Definition 2.1 ([35, Definitions 1.1]) *A* **k-graph** *(rank k graph or higher rank graph) (Λ, d) consists of a countable small category Λ (with range and source maps r and s respectively) together with a functor $d : \Lambda \longrightarrow \mathbb{N}^k$ satisfying the* **factorisation property:** *for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$. For $n \in \mathbb{N}^k$ and $v \in \Lambda^0$ we write $\Lambda^n := d^{-1}(n)$, $\Lambda(v) := r^{-1}(v)$ and $\Lambda^n(v) := \{\lambda \in \Lambda^n \,|\, r(\lambda) = v\}$. Let $\Sigma := \bigcup_{i=1}^k \Lambda^{e_i}$.*

We refer to [42] as an appropriate reference on category theory as we will be interested more in the combinatorial graph-like nature of higher rank graphs. As the name suggests a higher rank graph can be thought of as a higher rank analogue of a directed graph. By [35, Remarks 1.2] we may identify the objects of Λ with Λ^0. Thus we will refer to elements of Λ^0 as vertices and morphisms in Λ as paths. Moreover we will refer to paths of degree e_i (for some $i \in \{1, \ldots, k\}$) as i-edges or just edges. We will let (Λ, d) (or more succinctly Λ with the understanding that the degree functor will be denoted by d) stand for a generic k-graph. As in [35] we will assume throughout the thesis, unless otherwise stated (for example when Λ is a finite locally convex k-graph - see Chapter 3), that all k-graphs Λ are row finite and have no sources as defined below.

Definition 2.2 ([35, Definitions 1.4]) *The k-graph Λ is* **row finite** *if for each $m \in \mathbb{N}^k$ and $v \in \Lambda^0$ the set $\Lambda^m(v)$ is finite. We say that Λ has* **no**

sources *if* $\Lambda^m(v) \neq \emptyset$ *for all* $v \in \Lambda^0$ *and* $m \in \mathbb{N}^k$. *Similarly we say that* Λ *has* **no sinks** *if* $s^{-1}(v) \cap \Lambda^n \neq \emptyset$ *for all* $v \in \Lambda^0$ *and* $n \in \mathbb{N}^k$.

A justification that k-graphs may be thought of as a higher rank analogue to directed graphs is the following motivating example.

Let $E = (E^0, E^1, r_E, s_E)$ be a row finite directed graph with no sinks (as defined in [37, 1] for example) where we denote the range and source maps of E by r_E to s_E to distinguish them from the range and source maps of the resulting category. Then the finite path space E^* is a 1-graph where the degree functor is merely the length function on E^* and the range and source maps $r, s : E^* \longrightarrow E^0$ are given by $r = s_E,\ s = r_E$. Note that the direction of a path is inverted to agree with category theory notation. Consequently, the sinks, sources of a directed graph become the source, sinks of the associated 1-graph respectively. It is clear that up to isomorphism (see Definition 2.5), every 1-graph arises from a directed graph in the above way.

Loosely speaking, a k-graph can be thought of as a coloured directed graph, in the following sense. First, recall the definition of the vertex matrices of Λ from [35, §6].

Definition 2.3 *Define the vertex matrices of* Λ, M_i, *by the following: For* $u, v \in \Lambda$ *and* $i = 1, 2, \ldots, k$

$$M_i(u, v) := |\{\lambda \in \Lambda^{e_i} \mid r(\lambda) = u, s(\lambda) = v\}|\,.$$

Let $C := \{c_i\}_{i=1}^k$ be a finite set of colours. We may construct a coloured directed graph E from a k-graph Λ by letting the vertices of E be $E^0 := \Lambda^0$ and drawing $M_i(u, v)$ edges of colour c_i from u to v. So an edge of colour c_i represents an i-edge. In other words, a k-coloured directed graph E is a sextuple (E^0, E^1, r, s, C, c) where (E^0, E^1, r, s) is a directed graph, C is a set (of colours) of cardinality k and $c : E^1 \longrightarrow C$ is a colouring of the edges. The same construction was noticed in [49] and we adopt their fitting terminology for the directed graph E above but we choose to direct our edges in the opposite direction to theirs so that the skeleton of a 1-graph is just the underlying directed graph.

Definition 2.4 *We call any coloured directed graph constructed from a k-graph Λ as above, the* **skeleton** *of Λ. We adopt the convention that blue edges represent 1-edges and red edges represent 2-edges.*

Note that not every k-coloured directed graph (i.e. a directed graph with a mapping from a finite set of k colours into the set of edges of the graph) is a skeleton of a k-graph. There are compatability conditions that will be made clear later.

We may now visualise how paths are composed in Λ. However, similar to the fact that each Bratteli diagram gives rise to an unique AF-algebra but an AF-algebra gives rise to many Bratteli diagrams (see [4, 15]), is the fact that each k-graph gives rise to an unique skeleton but there may be more than one k-graph with the same skeleton. This is a consequence of the factorisation property on Λ. For example, we claim that the 2-coloured graph in Figure 2.1 is the skeleton of a 2-graph $O_{(2,2)}$ say.

Consider the degree (1,1) path $a\alpha$. By the factorisation property and commutativity of $\mathbb{N}^2$ we must have $a\alpha = \lambda\mu$ for some $\lambda \in O_{(2,2)}^{e_2}$, $\mu \in \Lambda^{e_1}$. There is a choice as to what λ and μ should be. The skeleton in Figure 2.1 gives us no information as to what that choice is.

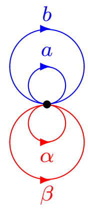

Figure 2.1: The skeleton of $O_{(2,2)}$

For example, let $O_{(2,2)}$ be the 2-graph generated (as a category) by $O^0_{(2,2)} = \{*\}$, $O^{e_1}_{(2,2)} = \{a, b\}$ and $O^{e_2}_{(2,2)} = \{\alpha, \beta\}$ satisfying the following so called factorisations of the degree (1,1) paths (a more precise definition of $O_{(2,2)}$ will be made later):

$$a\alpha = \alpha a \quad b\alpha = \beta a$$
$$a\beta = \alpha b \quad b\beta = \beta b$$

As well as being useful visualisation aids, skeletons can be used to construct k-graphs. Using their notion of a product system over right-angled Artin semigroups Fowler and Sims have shown that (up to isomorphism) every k-graph arises from a collection of k directed graphs each with a common vertex set, satisfying a certain compatability condition ([24, Theorem 2.1, Theorem 2.2 Remark 2.3]). Therefore, given a k-coloured graph we may form k directed graphs with a common vertex set in the obvious way and check if the compatatability condition is satisfied.

The compatability condition is satisfied if the vertex matrices of the directed graphs satisfy conditions (H0), (H1a), (H1b) and (H1c) from [52].

When $k = 2$ this construction reduces to Kumjian and Pask's construction of 2-graphs from two 1-graphs in [35, §6]. In this case the only requirements of the directed graphs are that they are row finite directed graphs with no sinks and their vertex matrices must commute. Let E, F be directed graphs and A, B be their associated 1-graphs respectively. If E, F satisfy the above requirements then there exist bijections from $A^1 * B^1 := \{(a,b) \in \mathcal{A}^1 \times B^1 \,|\, s(a) = r(b)\}$ onto $B^1 * A^1 := \{(b,a) \in \mathbb{B}^1 \times A^1 \,|\, s(b) = r(a)\}$. Let θ be such a bijection then we may form the 2-graph $A *_\theta B$ where $(A *_\theta B)^0 := A^0 = B^0$ and we may identify $(A *_\theta B)^{e_1}$ with A^1 and $(A *_\theta B)^{e_2}$ with B^1. Paths in $(A *_\theta B) \backslash (A *_\theta B)^0$ may be thought of as (non-empty) sequences $\{\lambda_i\}_{i=1}^n \subset A^1 \sqcup B^1$ where $s(\lambda_i) = r(\lambda_{i+1})$ for all $i = 1, \ldots, n-1$. We write such a sequence as $\lambda_1 \lambda_2 \cdots \lambda_n$. The degree of $\lambda_1 \lambda_2 \cdots \lambda_n$ is (m_1, m_2) where m_1 is the number of λ_i's in A^1 and m_2 is the number of λ_i's in B^1. The role of the bijection θ is to ensure that $A *_\theta B$ satisfies the factorisation property as indicated by the following. For $a \in (A *_\theta B)^{e_1}$ and $b \in (A *_\theta B)^{e_2}$ with $s(a) = r(b)$ we let $ab = b'a'$ in $(A *_\theta B)^{(1,1)}$ if $\theta(a,b) = (b', a')$. Henceforth we will refer to the bijections such as θ as factorisation rules. For a formal definition of $A *_\theta B$ see [35, §6].

An example of a k-graph is Ω_k [35, Examples 1.7 (ii)], which has the skeleton in Figure 2.2 when $k = 2$.

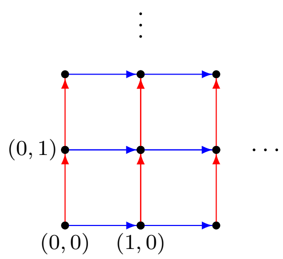

Figure 2.2: The skeleton of Ω_2.

The k-graph Ω_k is the prototype for an infinite path in Λ. The following definition makes the notion of an infinite path in Λ precise.

Definition 2.5 ([35, Definitions 2.1,4.1]) *Let Λ be a k-graph, then*

$$\Lambda^\infty = \{x : \Omega_k \longrightarrow \Lambda \,|\, x \text{ is a } k\text{-graph morphism}\}$$

is the infinite path space of Λ *where by a* k*-graph morphism we mean a map that respects concatenation, the range and source maps and the degree functor. We shall refer to elements of* Λ^∞ *as infinite paths in* Λ*. For each* $p \in \mathbb{N}^k$ *define* $\sigma^p : \Lambda^\infty \longrightarrow \Lambda^\infty$ *by* $\sigma^p(x)(m,n) = x(m+p, n+p)$ *for* $x \in \Lambda^\infty$ *and* $(m,n) \in \Omega_k$*. For* $x \in \Lambda^\infty$ *and* $p \in \mathbb{Z}^k$ *we say that* p *is a* **period** *of* x *if for every* $(m,n) \in \Omega_k$ *with* $m + p \geq 0$ *we have* $x(m+p, n+p) = x(m,n)$*. We say that* x *is* **periodic** *if it has a nonzero period.*

Motivated by their definition of a graph C^*-algebra in [37], to each k-graph Λ, Kumjian and Pask associated a C^*-algebra, called a k-graph C^*-algebra, that is generated by a family of partial isometries satisfying, amongst other relations, a Cuntz-Krieger type relation. Since we will be dealing with finite k-graphs we make the following equivalent definition of a k-graph C^*-algebra (see [35, Definitions 1.5, Remarks 1.6 (iii)]).

Definition 2.6 *Let* Λ *be a row finite* k*-graph. Then* $C^*(\Lambda)$ *is defined to be the universal* C^**-algebra generated by a family* $\{s_\lambda \,|\, \lambda \in \Lambda\}$ *of partial isometries satisfying:*

(i) $\{s_v \,|\, v \in \Lambda^0\}$ *is a family of mutually orthogonal projections,*

(ii) $s_{\lambda\mu} = s_\lambda s_\mu$ *for all* $\lambda, \mu \in \Lambda$ *such that* $s(\lambda) = r(\mu)$*,*

(iii) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ *for all* $\lambda \in \Lambda$*,*

(iv) $s_v = \sum_{\alpha \in \Lambda^{e_i}(v)} s_\lambda s_\lambda^*$ *for all* $i \in \{1, \ldots, k\}$ *and* $v \in \Lambda^0$ *such that* $\Lambda^{e_i}(v) \neq \emptyset$*.*

For $\lambda \in \Lambda$*, define* $p_\lambda := s_\lambda s_\lambda^*$ *(note that* $p_v = s_v$ *for all* $v \in \Lambda^0$*). A family of partial isometries satisfying (i)-(iv) above is called a* ***-representation** *of* Λ*.*

See [2, Definition 1.2] for the definition of an universal C^*-algebra.

Recall from [35, Lemma 3.1] that for $\lambda, \mu \in \Lambda$, $s_\lambda^* s_\mu = \sum_{\substack{\lambda\alpha = \mu\beta \\ d(\lambda\alpha) = q}} s_\alpha s_\beta^*$ for all $q \in \mathbb{N}^k$ such that $d(\lambda) \leq q$ and $d(\mu) \leq q$. Consequently, to reduce notation we make the following definition.

Definition 2.7 *For $\lambda, \mu \in \Lambda$ define $\mathcal{E}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda \,|\, \lambda\alpha = \mu\beta,\ d(\lambda\alpha) = \sup\{d(\lambda), d(\mu)\}\}$ to be the set of (minimal) extensions of λ with respect to μ.*

2.2 Hilbert modules

We take our notation and definitions from [61, Chapter 15] and [50, Chapter 2]. First we introduce the notion of a R-module, where R is a ring, taken from [41].

Definition 2.8 *Let R be a ring. A right* ***R*-module** *X is an additive abelian group together with a function $p : X \times R \longrightarrow X$, written $p(x, r) = x \cdot r$, such that*

(i) $x \cdot (r + r') = x \cdot r + x \cdot r'$ *for all* $x \in X,\ r, r' \in R$,

(ii) $(x + y) \cdot r = x \cdot r + y \cdot r$ *for all* $x, y \in X,\ r \in R$,

(iii) $x \cdot (rr') = (x \cdot r) \cdot r'$ *for all* $x \in X,\ r, r' \in R$.

Similarly, one may define a left R-module.

Note that an abelian group is a left $\mathbb{Z}$-module in the natural way.

Definition 2.9 *Let A be a C^*-algebra. A* **pre Hilbert A-module** *is a right A-module X (which is at the same time a complex vector space) equipped with an A-valued* **inner product** $\langle \cdot, \cdot \rangle : X \times X \longrightarrow A$ *that is sesquilinear, positive definite, and respects the module action. In other words:*

(i) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$ *for* $x, y_1, y_2 \in X$,

(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ *for* $x, y \in X,\ a \in A$,

(iii) $\langle x, zy \rangle = z\langle x, y \rangle$ *for* $x, y \in X,\ z \in \mathbb{C}$,

(iv) $\langle x, y \rangle = \langle y, x \rangle^*$ *for* $x, y \in X$,

(v) $\langle x, x \rangle \geq 0$ *for* $x \in X$, *and* $\langle x, x \rangle \iff x = 0$.

Definition 2.10 *The* **norm** *of an element $x \in X$ is defined as*

$$\|x\|_X := \sqrt{\|\langle x, x \rangle\|}.$$

If a pre Hilbert A-module is complete with respect to its norm, it is said to be a **Hilbert A-module***.*

Definition 2.11 *We call X a* **full Hilbert A-module** *when* $A = \overline{\text{span}}\{\langle x, y \rangle \mid x, y \in X\}$.

As the name suggests the norm on a Hilbert A-module behaves like the norm on a Hilbert space, that is

(i) $\|x\|_X \geq 0$ for all $x \in X$ and $\|x\|_X = 0 \iff x = 0$,

(ii) $\|zx\|_X = |z|\|x\|_X$ for all $x \in X,\ z \in \mathbb{C}$,

(iii) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$ for all $x, y \in X$.

The fact that (iii) holds follows from the Cauchy-Schwartz inequality satisfied by the norm on a Hilbert A-module.

Definition 2.12 *Let X be a Hilbert A-module. A map $T : X \longrightarrow X$ is said to be* **adjointable** *if there exists a map $T^* : X \longrightarrow X$ satisfying*

$$\langle x, Ty \rangle = \langle T^* x, y \rangle$$

for all $x, y \in X$. Such a map T^* is then called the **adjoint** *of T.*

By $\mathbb{B}(X)$ we denote the set of all adjointable maps on X.

The adjoint of a map is unique. If a map T is adjointable then it is a module map that is bounded with respect to the operator norm

$$\|T\| := \sup\{\|Tx\|_X \mid \|x\|_X = 1\}.$$

Equiped with the natural *-operation, norm, ring and vector space structure $\mathbb{B}(X)$ is in fact a C^*-algebra.

Definition 2.13 *If $x, y \in X$, define $\theta_{x,y} : X \longrightarrow X$ by*

$$\theta_{x,y}(z) := x\langle y, z \rangle.$$

We let $\mathbb{K}(X) := \overline{\text{span}}\{\theta_{x,y} \mid x, y \in X\}$ be the set of **compact adjointable operators** *on X.*

As the notation suggests, $\mathbb{K}(X)$ is a C^*-algebra, which is an essential ideal in $\mathbb{B}(X)$.

Let X be a Hilbert module over a C^*-algebra A and Y be a Hilbert module over a C^*-algebra B. If $\psi : A \longrightarrow \mathbb{B}(X)$ is a *-homomorphism then we may form the ψ-tensor product of X and Y denoted by $X \otimes_\psi Y$ [61, 15.M]. The tensor product is also a Hilbert module in the natural way and has the inner product given below

$$\langle x_1 \otimes x_2 \,|\, y_1 \otimes y_2 \rangle = \langle x_2 \,|\, \psi(\langle x_1, y_1 \rangle)(y_2) \rangle$$

for all $x_1, y_1 \in X,\ x_2, y_2 \in Y$.

If $\{X_i\}_{i=0}^\infty$ is a family of Hilbert modules over a C^*-algebra A then we may define their direct sum $\bigoplus_{i=0}^\infty X_i$ to be the set of all sequences $\{x_i\}_{i=0}^\infty$, with $x_i \in X_i$, such that $\sum_{i=0}^\infty \langle x_i, x_i \rangle$ converges in A endowed with coordinatewise addition and module action. Define an inner product on $\bigoplus_{i=0}^\infty X_i$ by $\langle x, y \rangle = \sum_{i=0}^\infty \langle x_i, y_i \rangle$ for all $x = \{x_i\}, y = \{y_i\} \in \bigoplus_{i=0}^\infty X_i$. Naturally $\bigoplus_{i=0}^\infty X_i$ is a Hilbert module (see [39] for details).

2.3 Homology

Definition 2.14 *For any ring R, a* **chain complex** *K of $R-$modules is a family $\{K_n, \partial_n\}$ of R-modules K_n and R-module maps $\partial_n : K_n \longrightarrow K_{n-1}$ called differentials, defined for all $n \in \mathbb{Z}$, such that $\partial_n \partial_{n+1} = 0$.*

A chain complex appears as a doubly infinite sequence

$$\cdots \longleftarrow K_{-1} \xleftarrow{\partial_0} K_0 \xleftarrow{\partial_1} K_1 \xleftarrow{\partial_2} K_2 \longleftarrow \cdots$$

Note that we may have $K_n = 0$ for all $n < M,\ n > N$ for some $N, M \in \mathbb{Z}$ with $M < N$.

Definition 2.15 *If $K = \{K_n, \partial_n\}$ is a chain complex of R-modules, then the* **homology** $\mathrm{Hom}(K)$ *is the family of R-modules $\{H_n(K)\}$ where $H_n(K) := \ker \partial_n / \operatorname{im} \partial_{n+1}$ for all $n \in \mathbb{Z}$.*

Chapter 3

Representations of k-graphs

We construct a *-representation of a k-graph Λ in section 3.1 that generalises the representation of Cuntz-Toeplitz algebras found in [17] and the representations on a subspace of the full Fock space found in [18] (c.f. [20, §2.10]). In section 3.2 we construct a faithful *-representation of Λ on a Cuntz-Pimsner algebra in the sense that $C^*(\Lambda) \cong \mathcal{O}_X$ for an appropriate Hilbert bimodule X.

3.1 Constructing k-graph C^*-algebras as quotients of concrete C^*-algebras

Let (Λ, d) be a k-graph and form the Hilbert space $\mathcal{H} := \mathbb{C}\Omega \oplus \ell^2(\Lambda \backslash \Lambda^0)$ where Ω is a unit vector. We shall identify a path λ in $\Lambda \backslash \Lambda^0$ with its corresponding canonical basis vector in $\mathcal{H}$.

Associate to each $\lambda \in \Lambda\backslash\Lambda^0$ a partial isometry $T_\lambda \in \mathbb{B}(\mathcal{H})$ defined by

$$\begin{aligned} T_\lambda\Omega &= \lambda \\ T_\lambda\mu &= \begin{cases} \lambda\mu & \text{if } r(\mu) = s(\lambda) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } \mu \in \Lambda\backslash\Lambda^0, \end{aligned}$$

and to each $v \in \Lambda^0$ a projection $T_v \in \mathbb{B}(\mathcal{H})$ defined by

$$\begin{aligned} T_v\Omega &= \Omega \\ T_v\mu &= \begin{cases} \mu & \text{if } r(\mu) = v \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } \mu \in \Lambda\backslash\Lambda^0. \end{aligned}$$

Furthermore, let $Q_\lambda := T_\lambda T_\lambda^*$ for all $\lambda \in \Lambda$. (Notice that $Q_v = T_v$ for all $v \in \Lambda^0$.) Define $R_{v,i} := T_v - \sum_{\beta\in\Lambda^{e_i}(v)} Q_\beta$ and $R_v := \prod_{i=1}^k R_{v,i}$. Then we have the following simple consequences.

Lemma 3.1 *Fix $i \in \{1, 2, \ldots, k\}$ then $\{R_{v,i} \mid v \in \Lambda^0\}$ is a family of projections and R_v is the rank 1 projection, R, onto $\mathbb{C}\Omega$ for any $v \in \Lambda^0$.*

Proof. We have that

$$\begin{aligned} R_{v,i}\Omega &= \Omega \\ R_{v,i}\mu &= \begin{cases} \mu & \text{if } r(\mu) = v, d(\mu) \not\geq e_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } \mu \in \Lambda\backslash\Lambda^0. \end{aligned}$$

From which we can clearly see that the first statement is true. Now

$$\begin{aligned} R_v\Omega &= \Omega \\ R_v\mu &= \begin{cases} \mu & \text{if } r(\mu) = v, d(\mu) \not\geq e_i \text{ for all } i \in \{1, 2, \ldots, k\} \\ 0 & \text{otherwise} \end{cases} \\ &= 0 \qquad \text{for all } \mu \in \Lambda\backslash\Lambda^0, \end{aligned}$$

so that R_v is indeed the projection onto $\mathbb{C}\Omega$ for all $v \in \Lambda^0$. □

Definition 3.1 *Let $\mathcal{K}_\Lambda$ be the closed two sided ideal of $\mathcal{T}_\Lambda := C^*(T_\lambda \mid \lambda \in \Lambda)$ generated by $\{R_{v,i} \mid v \in \Lambda^0, i = 1, \ldots, k\}$.*

In order to gain further insight on the structure of $\mathcal{T}_\Lambda$ and $\mathcal{K}_\Lambda$ we list some relations that are satisfied by their generators.

Lemma 3.2 *The following relations are satisfied in $\mathcal{T}_\Lambda$:*
Let $u, v \in \Lambda^0$, $\xi, \eta \in \Lambda$, $i, j = 1, 2, \ldots, k$, $I \subseteq \{1, 2, \ldots, k\}$ then

(i) $Q_u Q_v = \begin{cases} Q_u & \text{if } u = v \\ R & \text{otherwise} \end{cases},$

(ii) $R_{u,i} Q_v = \begin{cases} R_{u,i} & \text{if } u = v \\ R & \text{otherwise} \end{cases},$

(iii) $R_{u,i} R_{v,j} = R_{v,j} R_{u,i}$,

(iv) $R_{v,I} := \prod_{\iota \in I} R_{v,\iota}$ *is a projection,*

(v) for $d(\xi) > 0$ *we have* $R_{v,i} T_\xi = 0$ *if* $r(\xi) \neq v$ *or* $d(\xi) \geq e_i$,

(vi) $T_\xi R_{v,i} = T_\xi R$ *if* $s(\xi) \neq v$,

(vii) $R_{r(\xi),i} T_\xi = T_\xi R_{s(\xi),i}$ *if* $d(\xi) \ngeq e_i$,

(viii) $T_\xi T_\eta = \begin{cases} T_{\xi\eta} & \text{if } s(\xi) = r(\eta) \\ T_\xi R & \text{if } s(\xi) \neq r(\eta),\ d(\eta) = 0 \\ 0 & \text{otherwise} \end{cases},$

(ix) $T_\xi^* T_\eta = \sum_{(\alpha,\beta) \in \mathcal{E}(\xi,\eta)} T_\alpha T_\beta^*$ *if* $\sup\{d(\xi), d(\eta)\} \neq 0$.
(The sum is taken to be zero if $\mathcal{E}(\xi, \eta) = \emptyset$.)

Proof. We shall prove these relations between the generating operators of $\mathcal{T}_\Lambda$ and $\mathcal{K}_\Lambda$ by showing that they hold when the operators act on the canonical orthonormal basis of $\mathcal{H}$. The results will follow by the linearity and continuity of the operators involved. Let $u, v \in \Lambda^0$, $\mu \in \Lambda \backslash \Lambda^0$, $\xi, \eta \in \Lambda$, $i, j = 1, 2, \ldots, k$, $I \subseteq \{1, 2, \ldots, k\}$ then

(i) clearly,

$$\begin{aligned} Q_u Q_v \, \Omega &= \Omega \\ &= \begin{cases} Q_u \, \Omega & \text{if } u = v \\ R \, \Omega & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} Q_u Q_v \, \mu &= \begin{cases} Q_u \, \mu & \text{if } r(\mu) = v \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu & \text{if } r(\mu) = v = u \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} Q_u \, \mu & \text{if } u = v \\ R \, \mu & \text{otherwise} \end{cases} . \end{aligned}$$

The result follows.

(ii)

$$
\begin{aligned}
R_{u,i}Q_v\,\Omega &= \Omega = \begin{cases} R_{u,i}\,\Omega & \text{if } u = v \\ R_\mu\,\Omega & \text{otherwise} \end{cases} \\
R_{u,i}Q_v\,\mu &= \begin{cases} R_{u,i}\,\mu & \text{if } r(\mu) = v \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mu & \text{if } r(\mu) = v = u \text{ and } d(\mu) \not\geq e_i \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} R_{u,i}\,\mu & \text{if } u = v \\ R\,\mu & \text{otherwise} \end{cases} .
\end{aligned}
$$

(iii)

$$
\begin{aligned}
R_{u,i}R_{v,j}\,\Omega &= \Omega = R_{v,j}R_{u,i}\,\Omega \\
R_{u,i}R_{v,j}\,\mu &= \begin{cases} R_{u,i}\,\mu & \text{if } r(\mu) = v \text{ and } d(\mu) \not\geq e_j \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mu & \text{if } r(\mu) = v = u,\ d(\mu) \not\geq e_j \text{ and } d(\mu) \not\geq e_i \\ 0 & \text{otherwise} \end{cases} \\
&= R_{v,j}R_{u,i}\,\mu.
\end{aligned}
$$

(iv) First, note that by (iii) it makes sense to write $\prod_{\iota \in I} R_{u,\iota}$ since the order of the factors does not matter. Furthermore, the commutativity of the factors ensures that $\prod_{\iota \in I} R_{u,\iota}$ is indeed a projection. Formely one can prove this by induction on the cardinality of I.

(v)

$$
\begin{aligned}
R_{v,i}T_\xi\,\Omega &= R_{v,i}\,\xi = 0 \text{ if } r(\xi) \neq v \text{ or } d(\xi) \geq e_i \\
R_{v,i}T_\xi\,\mu &= \begin{cases} \xi\mu & \text{if } r(\mu) = s(\xi),\ r(\xi) = v,\ d(\xi\mu) \not\geq e_i \\ 0 & \text{otherwise} \end{cases} \\
&= 0 \text{ if } r(\xi) \neq v \text{ or } d(\xi) \geq e_i
\end{aligned}
$$

(vi)

$$\begin{aligned}
T_\xi R_{v,i}\,\Omega &= \xi = T_\xi R\,\Omega \\
T_\xi R_{v,i}\,\mu &= \begin{cases} T_\xi\,\mu & \text{if } r(\mu) = v \text{ and } d(\mu) \ngeq e_i \\ 0 & \text{otherwise} \end{cases} \\
&= 0 \text{ since } s(\xi) \neq v \\
&= T_\xi R\,\mu
\end{aligned}$$

(vii)

$$\begin{aligned}
R_{r(\xi),i} T_\xi\,\Omega &= \Omega = T_\xi R_{s(\xi)}\,\Omega \\
R_{r(\xi),i} T_\xi\,\mu &= \begin{cases} \xi\mu & \text{if } r(\mu) = s(\xi) \text{ and } d(\xi\mu) \ngeq e_i \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \xi\mu & \text{if } r(\mu) = s(\xi) \text{ and } d(\mu) \ngeq e_i \\ 0 & \text{otherwise} \end{cases} \\
&= T_\xi R_{s(\xi),i}\,\mu
\end{aligned}$$

The second equality follows because $d(\xi) \ngeq e_i$, therefore $d(\xi\mu) \ngeq e_i \iff d(\mu) \ngeq e_i$.

(viii)

$$T_\xi T_\eta\,\Omega \;=\; \begin{cases} T_\xi\,\Omega & \text{if } d(\eta)=0 \\ T_\xi\,\eta & \text{otherwise} \end{cases} = \begin{cases} \xi\eta & \text{if } s(\xi)=r(\eta) \\ \xi & \text{if } s(\xi)\neq r(\eta),\ d(\eta)=0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} T_{\xi\eta}\,\Omega & \text{if } s(\xi)=r(\eta) \\ T_\xi R\,\Omega & \text{if } s(\xi)\neq r(\eta),\ d(\eta)=0 \\ 0 & \text{otherwise} \end{cases}$$

$$T_\xi T_\eta\,\mu \;=\; \begin{cases} T_\xi\,\eta\mu & \text{if } s(\eta)=r(\mu) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \xi\eta\mu & \text{if } s(\xi)=r(\eta),\ s(\eta)=r(\mu) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} T_{\xi\eta}\,\mu & \text{if } s(\xi)=r(\eta) \\ T_\xi R\,\mu & \text{if } s(\xi)\neq r(\eta),\ d(\eta)=0 \\ 0 & \text{otherwise} \end{cases}$$

(ix) If $\xi,\eta\in\Lambda$ with $\sup\{d(\xi),d(\eta)\}\neq 0$ then for any $\mu\in\Lambda\backslash\Lambda^0$,

$$T_\xi^* T_\eta\,\Omega \;=\; \begin{cases} \Omega & \text{if } \xi=\eta \qquad (1) \\ \eta' & \text{if } \eta=\xi\eta' \text{ for some } \eta'\in\Lambda\backslash\Lambda^0 \qquad (2) \\ 0 & \text{otherwise} \end{cases},$$

$$T_\xi^* T_\eta\,\mu \;=\; \begin{cases} \nu & \text{if } \eta\mu=\xi\nu \text{ for some } \nu\in\Lambda \qquad (3) \\ 0 & \text{otherwise} \end{cases},$$

and

$$\sum_{(\alpha,\beta)\in\mathcal{E}(\xi,\eta)} T_\alpha T_\beta^*\,\Omega \;=\; \begin{cases} \Omega & \text{if } \mathcal{E}(\xi,\eta)=\{(\alpha_0,\beta_0)\} \text{ for some } \alpha_0,\beta_0\in\Lambda^0 \qquad (4) \\ \alpha_0 & \text{if } \mathcal{E}(\xi,\eta)=\{(\alpha_0,\beta_0)\} \text{ for some } \alpha_0\in\Lambda\backslash\Lambda^0, \beta\in\Lambda^0 \qquad (5) \\ 0 & \text{otherwise} \end{cases},$$

$$\sum_{(\alpha,\beta)\in\mathcal{E}(\xi,\eta)} T_\alpha T_\beta^*\,\mu \;=\; \begin{cases} \alpha_0\mu' & \text{if } \mu=\beta_0\mu' \text{ for some } (\alpha_0,\beta_0)\in\mathcal{E}(\xi,\eta),\ \mu'\in\Lambda \qquad (6) \\ 0 & \text{otherwise} \end{cases}.$$

We claim that we have the following equivalence of statements:

$$(1) \iff (4),\ (2) \iff (5),\ (3) \iff (6)$$

$(1) \iff (4)$: If $\xi = \eta$ then it is clear that $\mathcal{E}(\xi, \eta) = \{s(\xi), s(\eta)\}$ thus (4) holds. Conversely, if $\mathcal{E}(\xi, \eta) = \{(\alpha_0, \beta_0)\}$ for some $\alpha_0, \beta_0 \in \Lambda^0$ then $\xi\alpha_0 = \eta\beta_0 \Rightarrow \xi = \eta$ thus (1) holds.

$(2) \iff (5)$: If $\eta = \xi\eta'$ for some $\eta' \in \Lambda \backslash \Lambda^0$ then clearly $\mathcal{E}(\xi, \eta) = \{\eta', s(\eta)\}$ thus (5) holds.
Conversely, if $\mathcal{E}(\xi, \eta) = \{(\alpha_0, \beta_0)\}$ for some $\alpha_0 \in \Lambda \backslash \Lambda^0, \beta_0 \in \Lambda^0$ then $\xi\alpha_0 = \eta\beta_0 \Rightarrow \eta = \xi\alpha_0$ thus (2) holds.

$(3) \iff (6)$: If $\eta\mu = \xi\nu$ for some $\nu \in \Lambda$ then $d(\mu) = d(\xi\nu) - d(\eta) \geq \sup\{d(\xi), d(\eta)\} - d(\eta)$ since $d(\xi\nu) \geq d(\xi)$ and $d(\xi\nu) = d(\eta\mu) \geq d(\eta)$. Therefore there exists an unique $\beta_0 \in \Lambda$ such that $\mu = \beta_0\mu'$ for some $\mu' \in \Lambda$ and $d(\beta_0) = \sup\{d(\xi), d(\eta)\} - d(\eta)$. Similarly, $d(\nu) = d(\eta\nu) - d(\xi) \geq \sup\{d(\xi), d(\eta)\} - d(\xi)$. So there exists an unique $\alpha_0 \in \Lambda$ such that $\nu = \alpha_0\nu'$ for some $\nu' \in \Lambda$ and $d(\alpha_0) = \sup\{d(\xi), d(\eta)\} - d(\xi)$. Thus, $\eta\mu = \xi\nu \Rightarrow \eta\beta_0\mu' = \xi\alpha_0\nu'$ with $d(\eta\beta_0) = d(\xi\alpha_0) = \sup\{d(\xi), d(\eta)\} \Rightarrow \eta\beta_0 = \xi\alpha_0$ and $\mu' = \nu'$. Hence, (6) holds.
Conversely, if $\mu = \beta_0\mu'$ for some $\mu' \in \Lambda$, $(\alpha_0, \beta_0) \in \mathcal{E}(\xi, \eta)$ then $\eta\mu = \eta\beta_0\mu' = \xi\alpha_0\mu'$ so that (3) holds with $\nu = \alpha_0\mu'$.

It follows that,

$$T_\xi^* T_\eta = \begin{cases} \sum_{(\alpha,\beta)\in\mathcal{E}(\xi,\eta)} T_\alpha T_\beta^* & \text{if } \sup\{d(\xi), d(\eta)\} \neq 0 \\ Q_{s(\xi)} & \text{if } \xi = \eta \\ R & \text{otherwise} \end{cases}$$

□

Lemma 3.3 *The *-algebras*

$$\operatorname{span}\{T_\lambda T_\mu^*,\, T_{\lambda'} R T_{\mu'}^* \,|\, \lambda, \mu, \lambda', \mu' \in \Lambda\},$$
$$\operatorname{span}\{T_\xi R_{v,I} T_\eta^* \,|\, \xi, \eta \in \Lambda,\, v \in \Lambda^0,\, I \subset \{1, 2, \ldots, k\}\}$$

are dense in $\mathcal{T}_\Lambda, \mathcal{K}_\Lambda$ *respectively.*

Proof. Recall from, for example [55], that if for each $n = 1, 2, \ldots$ we set $W_n := \{w_1 w_2 \cdots w_n \,|\, w_i \in \{T_\lambda, T_\mu^* \,|\, \lambda, \mu \in \Lambda\}\}$ to be the set of all words of length n in the generators of $\mathcal{T}_\Lambda$ and their adjoints, then $\mathcal{T}_\Lambda = \overline{\operatorname{span}}\, W$ where $W := \bigcup_{n=1}^\infty W_n$. Now by repeated applications of Lemma 3.2 (i), (iii), (iv), (v), (viii) and (ix) we see that any word in W must be in $\operatorname{span}\{T_\lambda T_\mu^*,\, T_{\lambda'} R T_{\mu'}^* \,|\, \lambda, \mu, \lambda', \mu' \in \Lambda\}$ thus $\mathcal{T}_\Lambda = \overline{\operatorname{span}}\{T_\lambda T_\mu^*,\, T_{\lambda'} R T_{\mu'}^* \,|\, \lambda, \mu, \lambda', \mu' \in \Lambda\}$.

To see that $\mathcal{K}_\Lambda \subset \overline{\operatorname{span}}\{T_\xi R_{v,I} T_\eta^* \,|\, \xi, \eta \in \Lambda,\, v \in \Lambda^0,\, I \subset \{1, 2, \ldots, k\}\}$ first note that $\mathcal{K}_\Lambda = \overline{\operatorname{span}}\{xry \,|\, x, y \in \mathcal{T}_\Lambda,\ r \in C^*(R_{v,i} \,|\, v \in \Lambda^0,\, i \in \{1, 2, \ldots, k\}\}$. Thus it suffices to show that the set of monomials $\{xry \,|\, x, y \in \mathcal{T}_\Lambda,\ r \in C^*(R_{v,i} \,|\, v \in \Lambda^0,\, i \in \{1, 2, \ldots, k\})$ is a subset of $\{T_\xi R_{v,I} T_\eta^* \,|\, \xi, \eta \in \Lambda,\, v \in \Lambda^0,\, I \subset \{1, 2, \ldots, k\}\}$. Moreover, by Lemma 3.1, Lemma 3.2 (iii) we see that $C^*(R_{v,i} \,|\, v \in \Lambda^0,\, i \in \{1, 2, \ldots, k\}\} = \overline{\operatorname{span}}\{R_{v,I} \,|\, v \in \Lambda^0,\, I \subset \{1, 2, \ldots, k\}\}$. Therefore since $\mathcal{T}_\Lambda = \overline{\operatorname{span}}\{T_\lambda T_\mu^*,\, T_{\lambda'} R T_{\mu'}^* \,|\, \lambda, \mu, \lambda', \mu' \in \Lambda\}$ it suffices to show that $x R_{v,I} y$ is in $\{T_\xi R_{v,I} T_\eta^* \,|\, \xi, \eta \in \Lambda,\, v \in \Lambda^0,\, I \subset \{1, 2, \ldots, k\}\}$ for all $x, y \in \{T_\lambda T_\mu^*,\, T_{\lambda'} R T_{\mu'}^* \,|\, \lambda, \mu, \lambda', \mu' \in \Lambda\}$, $v \in \Lambda^0$ and $I \subset \{1, 2, \ldots, k\}$. But this follows from the relations listed in Lemma 3.2. Clearly the reverse inclusion holds, thus the result is proved. □

Lemma 3.4 *Let* $\pi : \mathcal{T}_\Lambda \longrightarrow \mathcal{T}_\Lambda / \mathcal{K}_\Lambda$ *be the quotient map then* $\{\pi(T_\lambda) \,|\, \lambda \in \Lambda\}$ *is a* $*$*-representation of* Λ*, i.e.*

(i) $\{\pi(T_v) \mid v \in \Lambda^0\}$ *is a family of mutually orthogonal projections,*

(ii) $\pi(T_\lambda)\pi(T_\mu) = \begin{cases} \pi(T_{\lambda\mu}) & \text{if } r(\mu) = s(\lambda) \\ 0 & \text{otherwise} \end{cases}$

(iii) $\pi(T_\lambda)^*\pi(T_\lambda) = \pi(T_{s(\lambda)})$

(iv) $\pi(T_v) = \sum_{\alpha\in\Lambda^n(v)} \pi(Q_\alpha)$ *for all* $n \in \mathbb{N}^k$

Proof. Follows from the definition of $R_{v,I}$ and Lemma 3.2 (i), (viii). □

Proposition 3.1 *With the notation as above*

$$C^*(\Lambda) \cong \mathcal{T}_\Lambda/\mathcal{K}_\Lambda$$

if $Q_v \notin \mathcal{K}_\Lambda$ *for all* $v \in \Lambda^0$.

Proof. For $z \in \mathbb{T}^k$ $\gamma_z(x) = F_z x F_z^*$, where F_z is the unitary defined by $F_z\Omega = \Omega$ and $F_z\mu = z^{d(\mu)}\mu$, defines an automorphism of $B(\mathcal{H})$ which leaves $\mathcal{T}_\Lambda$ invariant (c.f. [18, §2]). Note that $\gamma_z^{-1} = \gamma_{z^{-1}}$ therefore $\gamma_z(\mathcal{T}_\Lambda) = \mathcal{T}_\Lambda$ and $\gamma_{z|_{\mathcal{T}_\Lambda}}$ is an automorphism of $\mathcal{T}_\Lambda$. Clearly, $\gamma_z(R_{v,i}) = R_{v,i}$ so $\gamma_z(xR_{v,i}y) = \gamma_z(x)R_{v,i}\gamma_z(y) \in \mathcal{K}_\Lambda$ for all $x, y \in \mathcal{T}_\Lambda$. It follows that $\mathcal{K}_\Lambda$ is invariant under γ_z, hence γ_z is an automorphism of $\mathcal{K}_\Lambda$ also. If $\pi : \mathcal{T}_\Lambda \longrightarrow \mathcal{T}_\Lambda/\mathcal{K}_\Lambda$ is the quotient map then $\beta_z\pi = \pi\gamma_z$ defines an automorphism of $\mathcal{T}_\Lambda/\mathcal{K}_\Lambda$ for each $z \in \mathbb{T}^k$. By Lemma 3.4, the family of partial isometries $\{\pi(T_\lambda) \mid \lambda \in \Lambda\}$ is a *-representation of Λ therefore there exists a homomorphism $\phi : C^*(\Lambda) \longrightarrow \mathcal{T}_\Lambda/\mathcal{K}_\Lambda$ such that $\phi(s_\lambda) = T_\lambda$. The map $z \mapsto \beta_z$ defines an action $\beta : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{T}_\Lambda/\mathcal{K}_\Lambda)$ which satisfies $\phi\alpha_z = \beta_z\phi$ for all $z \in \mathbb{T}^k$, where α is the gauge action on $C^*(\Lambda)$. Now if $Q_v \notin \mathcal{K}_\Lambda$ for all $v \in \Lambda$ then $\phi(p_v) \neq 0$. Hence by the gauge invariant uniqueness theorem [35, Theorem 3.4] $\mathcal{T}_\Lambda/\mathcal{K}_\Lambda \cong C^*(\Lambda)$. □

Remark 3.1

- *It is clear that* $Q_v = \sum_{\alpha\in\Lambda(v)} T_\alpha R T_\alpha^*$ *- the series converges strongly.*

- *If* Λ *is a 1-graph then* $\mathcal{K}_\Lambda$ *is isomorphic to the* C^**-algebra of compact operators on an infinite dimensional separable Hilbert space. However if* Λ *is a* k*-graph with* $k \geq 2$ *then* $R_{v,i}$ *is a projection of infinite rank for all* $v \in \Lambda^0$ *and* $i = 1, 2, \ldots, k$.

- *If* $k = 1$ *then* $Q_v \notin \mathcal{K}_\Lambda$ *for all* $v \in \Lambda^0$*. It seems less straightforward to show that* Q_v *is not in* $\mathcal{K}_\Lambda$ *for all* $v \in \Lambda^0$ *when* Λ *is a* k*-graph,* $k \geq 2$.

- *As noted in the proof of Lemma 3.3,* $C^*(R_{v,i} \,|\, v \in \Lambda^0, i = 1, 2, \ldots, k) = \overline{\mathrm{span}}\{R_{v,I} \,|\, v \in \Lambda^0,\, I \subset \{1, 2, \ldots, k\}\}$*. Note also that the generators satisfy* $R_{u,I}R_{v,J} = \begin{cases} R_{u,I\cup J} & \text{if } u = v \\ R & \text{if } u \neq v \end{cases}$*. Therefore* $C^*(R_{v,i} \,|\, v \in \Lambda^0, i = 1, 2, \ldots, k)$ *is a commutative* C^**-algebra.*

- *Another formulation of* $\mathcal{T}_\Lambda$ *as the closure of the linear span of a set of monomials is*

$$\mathcal{T}_\Lambda = \overline{\mathrm{span}}\{T_\lambda T_v T_\mu^* \,|\, \lambda, \mu \in \Lambda,\ v \in \Lambda^0\}.$$

3.2 Cuntz-Pimsner algebras and k-graph C^*-algebras

First recall the definition of a Cuntz-Pimsner algebra from [47].

Definition 3.2 *A Hilbert bimodule* (X, ϕ) *is a pair consisting of a Hilbert module* X *over a* C^**-algebra* A*, together with an isometric *-homomorphism* $\phi : A \longrightarrow \mathbb{B}(X)$ *that provides the left* A*-module structure on* X.

Given a Hilbert bimodule (X, ϕ) we denote by $\otimes^m$ the m-fold tensor product of the module X with itself over the *-homomorphism ϕ (with the convention that $\otimes^0 = A$). Let

$$\mathcal{E}_+ := \bigoplus_{n=0}^{\infty} \otimes^n X$$

and by $\mathcal{T}_x \in \mathbb{B}(\mathcal{E}_+)$, the operator defined on an elementary tensor $y \in \otimes^n X$ by $\mathcal{T}_x(y) = x \otimes y \in \otimes^{n+1} X$. Let $J(\mathcal{E}_+)$ be the C^*-algebra generated in $\mathbb{B}(X)$ by

$$\mathbb{B}(\bigoplus_{n=0}^{N} \otimes^n X), \quad \text{N=0,1,...}$$

and S_x be the class of the operator $\mathcal{T}_x$ in the quotient algebra $M(\mathcal{E}_+)/J(\mathcal{E}_+)$, where $M(\mathcal{E}_+)$ is the multiplier algebra of $J(\mathcal{E}_+)$.

Definition 3.3 *The Cuntz-Pimsner algebra of the Hilbert bimodule X, denoted $\mathcal{O}_X$, is the C^*-algebra generated in $M(\mathcal{E}_+)/J(\mathcal{E}_+)$ by all the operators S_x, with $x \in X$.*

Before stating a proposed Hilbert bimodule whose Cuntz-Pimsner algebra is isomorphic to a k-graph C^*-algebra we state a few results that are needed for the construction and which will be of use elsewhere.

Let $c : \Lambda \longrightarrow G$ be a functor from Λ into a discrete abelian group G. Then, by [35, Corollary 5.3] there exists an action $\alpha^c : \hat{G} \longrightarrow \mathrm{Aut}(C^*(\Lambda))$ such that for $\chi \in \hat{G}$ and $\lambda \in \Lambda$,

$$\alpha^c_\chi(s_\lambda) = \langle \chi, c(\lambda) \rangle s_\lambda.$$

Unless otherwise stated we let $G = \mathbb{Z}^N$ for some $N = 1, 2, \ldots$, and identify $\hat{G}$ with $\mathbb{T}^N$ where $z \in \mathbb{T}^N$ is identified with the character $n \mapsto z^n$, $n \in G$. Also note that $\hat{\mathbb{T}}^N = G$ where now we identify $n \in G$ with the character

$z \mapsto z^n,\ z \in \hat{G} = \mathbb{T}^N$. With these conventions a simple calculation yields,

$$\alpha_z^c(s_\lambda s_\mu^*) = z^{c(\lambda)-c(\mu)} s_\lambda s_\mu^*.$$

It follows that α^c is a strongly continuous action.

Definition 3.4 ([44, Definition 3.1.2]) *Let β be a (strongly continuous) action of a compact abelian group H on a C^*-algebra B, and B^β its fixed point algebra. For a character $\chi \in \hat{H}$, we let $B^\beta(\chi)$ denote the spectral subspace $\{b \in B \mid \beta_t(b) = \chi(t)b \text{ for all } t \in H\}$. We say that β has large spectral subspaces if $\overline{B^\beta(\chi)^* B^\beta(\chi)} = B^\beta$ for each $\chi \in \hat{H}$.*

Lemma 3.5 *For each $n \in G$, $\Phi_n : C^*(\Lambda) \longrightarrow C^*(\Lambda)$ defined by*

$$\Phi_n(x) = \int_{\hat{G}} z^{-n} \alpha_z^c(x)\, dz \quad \text{for all } x \in C^*(\Lambda)$$

(where dz is the normalized Haar measure on $\hat{G}$) is a linear map of norm 1 which satisfies $\Phi_n(C^*(\Lambda)) = C^*(\Lambda)^{\alpha^c}(n)$ and $\Phi_n^2 = \Phi_n$.

Proof. These maps are well defined since $z \mapsto z^{-n}\alpha_z^c(x)$ is a continuous function for all $x \in C^*(\Lambda)$ and $n \in G$. It is clear that Φ_n is a linear map of norm 1 for each $n \in G$. To see that Φ_n maps onto $C^*(\Lambda)^{\alpha^c}(n)$ recall that each $x \in C^*(\Lambda)$ is the norm limit of a sequence of elements $\{x_m\}$ in $\operatorname{span}\{s_\lambda s_\mu^* \mid s(\lambda) = s(\mu)\}$. Now

$$\begin{aligned} \Phi_n(s_\lambda s_\mu^*) &= \int_{\hat{G}} z^{-n+c(\lambda)-c(\mu)} s_\lambda s_\mu^*\, dz \\ &= \begin{cases} s_\lambda s_\mu^* & \text{if } c(\lambda) - c(\mu) = n \\ 0 & \text{otherwise} \end{cases} \in C^*(\Lambda)^{\alpha^c}(n). \end{aligned}$$

Therefore $x_m \in C^*(\Lambda)^{\alpha^c}(n)$ for all $m \in \mathbb{N}$ and

$$\Phi_n(x) = \lim_{m\to\infty} \Phi_n(x_m) \in C^*(\Lambda)^{\alpha^c}(n).$$

Thus Φ_n maps into $C^*(\Lambda)^{\alpha^c}(n)$. Now given $x \in C^*(\Lambda)^{\alpha^c}(n)$ we see that $\Phi_n(x) = \int_{\hat{G}} z^{-n} z^n x \, dz = x$ which gives us that $\Phi_n(C^*(\Lambda)) = C^*(\Lambda)^{\alpha^c}(n)$ and $\Phi_n^2 = \Phi_n$. □

Lemma 3.6 *Let $n \in G$ then the linear span of $\{s_\lambda s_\mu^* \,|\, s(\lambda) = s(\mu),\, c(\lambda) - c(\mu) = n\}$ is dense in $C^*(\Lambda)^{\alpha^c}(n)$.*

Proof. It is clear that

$$\overline{\operatorname{span}}\{s_\lambda s_\mu^* \,|\, s(\lambda) = s(\mu),\, c(\lambda) - c(\mu) = n\} \subset C^*(\Lambda)^{\alpha^c}(n)$$

for all $n \in G$. To prove the reverse inclusion, choose any x in $C^*(\Lambda)^{\alpha^c}(n)$ then $x = \lim_{m\to\infty} x_m$ for some $x_m \in \overline{\operatorname{span}}\{s_\lambda s_\mu^* \,|\, \lambda, \mu \in \Lambda,\, s(\lambda) = s(\mu)\}$. Then by the proof of Lemma 3.5

$$x = \Phi_n(x) = \lim_{m\to\infty} \Phi_n(x_m)$$

so that x is in $\overline{\operatorname{span}}\{s_\lambda s_\mu^* \,|\, s(\lambda) = s(\mu),\, c(\lambda) - c(\mu) = n\}$. □

Lemma 3.7 *If Λ has no sinks then the action α^c of $\hat{G}$ on $C^*(\Lambda)$ has large spectral subspaces.*

Proof: It is clear that

$$\overline{(C^*(\Lambda)^{\alpha^c}(n))^* C^*(\Lambda)^{\alpha^c}(n)} \subset C^*(\Lambda)^{\alpha^c}.$$

By Lemma 3.6 we have

$$C^*(\Lambda)^{\alpha^c}(n) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* \,|\, c(\lambda) - c(\mu) = n\}$$

for all $n \in G$. Now since Λ has no sinks, given $n \in G$ and $\xi, \eta \in \Lambda$ with $c(\xi) = c(\eta)$ and $s(\xi) = s(\eta)$ we may choose $\lambda \in \Lambda$ such that $s(\lambda) = s(\xi)$ and $c(\lambda) - c(\eta) = n$. Then

$$s_\xi s_\eta^* = (s_\xi s_\lambda^*)(s_\lambda s_\eta^*) \in (C^*(\Lambda)^{\alpha^c}(n))^* C^*(\Lambda)^{\alpha^c}(n)$$

As $C^*(\Lambda)^{\alpha^c}$ is the closed linear span of monomials of the form $s_\xi s_\eta^*$ with $c(\xi) = c(\eta)$ it follows that

$$C^*(\Lambda)^{\alpha^c} \subset \overline{(C^*(\Lambda)^{\alpha^c}(n))^* C^*(\Lambda)^{\alpha^c}(n)}$$

for each $n \in G$. Thus the result has been proved. □

Definition 3.5 *We define the length functor* $|\cdot| : \Lambda \longrightarrow \mathbb{Z}$ *by* $|\lambda| = \sum_{i=1}^k n_i$ *if* $d(\lambda) = \sum_{i=1}^k n_i e_i$. *Also let* β *denote the action* $\alpha^{|\cdot|}$ *of* $\mathbb{T}$ *on* $C^*(\Lambda)$.

Note that $\beta_z = \alpha_{(z,z,\ldots,z)}$ (k z's) for all $z \in \mathbb{T}$.

Lemma 3.8 *Let* $A := C^*(\Lambda)^\beta$ *and* $X := C^*(\Lambda)^\beta(1) = \{x \in C^*(\Lambda) \,|\, \beta_z(x) = z(x)\}$. *If* Λ *has no sinks then* X *is a full Hilbert* A*-module with the right action given by multiplication in* $C^*(\Lambda)$ *and inner product* $\langle x, y\rangle = x^* y$ *for all* $x, y \in X$.

Proof. It is clear that X is a vector space - indeed equipped with the norm on $C^*(\Lambda)$, X is a Banach space. To see that the action is well-defined, let $x \in X$, $a \in A$ then $\beta_z(x \cdot a) = \beta_z(xa) = zxa = z(x \cdot a) \Rightarrow x \cdot a \in X$. Therefore there exists a pairing $(x, a) \mapsto x \cdot a = xa : X \times X \longrightarrow A$. It is obviously bilinear as multiplication is and $x \cdot (ab) = (x \cdot a) \cdot b$ for all $x \in X$, $a, b \in A$. Therefore X is a right A-module.

To see that $\langle\cdot,\cdot\rangle : X \times X \longrightarrow A$ is an inner product on X we must first check that $\langle\cdot,\cdot\rangle$ is well-defined i.e. maps into A. But this is clear since $\beta_z(x^* y) = \bar{z} z x^* y = x^* y$ for all $x, y \in X$ and $z \in \mathbb{T}$. A routine check shows that $\langle\cdot,\cdot\rangle$ satisfies the axioms of an inner product. Thus X is a pre-Hilbert A-module. Now for any $x \in X$, $\|x\|_X := \sqrt{\|\langle x, x\rangle\|} = \|x\|$ by the C^*-equation.

Therefore X is complete with respect to the inner product norm since it is complete with respect to the C^*-norm. Hence X is a Hilbert A-module.

Now, since β has large spectral subspaces by Lemma 3.4 we see that $A = C^*(\Lambda)^\beta = \overline{(C^*(\Lambda)^\beta(1))^* C^*(\Lambda)^\beta(1)} = \overline{\text{span}}\{\langle x, y\rangle \mid x, y \in X\}$ by definition. Thus X is a full Hilbert module. □

Let $\phi : A \longrightarrow \mathbb{B}(X)$ be defined by $\phi(a)x = ax$ for all $x \in X$. Then ϕ provides X with a left action, $a \cdot x := \phi(a)x = ax$ for all $a \in A$, $x \in X$. We will show that ϕ is injective thus X is a Hilbert bimodule in the sense of [47]. The proof of the injectivity of ϕ is inspired by proofs of the gauge invariant theorems [29, Theorem 2.3], [1, Theorem 2.1.1],[35, Theorem 3.4]. We prove now that there exists a gauge invariant action on $\phi(A)$.

Lemma 3.9 *For all $z \in \mathbb{T}^k$, we may define automorphisms, γ_z, of $\phi(A)$ by $\gamma_z\phi = \phi\alpha_z$ which in turn define an action, γ, of $\mathbb{T}^k$ on $\phi(A)$.*

Proof First we must check that the maps γ_z are well-defined for all $z \in \mathbb{T}^k$. To see this we note that,

$$\phi(a) = 0 \Rightarrow ax = 0\ \forall x \in X \Rightarrow \alpha_z(a)x = 0\ \forall x \in X \Rightarrow \phi(\alpha(a)) = 0.$$

Therefore, if $\phi(a) = \phi(b)$ then $\phi(a-b) = 0 \Rightarrow \phi(\alpha_z(a-b)) = 0 \Rightarrow \phi(\alpha_z(a)) = \phi(\alpha_z(b))$. Thus if $\phi(a) = \phi(b)$ for some $a, b \in A$ then $\gamma_z(\phi(a)) = \gamma_z(\phi(b))$ for all $z \in \mathbb{T}^k$.

It is clear that (since $\alpha_z(A) = A$ for all $z \in \mathbb{T}^k$) γ_z is a surjective *-homomorphism for all $z \in \mathbb{T}^k$ and a routine check shows that $\gamma : \mathbb{T}^k \longrightarrow \text{End}(\phi(A))$ is a homomorphism. Therefore, it follows that γ_z is also injective for all $z \in \mathbb{T}^k$. Thus γ is indeed an action of $\mathbb{T}^k$ on $\phi(A)$ as claimed. □

We repeat and prove our claim in the following Lemma.

Lemma 3.10 *Let X be the right Hilbert module over A as defined in Lemma 3.8. Then the *-homomorphism $\phi : A \longrightarrow \mathbb{B}(X)$ defined by $\phi(a) = ax$ for all $x \in X$ is injective. Thus (X, ϕ) is a Hilbert bimodule in the sense of [47]. Moreover $\phi(A) = \mathbb{K}(X)$.*

Proof. First we claim that ϕ is injective on A^α. Note that $A^\alpha = C^*(\Lambda)^\alpha$. By [35, Lemma 3.2], $C^*(\Lambda)^\alpha = \mathcal{F} := \overline{\bigcup_{n\in N^k} \mathcal{F}_n}$ where $\mathcal{F}_n = \bigoplus_{v\in\Lambda^0} \mathcal{F}_n(v),\ \mathcal{F}_n(v) = \overline{\text{span}}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \Lambda^0,\ s(\lambda) = s(\mu) = v\} \cong \mathbb{K}(\ell^2(s^{-1}(v) \cap \Lambda^n))$. Thus by for example [43, Lemma 1.3]

$$\ker \phi_{|_{A^\alpha}} = \overline{\bigcup_{n\in\mathbb{N}^k} \ker \phi_{|_{A^\alpha}} \cap \mathcal{F}_n}.$$

Thus it suffices to show that $\ker \phi_{|_{\mathcal{F}_n}} = \{0\}$ for all $n \in \mathbb{N}^k$. But this follows if $\ker_{|_{\mathcal{F}_n(v)}} = \{0\}$ for all $v \in \Lambda^0,\ n \in \mathbb{N}^k$. Now, for all $v \in \Lambda^0,\ n \in \mathbb{N}^k,\ \mathcal{F}_n(v)$ is a simple C^*-algebra, therefore it is enough to show that $\phi_{|_{\mathcal{F}_n(v)}} \neq 0$.

Let $v \in \Lambda^0,\ n \in \mathbb{N}^k$. If $n = 0$ then $\mathcal{F}_n(v) = \mathbb{C}p_v$. Choose any $i \in \{1, 2, \dots, k\}$ and any $\lambda \in \Lambda^{e_i}(v)$ then s_λ is in X and

$$\phi_{|_{\mathcal{F}_n(v)}}(p_v)s_\lambda = p_v s_\lambda = s_\lambda \neq 0$$

Thus $\phi_{|_{\mathcal{F}_0(v)}} \neq 0$. If $n \neq 0$ then since Λ has no sinks we may choose any $\lambda \in \Lambda^n$ with $s(\lambda) = v$. Now, we may write $\lambda = \lambda_1\lambda'$ with $|\lambda_1| = 1$, so s_{λ_1} is in X and

$$\phi_{|_{\mathcal{F}_n(v)}}(s_\lambda s_\lambda^*)s_{\lambda_1} = s_{\lambda_1} s_{\lambda'} s_{\lambda'}^* \neq 0.$$

Thus $\phi_{|_{\mathcal{F}_n(v)}} \neq 0$ for all $v \in \Lambda^0,\ n \in \mathbb{N}^k$. Hence ϕ is faithful on A^α as claimed.

Let $\Phi := \Phi_0$ and note that by [35, Lemma 3.3], Φ is faithful (in the sense that $\Phi(a) \neq 0$ if a is a non-zero positive element of $C^*(\Lambda)$). Thus let a be any non-zero positive element of A, then

$$0 \neq \phi(\Phi(a)) = \int_{T^k} \phi\alpha_z(a)\, dz = \int_{\mathbb{T}^k} \gamma_z\phi(a)\, dz.$$

Thus $\phi(a) \neq 0$. Hence ϕ is injective.

To see that $\phi(A) = \mathbb{K}(X)$ let $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ and $|\lambda| = |\mu|$. Then $s_\lambda s_\mu^* \in A$ and $x := s_\lambda s_{\lambda'}^*,\ y := s_\mu s_{\lambda'}^* \in X$ where $\lambda = \lambda_1 \lambda',\ |\lambda_1| = 1$. Now, for $w \in X$,

$$\begin{aligned}\theta_{x,y} w &= x\langle y, w\rangle = s_\lambda s_{\lambda'}^* s_{\lambda'} s_\mu^* w \\ &= s_\lambda s_\mu^* w\end{aligned}$$

Thus $\phi(s_\lambda s_\mu^*) = \theta_{x,y}$ and it follows that ϕ maps into $\mathbb{K}(X)$.

Conversely, let $x, y, w \in X$, then $\theta_{x,y}(w) = x\langle y, z\rangle = xy^* z$. But $\beta_z(xy^*) = xy^*$ for all $z \in \mathbb{T}$ thus $xy^* \in A$. Therefore,

$$\begin{aligned}\phi(xy^*)w = xy^* w &= \theta_{x,y} w \qquad \forall\, w \in X \\ \Rightarrow \theta_{x,y} &= \phi(xy^*)\end{aligned}$$

Hence ϕ maps onto $\mathbb{K}(X)$. $\square$

Definition 3.6 *We let* $\tau_\lambda := S_{s_\lambda}$ *for all* $\lambda \in \Sigma := \{\lambda \in \Lambda \,|\, |\lambda| = 1\} = \bigcup_{i=1}^k \Lambda^{e_i}$ *and* $\tau_{v,i} := \sum_{\alpha \in \Lambda^{e_i}(v)} \tau_\alpha \tau_\alpha^*$ *for all* $v \in \Lambda^0,\ i \in \{1, 2, \ldots, k\}$.

Lemma 3.11 *The following relations are satisfied in* $\mathcal{O}_X$. *For all* $\lambda, \mu, \xi, \eta \in \Sigma$,

1. $\tau_\lambda^* \tau_\mu = \displaystyle\sum_{\substack{\lambda\alpha = \mu\beta \\ d(\alpha) = d(\mu)}} \tau_\alpha \tau_\beta^*$,
2. τ_λ *is a partial isometry. Also,* $\tau_\lambda^* \tau_\lambda = \tau_\mu^* \tau_\mu$ *if* $s(\lambda) = s(\mu)$,
3. $\tau_{v,i} \tau_\lambda = \tau_\lambda$ *if* $r(\lambda) = v$, *for all* $i \in \{1, 2, \ldots, k\}$,
4. $\tau_\lambda \tau_\mu = \tau_\xi \tau_\eta$ *if* $\lambda\mu = \xi\eta$,

5. *$\tau_{v,i}$ are projections and $\tau_{v,i} = \tau_{v,j}$ for all $i, j \in \{1, 2, \ldots, k\}$,*

Proof. First, following the convention set in [47], we identify A and $\mathbb{B}(X)$ with their image in $M(\mathcal{E}_+)/J(\mathcal{E}_+)$ given by

$$\begin{aligned} a(x_1 \otimes \cdots \otimes x_n) &= \phi(a)(x_1) \otimes \cdots \otimes x_n \\ T(x_1 \otimes \cdots \otimes x_n) &= T(x_1) \otimes \cdots \otimes x_n \end{aligned}$$

for all $n \geq 1$, $x_1, \ldots, x_n \in X, a \in A$ and $T \in \mathbb{B}(X)$. Under these identifications and by [47, Proposition 1.3] (or by direct calculations) we see that $s_\lambda s_\mu^* \equiv \phi(s_\lambda s_\mu^*) = \theta_{s_\lambda, s_\mu} \equiv \tau_\lambda \tau_\mu^*$ for all $\lambda, \mu \in \Sigma$. Therefore, for all $\lambda, \mu, \xi, \eta \in \Sigma$,

1. $\tau_\lambda^* \tau_\mu = \langle s_\lambda, s_\mu \rangle = s_\lambda^* s_\mu = \sum_{\substack{\lambda\alpha = \mu\beta \\ d(\alpha) = d(\mu)}} s_\alpha s_\beta^* = \sum_{\substack{\lambda\alpha = \mu\beta \\ d(\alpha) = d(\mu)}} \tau_\alpha \tau_\beta^*,$

2. $\tau_\lambda^* \tau_\lambda = \langle s_\lambda, s_\lambda \rangle = p_{s(\lambda)}$. Therefore $\tau_\lambda^* \tau_\lambda$ is a projection and τ_λ is a partial isometry. Also

   $$\tau_\lambda^* \tau_\lambda = \langle s_\lambda, s_\lambda \rangle = s_\lambda^* s_\lambda = s_\mu^* s_\mu = \langle s_\mu, s_\mu \rangle = \tau_\mu^* \tau_\mu$$

   if $s(\lambda) = s(\mu)$,

3. $\tau_{v,i} \tau_\lambda = \sum_{\alpha \in \Lambda^{e_i}(v)} \tau_\alpha \tau_\alpha^* \tau_\lambda = \sum_{\alpha \in \Lambda^{e_i}(v)} \tau_\alpha \langle s_\alpha, s_\lambda \rangle = \sum_{\alpha \in \Lambda^{e_i}(v)} \tau_\alpha s_\alpha^* s_\lambda = S_{\sum_{\alpha \in \Lambda^{e_i}(v)} s_\alpha s_\alpha^* s_\lambda}$
   $= S_{s_\lambda} = \tau_\lambda$ if $r(\lambda) = v$,

4. $\tau_\lambda \tau_\mu = \sum_{\alpha \in \Lambda^{d(\mu)}(r(\lambda))} \tau_\alpha \tau_\alpha^* \tau_\lambda \tau_\mu = \sum_{\alpha \in \Lambda^{d(\mu)}(r(\lambda))} \tau_\alpha \sum_{\substack{\alpha\beta = \mu\gamma \\ d(\beta) = d(\mu)}} \tau_\beta \tau_\gamma^* \tau_\mu$
   $= \sum_{\alpha \in \Lambda^{d(\mu)}(r(\lambda))} \tau_\alpha \sum_{\substack{\alpha\beta = \mu\gamma \\ d(\beta) = d(\mu)}} \tau_\beta \langle s_\gamma, s_\mu \rangle = \tau_\xi \tau_\eta \tau_\mu^* \tau_\mu^* = \tau_\xi \tau_\eta \tau_\eta^* \tau_\eta = \tau_\xi \tau_\eta,$

5. Note that if Λ has no sinks then given $v \in \Lambda^0$ we may choose any $\lambda \in \Lambda^{e_i}$, $\mu \in \Lambda^{e_j}$ such that $s(\lambda) = s(\mu) = v$ and from 1. and 2.

   $$\tau_{v,i} = \sum_{\alpha \in \Lambda^{e_i}(v)} \tau_\alpha \tau_\alpha^* = \tau_\lambda^* \tau_\lambda = \tau_\mu^* \tau_\mu = \sum_{\alpha \in \Lambda^{e_j}(v)} \tau_\alpha \tau_\alpha^* = \tau_{v,j}.$$

Therefore the result is proved. However if we do not assume that Λ has no sinks then the following argument shows that the result holds.

It is clear that $\tau_{v,i}$ is self-adjoint. Now

$$\begin{aligned}\tau_{v,i}^2 &= \sum_{\alpha,\beta\in\Lambda^{e_i}(v)} \tau_\alpha\tau_\alpha^*\tau_\beta\tau_\beta^* = \sum_{\alpha,\beta\in\Lambda^{e_i}(v)} \tau_\alpha s_\alpha^* s_\beta \tau_\eta^* \\ &= \sum_{\alpha,\in\Lambda^{e_i}(v)} \tau_\alpha\tau_\alpha^*\tau_\alpha\tau_\alpha^* = \sum_{\alpha\in\Lambda^{e_i}(v)} \tau_\alpha\tau_\alpha^* \\ &= \tau_{v,i}\end{aligned}$$

Thus $\tau_{v,i}$ is a projection for all $v \in \Lambda^0,\ i \in \{1,2,\ldots,k\}$. Fix $v \in \Lambda^0$. To see that $\tau_{v,i} = \tau_{v,j}$ for all $i,j \in \{1,2,\ldots,k\}$ consider

$$\begin{aligned}\tau_{v,i}\tau_{v,j} &= \sum_{\alpha\in\Lambda^{e_j}(v)} \tau_{v,i}\tau_\alpha\tau_\alpha^* = \sum_{\alpha\in\Lambda^{e_j}(v)} \tau_\alpha\tau_\alpha^* \\ &= \tau_{v,j}.\end{aligned}$$

Therefore $\tau_{v,j} = \tau_{v,j}^* = \tau_{v,j}\tau_{v,i} = \tau_{v,i}$ for all $v \in \Lambda^0,\ i,j \in \{1,2,\ldots,k\}$. □

Following the above Lemma we are able to make the following definition.

Definition 3.7 *For $v \in \Lambda^0$ define $\tau_v := \tau_{v,i}$ for any $i = 1,2,\ldots k$. For $\lambda \in \Lambda\backslash\Lambda^0$ define $\tau_\lambda := \tau_{\lambda_1}\tau_{\lambda_2}\cdots\tau_{\lambda_n}$ if $\lambda = \lambda_1\lambda_2\cdots\lambda_n$ for some $n \geq 1$ with $\lambda_j \in \Sigma,\ j = 1,2,\ldots,n$.*

Lemma 3.12 *The family $\{\tau_\lambda \mid \lambda \in \Lambda\}$ is a *-representation of Λ. i.e.*

(i) *$\{\tau_v\}_{v\in\Lambda^0}$ are mutually orthogonal projections,*

(ii) *$\tau_{\lambda\mu} = \tau_\lambda\tau_\mu$ for all $\lambda,\mu \in \Lambda$,*

(iii) *$\tau_\lambda^*\tau_\lambda = \tau_{s(\lambda)}$ for all $\lambda \in \Lambda$,*

(iv) *$\tau_v = \sum_{\xi\in\Lambda^{e_i}(v)} \tau_\xi\tau_\xi^*$ for all $i = 1,2,\ldots,k$.*

Proof.

(i) For any $u, v \in \Lambda^0$ we have

$$\begin{aligned}
\tau_u \tau_v &= \sum_{\xi \in \Lambda^{e_i}(u)} \tau_\xi \tau_\xi^* \sum_{\eta \in \Lambda^{e_i}(v)} \tau_\eta \tau_\eta^* \\
&= \sum_{\substack{\xi \in \Lambda^{e_i}(u) \\ \eta \in \Lambda^{e_i}(v)}} \tau_\xi \langle s_\xi, s_\eta \rangle \tau_\eta^* \\
&= \begin{cases} \tau_u & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}
\end{aligned}$$

(ii) Follows from Lemma 3.11 4.

(iii) For any $\lambda \in \Lambda$ and $i = 1, 2, .., k$ we have

$$\begin{aligned}
\tau_\lambda^* \tau_\lambda &= \langle s_\lambda, s_\lambda \rangle = s_\lambda^* s_\lambda = \sum_{\alpha \in \Lambda^{e_i}(s(\lambda))} s_\alpha s_\alpha^* \\
&= \sum_{\alpha \in \Lambda^{e_i}(s(\lambda))} \tau_\alpha \tau_\alpha^* = \tau_{s(\lambda)}
\end{aligned}$$

(iv) By definition. □

Lemma 3.13

If $x = \sum_{i=1}^{\infty} c_i s_{\lambda_i} s_{\mu_i}^* \in X$ *then* $S_x = \sum_{i=1}^{\infty} c_i \tau_{\lambda_i} \tau_{\mu_i}^*$.

Proof. As noted in the paragraph preceding [47, Definition 1.1], the norm of S_x is equal to the norm of x for all $x \in X$. Therefore the map $x \mapsto S_x$ is a continuous linear mapping from X into $\mathcal{O}_X$. Thus it suffices to show that $S_{s_\lambda s_\mu^*} = \tau_\lambda \tau_\mu$ if $\lambda, \mu \in \Lambda$ with $|\lambda| - |\mu| = 1$ since the result follows from the continuity and linearity of the map $x \mapsto S_x$, and the fact that $X_0 := \text{span}\{s_\lambda s_\mu^* \,|\, |\lambda| - |\mu| = 1\}$ is dense in X by Lemma 3.6.

Let $\lambda, \mu \in \Lambda$ with $|\lambda| - |\mu| = 1$. Then we may write $\lambda = \lambda_1 \cdots \lambda_{n+1},\ \mu = \mu_1 \cdots \mu_n$ for some $\lambda_1, \ldots, \lambda_{n+1}, \mu_1, \ldots, \mu_n \in \Sigma,\ n \geq 1$. Consider the action of $\mathcal{T}_{\lambda_1} \cdots \mathcal{T}_{\lambda_{n+1}} \mathcal{T}_{\mu_n}^* \cdots \mathcal{T}_{\mu_1}^*$ on the elementary tensor vectors in $\mathcal{E}_+$. Let $x = x_1 \otimes \cdots \otimes x_m \in \otimes^m X$. By noticing that $s_{\lambda_i} \cdots s_{\lambda_{n+1}} s_{\mu_n}^* \cdots s_{\mu_1} x_1 \cdots x_{i-2}$ is in A for all $i = 3, \ldots, n+1$ and using the fact that $w \otimes \phi(a) y = wa \otimes y$ for all $w, y \in X$ and $a \in A$ we see that if $m \geq n$ then

$$\begin{aligned}
\mathcal{T}_{\lambda_1} \cdots \mathcal{T}_{\lambda_{n+1}} \mathcal{T}_{\mu_n}^* \cdots \mathcal{T}_{\mu_1}^* x &= s_{\lambda_1} \otimes \cdots \otimes (s_{\lambda_{n+1}} s_{\mu_n}^* \cdots s_{\mu_1}^* x_1 \cdots x_{n-1}) x_n \otimes \cdots \otimes x_m \\
&= s_\lambda s_\mu^* \otimes x \\
&= \mathcal{T}_{s_\lambda s_\mu^*} x
\end{aligned}$$

and if $m < n$ then $\mathcal{T}_{\lambda_1} \cdots \mathcal{T}_{\lambda_{n+1}} \mathcal{T}_{\mu_n}^* \cdots \mathcal{T}_{\mu_1}^* x = 0$.

From this we see that $\mathcal{T}_{s_\lambda s_\mu^*} - \mathcal{T}_{\lambda_1} \cdots \mathcal{T}_{\lambda_{n+1}} \mathcal{T}_{\mu_n}^* \cdots \mathcal{T}_{\mu_1}^*$ is in $J(\mathcal{E}_+)$. Indeed since $\phi(A) = \mathbb{K}(X)$, by [47, Remark 1.2 (4)] $J(\mathcal{E}_+) \cap \mathcal{T}_X = \mathbb{K}(\mathcal{E}_+)$ and the above difference is $\mathcal{T}_{s_\lambda s_\mu^*} Q_n \in \mathbb{K}(\mathcal{E}_+)$ where Q_n is the orthogonal projection onto the first n summands of $\mathcal{E}_+$. Thus $S_{s_\lambda s_\mu^*} = S_{s_{\lambda_1}} \cdots S_{s_{\lambda_{n+1}}} S_{s_{\mu_n}}^* \cdots S_{s_{\mu_1}}^*$. □

Proposition 3.2 *If Λ is a k-graph with no sinks and (X, ϕ) is the Hilbert bimodule over A defined above then $C^*(\Lambda) \cong \mathcal{O}_X$.*

Proof. We check that, for $x \in X$, the elements $t_x := x \in C^*(\Lambda)$ and *-homomorphism $\sigma := \mathrm{id}_A$ satisfy relations (1)-(4) in [47, Theorem 3.12]. The only relation that might be a little unclear is (4).

It suffices to show that the case $a = s_\lambda s_\mu^*$ with $\lambda, \mu \in \Lambda \backslash \Lambda^0$ holds because $\mathrm{span}\{s_\lambda s_\mu^* \,|\, \lambda, \mu \in \Lambda,\ |\lambda| = |\mu|\}$ is dense in A by Lemma 3.6 and $s_v s_v^* = s_v = \sum_{\alpha \in \Lambda^{e_i}(v)} s_\alpha s_\alpha^*$ for any $i \in \{1, 2, \ldots, k\}$. Let $\lambda, \mu \in \Lambda \backslash \Lambda^0$ with $|\lambda| = |\mu|$, then

$$\sigma^{(1)} \phi(s_\lambda s_\mu^*) = \sigma^{(1)}(\theta_{s_\lambda, s_\mu}) = t_{s_\lambda} t_{s_\mu}^* = s_\lambda s_\mu^* = \sigma(s_\lambda s_\mu^*).$$

Hence by [47, Theorem 3.12] there exists a *-homomorphism $\tilde{\sigma} : \mathcal{O}_X \longrightarrow C^*(\Lambda)$ such that $\tilde{\sigma}(S_x) = t_x = x$ and $\tilde{\sigma}(a) = \sigma(a) = a$ for all $x \in X$ and $a \in A$.

Conversely, by Lemma 3.12, the family of partial isometries $\{\tau_\lambda \mid \lambda \in \Lambda^0\}$ is a *-representation of Λ in $\mathcal{O}_X$. Therefore by the universal property of $C^*(\Lambda)$ ([35, Remarks 1.6]) there exists a *-homomorphism $\pi : C^*(\Lambda) \longrightarrow \mathcal{O}_X$.

Now by Lemma 3.13 the family $\{\tau_\lambda \mid \lambda \in \Lambda\}$ generates $\mathcal{O}_X$ therefore it is clear that $\tilde{\sigma}\pi = \mathrm{id}_{C^*(\Lambda)}$ and $\pi\tilde{\sigma} = \mathrm{id}_{\mathcal{O}_X}$ hence $C^*(\Lambda)$ is isomorphic to $\mathcal{O}_X$. □

Proposition 3.3 *The C^*-algebra A, as defined above, is stably isomorphic to $C^*(\mathbb{Z} \rtimes_{|\cdot|} \Lambda)$.*

Proof. By [33, Theorem 2], Lemma 3.4 and the fact that β is a strongly continuous action of the compact abelian group $\mathbb{T}$ on $C^*(\Lambda)$, $(C^*(\Lambda) \rtimes_\beta \mathbb{T}) \otimes \mathbb{K} \cong C^*(\Lambda)^\beta \otimes \mathbb{K}$. By [35, Corollary 5.3], $C^*(\mathbb{Z} \times_{|\cdot|} \Lambda) \cong C^*(\Lambda) \rtimes_\beta \mathbb{T}$. Therefore,

$$\begin{aligned} C^*(\mathbb{Z} \times_{|\cdot|} \Lambda) \otimes \mathbb{K} &\cong (C^*(\Lambda) \rtimes_\beta \mathbb{T}) \otimes \mathbb{K} \\ &\cong C^*(\Lambda)^\beta \otimes \mathbb{K}. \end{aligned}$$

□

Conjecture 3.1 *If Λ satisfies* **(S)** *(as defined in Definition 4.3) then A is an AF-algebra*

Proof. By Proposition 3.3 A is stably isomorphic to $C^*(\mathbb{Z} \times_{|\cdot|} \Lambda)$. Thus it is enough to show that $C^*(\mathbb{Z} \times_{|\cdot|} \Lambda)$ is an AF-algebra if Λ satisfies **(S)**.

Suppose that $(n, \lambda), (m, \mu) \in \mathbb{Z} \times_{|\cdot|} \Lambda$ with $r(n, \lambda) = r(m, \mu)$, $\langle d(n, \lambda), d(m, \mu) \rangle = 0$ then $(n, r(\lambda)) = (m, r(\mu))$ and $\langle d(\lambda), d(\mu) \rangle = 0 \Rightarrow r(\lambda) = r(\mu)$ and

$\langle d(\lambda), d(\mu)\rangle = 0 \Rightarrow s(\lambda) \neq s(\mu)$ (since Λ satisfies **(S)**) $\Rightarrow s(n,\lambda) \neq s(m,\mu)$. Therefore $\mathbb{Z} \times_{|\cdot|} \Lambda$ satisfies **(S)**.

If $(n,\lambda) \in \mathbb{Z} \times_{|\cdot|} \Lambda$, $d(n,\lambda) > 0$ then $r(n,\lambda) = (n, r(\lambda))$ and $s(n,\lambda) = (n + |\lambda|, s(\lambda))$. But $n + |\lambda| > n$ therefore $r(n,\lambda) \neq s(n,\lambda)$ for all $(n,\lambda) \in \mathbb{Z} \times_{|\cdot|} \Lambda$ with $d(n,\lambda) > 0$. Hence, $Z \times_{|\cdot|} \Lambda$ does not have any loops.

Therefore if Conjecture 4.1 is true, $C^*(\mathbb{Z} \times_{|\cdot|} \Lambda)$ is an AF-algebra and hence $C^*(\Lambda)^\beta$ is also. □

If the above conjecture is true then we may use [47, Theorem 4.9] to calculate the K-theory of $C^*(\Lambda)$ under condition **(S)**:

$$K_0(C^*(\Lambda)) = \operatorname{coker}(\beta_X)$$

$$K_1(C^*(\Lambda)) = \ker(\beta_X)$$

where $\beta_X = \otimes(\mathrm{id}_A - [X])$ in the notation of [47].

Remark 3.2

(1) If Λ is a 1-graph then A is an AF-algebra since $A = C^*(\Lambda)^\alpha$ so the K-theory can be recovered from the Cuntz-Pimsner six term exact sequence in [47, Theorem 4.9].

(2) It was shown in [59] that for any infinite matrix, M, with no zero row nor column and index set I, the Exel-Laca algebra $\mathcal{O}_M$ ([21]) is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{\tilde{X}}$ where $\tilde{X}$ is a Hilbert bi-module over the C^*-algebra $\tilde{D} := C^*(S_i^* S_i \,|\, i \in I)$. The analogue of this construction for k-graph C^*-algebras is the Hilbert bimodule $Y := \overline{\operatorname{span}}\{s_\lambda a \,|\, \lambda \in \Sigma,\ a \in D\}$ over the C^*-algbra $D := C^*(s_\lambda^* s_\mu \,|\, \lambda, \mu \in \Sigma\}$ resulting in $\mathcal{O}_Y \cong C^*(\Lambda)$, the details of which are shown in Appendix

A. We do not need to assume that Λ has no sinks for this construction. Also note that $D \subset A$.

(3) *We require Λ to not have sinks so that A is stably isomorphic to a k-graph C^*-algebra and is therefore open to further analysis. If we allow Λ to have sinks then by [47, Remark 1.2 (3)], all of the results hold if we replace A by $\tilde{A} := \overline{\text{span}}\{\langle x, y\rangle \mid x, y \in X\}$ - an essential ideal in A apart from Proposition 3.3 and Conjecture 3.1.*

Chapter 4

AF k-graph C^*-algebras

In this chapter we investigate into which criteria should be put on a k-graph Λ so that $C^*(\Lambda)$ is an AF-algebra [4, 15]. First we recall a Lemma from [35] stating a sufficient condition for $C^*(\Lambda)$ to be AF.

Lemma 4.1 ([35, Lemma 5.4]) *Let Λ be a k-graph and suppose there is a map $b : \Lambda^0 \longrightarrow \mathbb{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$, then $C^*(\Lambda)$ is AF.*

The existence of such a map $b : \Lambda^0 \longrightarrow \mathbb{Z}^k$ is a rather strong condition on Λ. For example, there are many 1-graphs for which there is no such map, one of which is shown in Figure 4.1, yet their C^*-algebra is an AF-algebra (see [35, Theorem 2.4], which says that a row-finite directed graph C^*-algebra is AF precisely when it contains no loops).

In the following section we explore the possibilities of other sufficient conditions on Λ so that $C^*(\Lambda)$ is an AF-algebra.

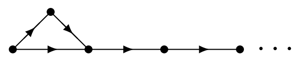

Figure 4.1: The skeleton of a 1-graph Λ with $C^*(\Lambda)$ AF.

4.1 Sufficient conditions

As an analogue to [37, Theorem 2.4], it might be reasonable to conjecture that a k-graph C^*-algebra $C^*(\Lambda)$ is an AF-algebra if and only if Λ has no periodic infinite paths i.e. there does not exist a periodic $x \in \Lambda^\infty$. However we believe that there exists a counter example to:

if Λ has no infinite periodic paths then $C^*(\Lambda)$ is an AF-algebra.

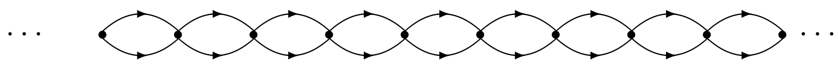

Figure 4.2: The skeleton of A.

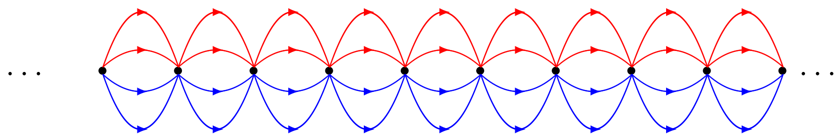

Figure 4.3: The skeleton of $\mathcal{L}$.

Let A be the 1-graph with skeleton as depicted in Figure 4.2. So $A^0 := \mathbb{Z}$, $A^1 := \{a_n, b_n \,|\, n \in \mathbb{Z}\}$ with $r(a_n) = r(b_n) = n$ and $s(a_n) = s(b_n) = n+1$. Let $(A^1 * A^1)_n := \{(a_n, a_{n+1}), (a_n, b_{n+1}), (b_n, a_{n+1}), (b_n, b_{n+1})\}$. A factorisation rule $\phi : A^1 * A^1 \longrightarrow A^1 * A^1$ may be defined in terms of its *sections* (i.e. bijections ϕ_n on $(A^1 * A^1)_n$) by setting $\phi(x) = \phi_n(x)$ if $x \in (A^1 * A^1)_n$. Note that ϕ is well-defined since $A^1 * A^1 = \bigsqcup_{n\geq 0}(A^1 * A^1)_n$, and that ϕ_n is a member of the symmetric group S_4. It is conceivable that a careful choice of ϕ_n will result in a 2-graph $\mathcal{L} := A *_\phi A$ (see Figure 4.3) with no infinite periodic paths.

Assuming that such a ϕ exists, one sees that $\mathcal{L}$ is cofinal and satisfies the aperiodicity condition. Therefore by [35, Proposition 4.8] $C^*(\mathcal{L})$ is simple. Using Proposition 5.1 it is easy to calculate the K-theory as being $K_0(C^*(\mathcal{L})) \cong K_1(C^*(\mathcal{L})) \cong \mathbb{Z}[\frac{1}{2}]$. Thus, in particular, $C^*(\mathcal{L})$ is not an AF-algebra. However, $\mathcal{L}$ does not have any infinite periodic paths.

Also note that $C^*(\mathcal{L})$ may not be purely infinite (i.e. every hereditary subalgebra contains an infinite projection). Consequently, the dichotomy satisfied by simple 1-graph C^*-algebras [37, Corollary 3.10] - simple 1-graph C^*-algebras are either AF or purely infinite - may not carry over to k-graph C^*-algebras, $k \geq 2$.

Definition 4.1 ([49, Definition 3.9]) *A k-graph Λ is said to be locally convex if whenever there is a $\lambda \in \Lambda^{e_i}(u)$ and a $\mu \in \Lambda^{e_j}(u)$ for some $u \in \Lambda^0$ then $\Lambda^{e_j}(s(\lambda)) \neq \emptyset$ and $\Lambda^{e_i}(s(\mu)) \neq \emptyset$.*

Remark 4.1 *It is clear that a locally convex k-graph, Λ, is finite(i.e. $|\Lambda| < \infty$) if and only if Σ is finite and Λ has no loops.*

Definition 4.2 *We let $\Lambda^0_{s,i} := \{v \in \Lambda^0 \mid \Lambda^{e_i}(v) = \emptyset\}$ be the set of i-sources of Λ and $\Lambda^0_s := \{v \in \Lambda^0 \mid \Lambda(v) = \emptyset\}$ be the set of sources of Λ.*

The following Lemma generalises [37, Corollary 2.3].

Lemma 4.2 *If Λ is a finite locally convex k-graph then $C^*(\Lambda) \cong \bigoplus_{u \in \Lambda^0_s} M_{n_u}$ where $n_u := |\{\lambda \in \Lambda \mid s(\lambda) = u\}|$ is the number of paths ending at u.*

Proof. The Lemma has been proved for the case $k = 1$ in [37, Corollary 2.3], therefore assume that $k > 1$. By [49, Propsition 3.11, Proposition 3.5, Remarks 3.8] we see that $C^*(\Lambda) = \text{span}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$. Thus $C^*(\Lambda)$ is obviously finite dimensional.

Now given $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu) \notin \Lambda^0_s$,

$$\begin{aligned}
s_\lambda s_\mu^* &= \sum_{\alpha_1 \in \Lambda^{e_{i_1}}(v)} s_{\lambda\alpha_1} s_{\mu\alpha_1}^* \qquad \text{for some } i_1 \in \{1, 2, \ldots, k\} \\
&= \sum_{\substack{\alpha_1 \in \Lambda^{e_{i_1}}(v) \\ s(\alpha_1) \notin \Lambda^0_s}} s_{\lambda\alpha_1} s_{\mu\alpha_1}^* + \sum_{\substack{\alpha_1 \in \Lambda^{e_{i_1}}(v) \\ s(\alpha_1) \in \Lambda^0_s}} s_{\lambda\alpha_1} s_{\mu\alpha_1}^* \\
&= \sum_{\substack{\alpha_1 \in \Lambda^{e_{i_1}}(v) \\ s(\alpha_1) \notin \Lambda^0_s \\ \alpha_2 \in \Lambda^{e_{i_2}}(s(\alpha_1))}} s_{\lambda\alpha_1\alpha_2} s_{\mu\alpha_1\alpha_2}^* + \sum_{\substack{\alpha_1 \in \Lambda^{e_{i_1}}(v) \\ s(\alpha_1) \in \Lambda^0_s}} s_{\lambda\alpha_1} s_{\mu\alpha_1}^* \\
&\vdots \\
&= \sum_{(\alpha,\beta) \in S} s_\alpha s_\beta^* \qquad \text{for some } S \subset \{\lambda \in \Lambda \mid,\ s(\lambda) \in \Lambda^0_s\}^2
\end{aligned}$$

where all the sums are finite. Therefore, $C^*(\Lambda) = \text{span}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \Lambda, s(\lambda) = s(\mu) \in \Lambda^0_s\}$.

If $\mu\xi \in \Lambda$ with $s(\mu), s(\xi) \in \Lambda^0_s$, notice that $\mu = \mu_1\mu'$, $\xi = \xi_1\xi'$ where $d(\mu_1) = d(\xi_1) = \inf\{d(\mu), d(\xi)\}$ and $\langle d(\mu'), d(\xi') \rangle = 0$. Therefore,

$$s_\mu^* s_\xi = \delta_{\mu_1,\xi_1} s_{\mu'}^* s_{\xi'}.$$

We claim that

$$s_{\mu'}^* s_{\xi'} = \begin{cases} s_{\xi'} & \text{if } d(\mu') = 0,\ r(\xi') = \mu' \\ s_{\mu'}^* & \text{if } d(\xi') = 0,\ r(\mu') = \xi' \\ 0 & \text{otherwise} \end{cases}$$

The first two cases above are clear, but the case $d(\xi') \neq 0,\ d(\mu') \neq 0$ deserves some explanation. Since $\langle d(\mu'), d(\xi')\rangle = 0$, there exists a $j \in \{1, 2, \ldots, k\}$ such that $d(\xi') \geq e_j$ but $d(\mu') \ngeq e_j$. If $r(\mu') \neq r(\xi')$ then $s^*_{\mu'} s_{\xi'} = 0$. If $r(\mu') = r(\xi')$ let $\mu' = \mu'_1 \cdots \mu'_n$, for some $\mu'_1, \ldots, \mu'_n \in \Sigma,\ n \in \{1, 2, \ldots\}$, and $\xi' = \xi'_1 \xi''$ with $d(\xi'_1) = e_j$. Then

$$r(\mu'_1) = r(\xi'_1),\ d(\mu'_1) = e_{i_1},\ d(\xi'_1) = e_j,\ i_1 \neq j \Rightarrow \Lambda^{e_j}(s(\mu'_1)) \neq \emptyset$$

since Λ is locally convex by hypothesis. Choose $\eta \in \Lambda^{e_j}(s(\mu'_1))$ then a similar argument shows that $\Lambda^{e_j}(s(\mu'_2)) \neq \emptyset$ and so on until we come to $\Lambda^{e_j}(s(\mu'_n)) \neq \emptyset$ a contradiction since $s(\mu'_n) = s(\mu) \in \Lambda^0_s$. Thus we cannot have a pair $\mu', \xi' \in \Lambda^0_s$ such that $\langle d(\mu'), d(\xi')\rangle = 0,\ r(\xi') = r(\mu')$ unless $d(\xi') = d(\mu') = 0$.

If in addition $\lambda, \eta \in \Lambda^0$ with $s(\lambda) = s(\mu)$ and $s(\eta) = s(\xi)$, it follows that

$$s_\lambda s^*_\mu s_\xi s^*_\eta = \begin{cases} s_\lambda s^*_\mu & \text{if } \mu = \xi \\ s_\lambda s_{\xi'} s^*_\eta & \text{if } \xi = \mu\xi' \text{ for some } \xi' \in \Lambda\backslash\Lambda^0 \\ s_\lambda s^*_{\mu'} s^*_\eta & \text{if } \mu = \xi\mu' \text{ for some } \mu' \in \Lambda\backslash\Lambda^0 \\ 0 & \text{otherwise} \end{cases}.$$

But since $s(\lambda), s(\eta) \in \Lambda^0_s$ we cannot have that $\xi = \mu\xi',\ \mu = \xi\mu'$ for some $\xi', \mu' \in \Lambda\backslash\Lambda^0$. Hence,

$$s_\lambda s^*_\mu s_\xi s^*_\eta = \begin{cases} s_\lambda s^*_\eta & \text{if } \mu = \xi \\ 0 & \text{otherwise} \end{cases}$$

therefore $\{s_\lambda s^*_\mu \,|\, \lambda, \mu \in \Lambda, s(\lambda) = s(\mu) \in \Lambda^0_s\}$ is a complete system of matrix units. In particular $s_\lambda s^*_\mu s_\xi s^*_\eta = 0$ if $s(\lambda) \neq s(\xi)$ therefore it follows that

$$\begin{aligned} C^*(\Lambda) &= \bigoplus_{u \in \Lambda^0_s} \operatorname{span}\{s_\lambda s^*_\mu \,|\, \lambda, \mu \in \Lambda,\ s(\lambda) = s(\mu) = u\} \\ &= \bigoplus_{u \in \Lambda^0_s} M_{n_u} \end{aligned}$$

where $n_u := |\{\lambda \in \Lambda \,|\, s(\lambda) = u\}|$ as claimed. □

Consequently if we can ensure that given any finite subset, Δ_0, of a k-graph Λ, there exists a finite locally convex sub-k-graph Δ containing Δ_0 then we may show that $C^*(\Lambda)$ is AF. For example, note that the coloured directed graph depicted in Figure 4.4 is the skeleton of an unique 2-graph, which we will call $\mathcal{C}_2$.

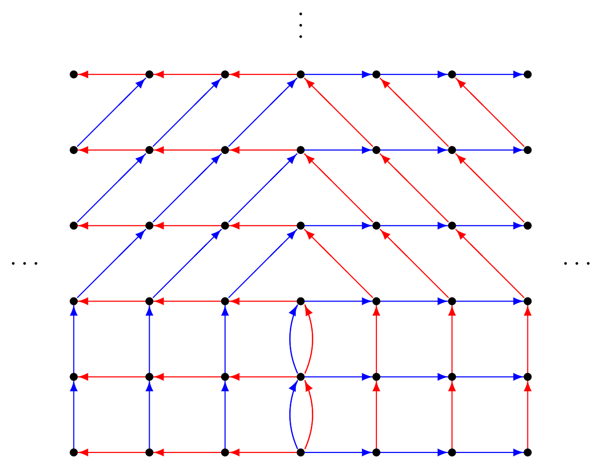

Figure 4.4: The skeleton of $\mathcal{C}_2$

Also note that there is no map $b : \mathcal{C}_2^0 \longrightarrow \mathbb{Z}^2$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \mathcal{C}_2$ so we cannot apply Lemma 4.1. However, $C^*(\mathcal{C}_2)$ is the inductive limit of a sequence of finite dimensional C^*-algebras as shown below.

By applications of the gauge invariant theorem [35, Theorem 3.4] it is clear that $C^*(\mathcal{C}_2) \cong \lim_{\rightarrow}(C^*(\mathcal{C}_2^{(n)}), \iota_n)$, where $\{\mathcal{C}_2^{(n)}\}_{n=1}^{\infty}$ is the sequence of finite locally convex sub-2-graphs depicted in Figure 4.5 and $\iota_n : C^*(\mathcal{C}_2^{(n)}) \longrightarrow C^*(\mathcal{C}_2^{(n+1)})$ are the connecting maps defined by $\iota_n(s_\lambda^{(n)}(s_\mu^{(n)})^*) = s_\lambda^{(n+1)}(s_\mu^{(n+1)})^*$ where $\{s_\lambda^n \,|\, \lambda \in \mathcal{C}_2^{(n)}\}$ is the universal $*$-representation of $C^*(\mathcal{C}_2^{(n)})$.

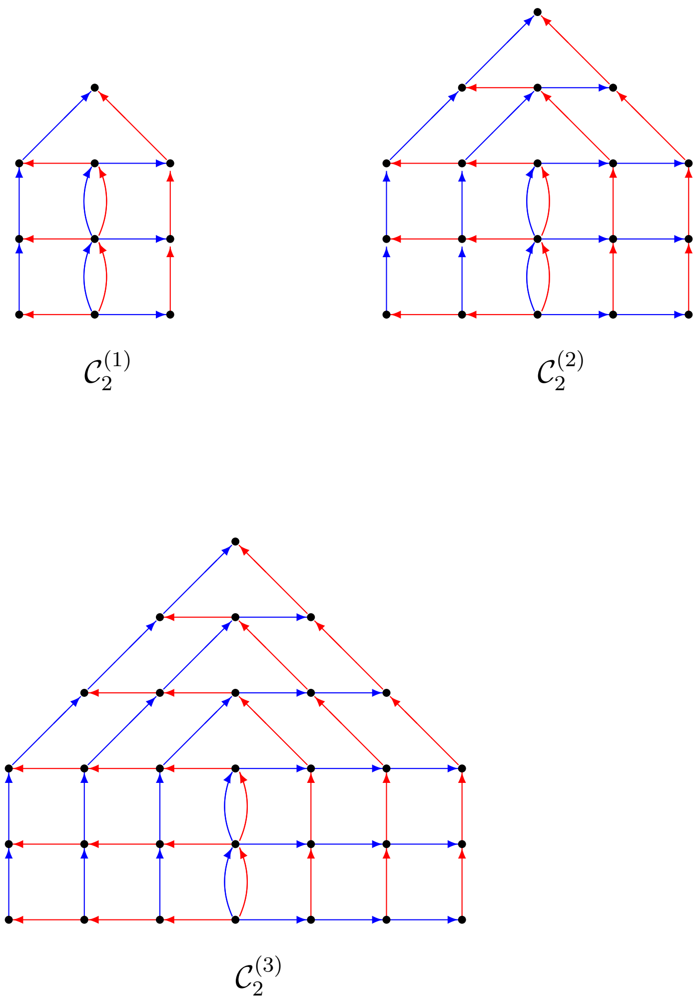

Figure 4.5: Sequence of finite locally convex 2-graphs

Definition 4.3 *Say that* Λ *satisfies condition* **(S)** *if given any* $\lambda, \mu \in \Lambda \backslash \Lambda^0$ *such that* $r(\lambda) = r(\mu)$ *and* $\langle d(\lambda), d(\mu) \rangle = 0$ *then* $s(\lambda) \neq s(\mu)$.

Definition 4.4 *For a subset* Γ *of a* k*-graph* Λ *we let* $\Sigma(\Gamma) := \{\alpha \in \Sigma \mid \exists\, \eta, \nu \in \Lambda,\ \xi \in \Gamma \text{ such that } \xi = \eta\alpha\nu\}$ *be the edges of* Γ *and* $V(\Gamma) := \{v \in \Lambda^0 \mid v = r(\alpha) \text{ or } v = s(\alpha) \text{ for some } \alpha \in \Sigma(\Gamma) \cup \Gamma\}$ *be the vertices of* Γ. *We also let* Γ^* *be the subcategory of* Λ *generated by* Γ.

Definition 4.5 *Let* Γ *be a subset of the edges,* Σ*, of a* k*-graph* Λ. *Then a* *vertex $v \in V(\Gamma)$ is said to be open in Γ if there exists $\lambda \in \Lambda^{e_i} \cap \Gamma,\ \mu \in \Lambda^{e_j} \cap \Gamma$, for some $i, j \in \{1, \ldots, k\}$, $i \neq j$, with $r(\lambda) = r(\mu) = u$ and $\mathcal{E}(\lambda, \mu) \not\subset \Gamma \times \Gamma$. Denote the set of open vertices in Γ by $O(\Gamma)$.*

Conjecture 4.1 *If Λ is a k-graph that satisfies* **(S)** *and has no loops then $C^*(\Lambda)$ is an AF-algebra.*

Strategy of proof. Since every element in $C^*(\Lambda)$ can be approximated by a finite linear combination of monomials in $\{s_\lambda s_\mu^* \mid \lambda, \mu \in \Lambda,\ s(\lambda) = s(\mu)\}$ it is enough to show that given any finite set of monomials, $\{s_\lambda s_\mu^*\}$, there exists a finite dimensional C^*-algebra of $C^*(\Lambda)$ that contains $\{s_\lambda s_\mu^*\}$. Thus it is enough to show that given any finite subset, F, of Λ there exists a finite sub-category, Δ, of Λ such that $F \subset \Delta$ and $\mathcal{E}(\lambda, \mu) \subset \Delta \times \Delta$ whenever $\lambda, \mu \in \Delta \cap \Sigma$. For then $C := \text{span}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \Delta\}$ is a finite dimensional C^*-subalgebra of $C^*(\Lambda)$ that contains $\{s_\lambda s_\mu^*\}_{\lambda,\mu \in F}$, as shown in the following.

It is clear that C is self-adjoint. To see that C is closed under multiplication, first note that if $\lambda, \mu, \xi, \eta \in \Delta \cap \Sigma$ then

$$s_\lambda s_\mu^* s_\xi s_\eta^* = \sum_{(\alpha,\beta) \in \mathcal{E}(\mu,\xi)} s_\lambda s_\alpha s_\beta^* s_\eta^* = \sum_{(\alpha,\beta) \in \mathcal{E}(\mu,\xi)} s_{\lambda\alpha} s_{\eta\beta}^* \in C$$

since $\mathcal{E}(\mu, \xi) \subset \Delta \times \Delta$ and Δ is a subcategory of Λ. Now let $\lambda, \mu, \xi, \eta \in \Delta \backslash \Lambda^0$ then $\mu = \mu_1 \cdots \mu_n,\ \xi = \xi_1 \cdots \xi_m$ for some $\mu_1, \ldots, \mu_n, \xi_1, \ldots, \xi_m \in \Delta \cap \Sigma,\ n, m \in \{1, 2, \ldots\}$. Thus

$$s_\mu^* s_\xi = s_{\mu_n}^* \cdots s_{\mu_1}^* s_{\xi_1} \cdots s_{\xi_m} = \sum s_{\alpha_1} \cdots s_{\alpha_m} s_{\beta_n}^* \cdots s_{\beta_1}^*$$

where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \Delta \cap \Sigma$. Therefore

$$s_\lambda s_\mu^* s_\xi s_\eta^* = \sum s_{\lambda\alpha} s_{\eta\beta}^*$$

where the sum is finite and every α, β and thus every $\lambda\alpha, \eta\beta$ is in Δ. It follows that C is closed under multiplication and is a finite dimensional C^*-subalgebra of $C^*(\Lambda)$ that clearly contains $\{s_\lambda s_\mu^*\}_{\lambda,\mu \in F}$.

Moreover, if we also insist that whenever $\lambda \in \Delta \cap \Sigma$ then $\Lambda^{d(\lambda)}(r(\lambda)) \subset \Delta$ then C is isomorphic to $\bigoplus_{u \in \Delta_s} M_{n_u}$ where $\Delta_s = \{v \in \Delta \mid d(v) = 0,\ \Lambda^{e_i}(v) \cap \Delta = \emptyset \text{ for all } i \in \{1, \ldots, k\}\}$ and $n_u := |\{\lambda \in \Delta \mid s(\lambda) = u\}|$; the proof of which is similar to the proof of Lemma 4.2.

To construct a required Δ from a finite subset, F, of Λ we must, in some sense, close the open vertices of $\Sigma(F)$. Thus a likely candidate for Δ is given below.

Let $\Delta_0 := \bigcup_{\alpha \in \Sigma(F)} \Lambda^{d(\alpha)}(r(\alpha)),\ V := V(F)$ and

$$\Delta_{j+1} := \Delta_j \cup \left(\bigcup_{u \in O(\Delta_j)} \Sigma(\Lambda^p(u)) \right)$$

where $p = (1, 1, \ldots, 1) \in \mathbb{N}^k$ and $j = 0, 1, \ldots$. Then set $\Delta := (\tilde{\Delta} \cup V)^*$ where $\tilde{\Delta} := \bigcup_{j=0}^{\infty} \Delta_j$. We claim that Δ satisfies (i) if $\lambda, \mu \in \Delta \cap \Sigma$ then $\mathcal{E}(\lambda, \mu) \subset \Delta \times \Delta$, and (ii) if $\lambda \in \Delta \cap \Sigma$ then $\Lambda^{d(\lambda)}(r(\lambda)) \subset \Delta$.

To prove (i), note that if $\lambda, \mu \in \Lambda \cap \Sigma$ then $\lambda, \mu \in \tilde{\Delta}$ and thus $\lambda, \mu \in \Delta_j$ for large enough j. If $r(\lambda) \neq r(\mu)$ then $\mathcal{E}(\lambda, \mu) = \emptyset \subset \Delta \times \Delta$. If $r(\lambda) = r(\mu)$ and $d(\lambda) = d(\mu)$ then either $\lambda \neq \mu \Rightarrow \mathcal{E}(\lambda, \mu) = \emptyset \subset \Delta \times \Delta$ or $\lambda = \mu \Rightarrow \mathcal{E}(\lambda, \mu) = \{(s(\lambda), s(\mu))\} \subset \Delta \times \Delta$. Finally, if $r(\lambda) = r(\mu)$ and $d(\lambda) \neq d(\mu)$ then $\mathcal{E}(\lambda, \mu) \subset \Sigma(\Lambda^p(r(\lambda))) \times \Sigma(\Lambda^p(r(\lambda))) \subset \Delta \times \Delta$. Hence, in all cases $\lambda, \mu \in \Lambda \cap \Sigma \Rightarrow \mathcal{E}(\lambda, \mu) \subset \Delta \times \Delta$.

To prove (ii), note that if $\lambda \in \Delta \cap \Sigma$ then $\lambda \in \tilde{\Delta}$ and thus $\lambda \in \Delta_j$ for some j. Let $j_0 = \min\{j \mid \lambda \in \Delta_j\}$. If $j_0 = 0$ then $\lambda \in \Delta_0 = \bigcup_{\alpha \in \Sigma(F)} \Lambda^{d(\alpha)}(r(\alpha)) \Rightarrow \Lambda^{d(\lambda)}(r(\lambda)) \subset \Delta$. If $j_0 > 0$ then $\lambda \in \bigcup_{u \in O(\Delta_{j_0 - 1})} \Sigma(\Lambda^p(u))$ thus $\Lambda^{d(\lambda)} \subset \Delta$ as required.

Since Λ satisfies **(S)** we expect $\tilde{\Delta}$ to be finite. If this is true then since Λ has no loops Δ will be a suitable finite subcategory of Λ as described above. $\square$

Consider the coloured directed graph in Figure 4.6, which is the skeleton of an unique 2-graph $\mathcal{X}$.

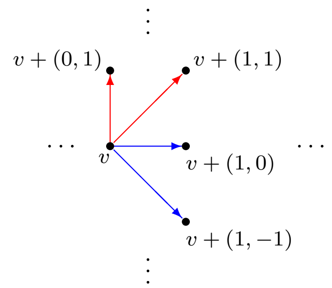

Figure 4.6: The skeleton of $\mathcal{X}$.

The edges, along with their respective sources, emitted by a typical vertex $v \in \mathcal{X}^0 := \mathbb{Z}^2$ *are shown.*

Note that $\mathcal{X}$ satisfies **(S)** but there exists a finite subset, F, of $\mathcal{X}$ for which there is no finite locally convex sub-2-graph that contains F. We give an example that shows that given a finite subset, F, of $\mathcal{X}$ then $\tilde{\Delta}$, as defined in the strategy of proof of Conjecture 4.1, is finite.

We may assume that $F = \Delta_0$. The sequence $\{\Delta_0\}_{j=0}^5$, with $\tilde{\Delta} = \Delta_5$, is illustrated in Figure 4.7 where an open circle represents an open vertex in the corresponding Δ_j.

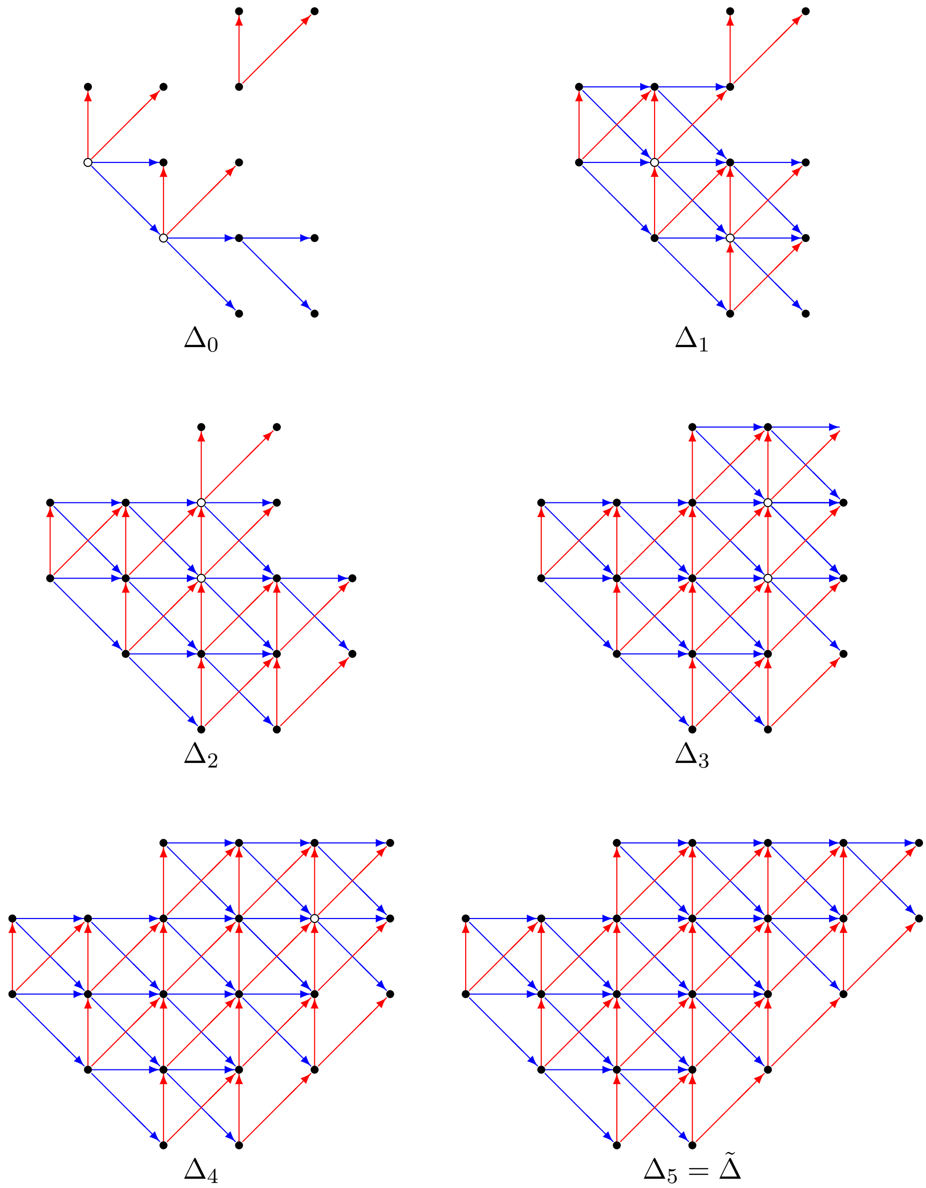

Figure 4.7: The sequence $\{\Delta_j\}_{j=0}^{5}$.

Consider the condition **(Γ)** defined below, which is weaker than **(S)**.

Definition 4.6 *The k-graph Λ is said to satisfy condition* **(Γ)** *if given any $\lambda, \mu \in \Lambda\backslash\Lambda^0$ such that $r(\lambda) = r(\mu),\ s(\lambda) = s(\mu)$ and $\langle d(\lambda), d(\mu)\rangle = 0$ then $\tilde{\mathcal{E}}(\lambda, \mu) := \{\alpha, \beta \,|\, \lambda\alpha = \mu\beta,\ d(\lambda\alpha) = \sup\{d(\lambda), d(\mu)\}\} = \emptyset$.*

Conjecture 4.2 *If the k-graph Λ satisfies condition* **(Γ)** *then $C^*(\Lambda)$ is an AF-algebra.*

Strategy of proof. As in the strategy of proof of Conjecture 4.1, given any finite subset, Δ_0, of Λ we hope to find a finite spanning set S such that $C^*(s_\lambda \,|\, \lambda \in \Delta_0)$ is a subalgebra of $C := \text{span}S$ when Λ satisfies **(Γ)**. However we do not expect that S will be a subset of monomials as was the case in the strategy of proof of Conjecture 4.1. □

An intriguing example of a 2-graph satisfying **(Γ)** but not **(S)** is the following.

Let A, B be 1-graphs with $V = A^0 = B^0 := [2\mathbb{N} \times 2\mathbb{N}] \cup [(2\mathbb{N}+1) \times (2\mathbb{N}+1)]$

$$A^1 := \{a(m), b(n) \mid m, n \in A^0\} \quad B^1 := \{\alpha(m), \beta(n) \mid m, n \in B^0\}$$

$$r(a(m)) = m \quad s(a(m)) = m + (1,1) \quad r(\alpha(m)) = m \quad s(\alpha(m)) = m + (1,1)$$
$$r(b(m)) = m \quad s(b(m)) = m + (2,0) \quad r(\beta(m)) = m \quad s(\beta(m)) = m + (0,2)$$

Thus

$$\begin{aligned} A^1 * B^1 &= \{(a(m), \alpha(m+(1,1))), (a(m), \beta(m+(1,1)), (b(m), \alpha(m+(2,0))), \\ &\quad (b(m), \beta(m+(2,0))) \mid m \in V\} \\ B^1 * A^1 &= \{(\alpha(m), a(m+(1,1))), (\alpha(m), b(m+(1,1))), (\beta(m), a(m+(0,2))), \\ &\quad (\beta(m), b(m+(0,2))) \mid m \in V\} \end{aligned}$$

Define $\mathcal{P}_\theta$ to be the 2-graph $A *_\theta B$ where θ ranges over all the possible factorisation rules. The skeleton of $\mathcal{P}_\theta$ is shown in Figure 4.8.

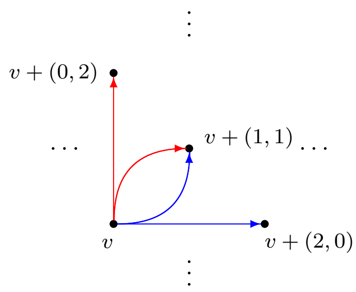

Figure 4.8: The skeleton of $\mathcal{P}_\theta$.

The edges, along with their respective sources, emitted by a typical vertex $v \in \mathcal{P}_\theta^0$ *are shown.*

Define a particular factorisation rule $\theta_1 : A^1 * B^1 \longrightarrow B^1 * A^1$ by

$$(a(u), \alpha(u+(1,1))) \mapsto (\beta(u), b(u+(0,2)))$$
$$(b(u), \alpha(u+(2,0))) \mapsto (\alpha(u), b(u+(1,1)))$$
$$(a(u), \beta(u+(1,1))) \mapsto (\beta(u), a(u+(0,2)))$$
$$(b(u), \beta(u+(2,0))) \mapsto (\alpha(u), a(u+(1,1)))$$

for all $u \in V$. Then one checks that $\mathcal{P}_{\theta_1}$ satisfies $(\mathbf{\Gamma})$ but not $(\mathbf{S})$. Now, $K_1(\mathcal{P}_{\theta_1}) = \{0\}$ (see Appendix B) in keeping with Conjecture 4.2. Also note that $\mathcal{P}_{\theta_1}$ is cofinal and satisfies the aperiodicity condition therefore $C^*(\mathcal{P}_{\theta_1})$ is simple by [35, Proposition 4.8].

4.2 Necessary conditions

By [37, Theorem 2.4] a 1-graph C^*-algebra is an AF-algebra if and only if the 1-graph has no loops, i.e. a morphism whose source is equal to its range. In stark contrast, there exists a k-graph Λ with no loops such that $C^*(\Lambda)$ is not an AF-algebra. Indeed there are an abundance of such examples, one of which is constructed from the 1-graph K^* where K is the directed graph shown in Figure 4.9.

Figure 4.9: The directed graph K.

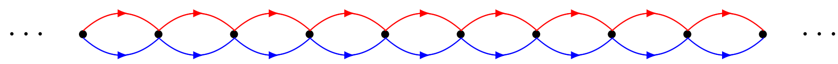

Figure 4.10: The skeleton of $\mathcal{K}_1$.

Let $\mathcal{K}_1 = f^*(K^*)$ (as defined in [35, Definition 1.9]) where $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by $f(m,n) = m+n$. Since f is surjective we see that an application of [35, Corollary 3.5] yields that $C^*(\Lambda) \cong C^*(K^*) \otimes C(\mathbb{T}) \cong \mathbb{K} \otimes C(\mathbb{T})$.

The following Lemma is a further confirmation that a k-graph C^*-algebra need not contain a loop in order for its C^*-algebra to be AF.

Definition 4.7 *Given $\lambda, \mu \in \Lambda$ and $q \in \mathbb{N}^k, q > d(\lambda)$ define the following to be the set of all extensions of λ of degree $q - d(\lambda)$ with respect to μ*

$$E^q_{\lambda,\mu} := \{\alpha \in \Lambda \mid \lambda\alpha = \mu\beta \text{ for some } \beta \in \Lambda \text{ and } d(\lambda\alpha) = q\}.$$

Lemma 4.3 *If there are $\lambda, \mu \in \Lambda \backslash \Lambda^0$ with $r(\lambda) = r(\mu)$, $s(\lambda) = s(\mu) =: v$ and if $E^q_{\lambda,\mu} \neq \Lambda^{q-d(\lambda)}(v)$, $E^q_{\mu,\lambda} = \Lambda^{q-d(\mu)}(v)$ for some $q \in \mathbb{N}^k$ with $q > d(\lambda)$ and $q > d(\mu)$ then $C^*(\Lambda)$ is not an AF-algebra.*

Proof. Let $s := s_\lambda s_\mu^*$. Then

$$\begin{aligned} s^*s &= p_\mu = \sum_{\beta \in \Lambda^{q-d(\mu)}(v)} p_{\mu\beta} = \sum_{\beta \in E^q_{\mu,\lambda}} p_{\mu\beta} \\ &= \sum_{\alpha \in E^q_{\lambda,\mu}} p_{\lambda\alpha} < \sum_{\alpha \in \Lambda^{q-d(\lambda)}(v)} p_{\lambda\alpha} = p_\lambda = ss^* \end{aligned}$$

Thus $C^*(\Lambda)$ contains an infinite projection, p_λ, and so cannot be an AF-algebra. □

As one would expect, the presence (at least of a certain type) of loop in a k-graph Λ ensures that $C^*(\Lambda)$ is not an AF algebra.

Definition 4.8 *Call α a n-loop in Λ where $n \in \mathbb{N}^k \backslash \{0\}$ if $\alpha \in \Lambda^n$ and $s(\alpha) = r(\alpha)$ (i.e. α is a loop of degree n in Λ). If α is a n-loop then we say that α has no n-exit if $\Lambda^n(r(\alpha)) = \{\alpha\}$.*

Lemma 4.4 *If Λ contains a n-loop with a n-exit, for some $n \in \mathbb{N}^k \backslash \{0\}$, then $C^*(\Lambda)$ is not an AF-algebra.*

Proof. Let α be a n-loop in Λ with n-exit β. Then

$$s_\alpha^* s_\alpha = p_{s(\alpha)} \geq p_\alpha + p_\beta > s_\alpha s_\alpha^*.$$

Thus $p_{s(\alpha)}$ is an infinite projection in $C^*(\Lambda)$, therefore $C^*(\Lambda)$ cannot be an AF-algebra. □

We suspect that if a k-graph Λ contains a simple loop, i.e a loop α such that $\alpha = \alpha_1 \alpha_2 \ldots \alpha_m$ for some $m \in \mathbb{N}\backslash\{0\}$, $\alpha_j \in \Lambda^{e_i}$ for some fixed $i \in \{1, \ldots, k\}$ with $r(\alpha_j)$ pairwise distinct for all $j = 1, \ldots, m$, then $C^*(\Lambda)$ is not an AF-algebra. We show that this is indeed the case when α is an e_i-loop for some $i \in \{1, \ldots, k\}$.

Until further notice we assume the following.

Assumption 4.1 *Fix $i \in \{1, 2, \ldots, k\}$*

- *Every me_i-loop in Λ has no me_i-exit.*
- *α is a e_i-loop in Λ with $r(\alpha) = v$.*

Definition 4.9 *Let $\mathbb{N}^k_{\neq i} := \{\sum_{j=1}^k n_j e_j \,|\, n_j \in \mathbb{N},\ n_i = 0\}$. For $m \in \mathbb{N}^k_{\neq i}$ let $\mathcal{B}_m := \Lambda^m(v),\ \Gamma_m := \Lambda^{e_i}(s(\mathcal{B}_m))$ and for $\beta \in \mathcal{B}_m$ let $f_m(\beta)$ be the (unique) element of $\mathcal{B}_m$ such that*

$$\alpha\beta = f_m(\beta)\gamma \qquad \text{for some } \gamma \in \Gamma_m$$

The factorisation property of Λ ensures that f_m is a well-defined mapping on $\mathcal{B}_m$.

Lemma 4.5 *The map $f_m : \mathcal{B}_m \longrightarrow \mathcal{B}_m$ defined above is a bijection*

Proof. Let $\beta \in \mathcal{B}_m$ then choose any $\gamma \in \Gamma_m$ with $r(\gamma) = s(\beta)$. Then there exists a (unique) $\beta' \in \mathcal{B}_m$ such that

$$\alpha\beta' = \beta\gamma$$

Thus $f_m(\beta') = \beta$ and f_m is surjective. Furthermore, since $\mathcal{B}_m$ is a finite set, f_m is a bijection. □

Remark 4.2 *Given $\beta \in \mathcal{B}_m$ there is exactly one $\gamma \in \Gamma_m$ such that $\beta\gamma \in \Lambda$ otherwise if γ' were another such edge of Γ_m then we would have*

$$\alpha\beta_1 = \beta\gamma$$

$$\alpha\beta_2 = \beta\gamma'$$

for some $\beta_1, \beta_2 \in \mathcal{B}_m,\ \beta_1 \neq \beta_2$ contradicting the injectivity of f_m.

Lemma 4.6 *Every $\gamma \in \Gamma_m$ is part of a ne_i-loop for some $n \in \mathbb{N}$.*

Proof. Let $\gamma_1 := \gamma \in \Gamma_m$. Then there exists a $\beta \in \mathcal{B}_m$ such that $\alpha\beta = f_m(\beta)\gamma$. As f_m is a permutation on the finite set $\mathcal{B}_m$, there is an $n \in \mathbb{N}$ such that $f_m^{(n)}(\beta) = \beta$. If $n = 0$ then $\gamma = \gamma_1$ is a loop. If not then we have the following situation

$$\begin{aligned}
\alpha\beta &= f_m(\beta)\gamma_1 \\
\alpha f_m(\beta) &= f_m^{(2)}(\beta)\gamma_2 \\
&\vdots \\
\alpha f_m^{(n-1)}(\beta) &= \beta\gamma_n \\
\alpha\beta &= f_m(\beta)\gamma_{n+1}
\end{aligned}$$

for some $\gamma_i \in \Gamma_m$. From this we see that $r(\gamma_i) = s(\gamma_{i+1})$ and $\gamma_{n+1} = \gamma_1$. Therefore $\gamma_n\gamma_{n-1}\cdots\gamma_1$ is a loop. Hence $\gamma = \gamma_1$ is part of a loop. □

Definition 4.10 *Let $\Lambda^{\neq i} := \bigcup_{m \in \mathbb{N}^k_{\neq i}} \Lambda^m$ and $\Lambda^{\neq i}(v) := \bigcup_{m \in \mathbb{N}^k_{\neq i}} \Lambda^m(v)$. Define the following C^*-subalgebras of $C^*(\Lambda)$:*

$$\begin{aligned}
A &:= p_v C^*(\Lambda) p_v \\
B &:= p_v C^*(s_\lambda \,|\, \lambda \in \Lambda^{\neq i}) p_v \\
&= \overline{\operatorname{span}}\{s_\lambda s_\mu^* \,|\, \lambda, \mu \in \Lambda^{\neq i}(v),\ s(\lambda) = s(\mu)\}
\end{aligned}$$

Lemma 4.7 *The map $\tau : B \longrightarrow B$ defined by*

$$\tau(b) = s_\alpha b s_\alpha^* \qquad b \in B$$

is an automorphism of B.

Proof. Let $\lambda, \mu \in \Lambda^{\neq i}(v)$ with $s(\lambda) = s(\mu)$. Then

$$\tau(s_\lambda s_\mu^*) = s_\alpha s_\lambda s_\mu^* s_\alpha^* = s_{\alpha\lambda} s_{\alpha\mu}^* = s_{f_{m_1}(\lambda)\gamma_1} s_{f_{m_2}(\mu)\gamma_2}^*$$

for some $m_1, m_2 \in \mathbb{N}^k_{\neq i}$, $\gamma_1 \in \Gamma_{m_1}$, $\gamma_2 \in \Gamma_{m_2}$. Therefore γ_1 is a part of a $n_1 e_i$-loop and γ_2 is a part of a $n_2 e_i$-loop by Lemma 4.6. The fact that $s(\gamma_1) = s(\gamma_2)$ coupled with our standing assumption forces $\gamma_1 = \gamma_2$. Thus

$$\begin{aligned}
\tau(s_\lambda s_\mu^*) &= s_{f_{m_1}(\lambda)} s_{\gamma_1} s_{\gamma_1}^* s_{f_{m_2}(\mu)}^* \\
&= s_{f_{m_1}(\lambda)} p_{r(\gamma)} s_{f_{m_2}(\mu)}^* \\
&= s_{f_{m_1}(\lambda)} s_{f_{m_2}(\mu)}^* \in B
\end{aligned}$$

It follows that τ maps B into B. Clearly τ is a *-homomorphism and it is easy to check that τ is surjective. □

Proposition 4.1 *With τ defined above we have the following *-isomorphism*

$$A \cong B \rtimes_\tau \mathbb{Z}$$

Proof. Now $A = \overline{\text{span}}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \Lambda(v), s(\lambda) = s(\mu)\}$. If $\lambda, \mu \in \Lambda(v)$ with $s(\lambda) = s(\mu)$ then

$$\lambda = \alpha^l \lambda' \qquad \mu = \alpha^m \mu'$$

for some $l, m \in \mathbb{N}$, $\lambda', \mu' \in \Lambda^{\neq i}$. Therefore

$$s_\lambda s_\mu^* = s_\alpha^l s_{\lambda'} s_{\mu'}^* (s_\alpha^*)^m \in C^*(B, s_\alpha) \subset C^*(\Lambda)$$

Therefore $A \subset C^*(B, s_\alpha)$. Clearly $C^*(B, s_\alpha) \subset A$. Thus $A = C^*(B, s_\alpha)$. Now, by consruction, s_α is an unitary in A which implements τ, therefore there exists a *-homomorphism π of $B \rtimes_\tau \mathbb{Z}$ onto A such that $\pi(b) = b$ for all $b \in B$ and $\pi(u) = s_\alpha$ where u is the universal unitary implementing τ.

There is an action ρ of $\mathbb{T}$ on $B \rtimes_\tau \mathbb{Z}$ such that $\rho_z(b) = b,\ \rho_z(u) = zu$ which induces a faithful conditional expectaion Φ of A onto B by the equation

$$\Phi(a) = \int_{\mathbb{T}} \rho_z(a)\, dz\,.$$

Now the gauge action ω on $C^*(\Lambda)$ induces an action θ of $\mathbb{T}$ on A by $\theta_z = \omega_{(1,1,\ldots,1,z,1,1,\ldots,1)}$ (z in the i^{th} place, 1's everywhere else) with $\pi\theta_z = \rho_z\pi$. Clearly π is faithful on $B = \Phi(A)$, so let a be a non-zero positive element of $B \rtimes_\tau \mathbb{Z}$ then

$$0 \neq \pi(\Phi(a)) = \pi\left(\int_{\mathbb{T}} \rho_z(a)\, dz\right) = \int_{\mathbb{T}} \pi\rho_z(a)\, dz = \int_{\mathbb{T}} \theta_z(\pi(a))\, dz$$

Hence $\pi(a) \neq 0$ and π is faithful on $B \rtimes_\tau \mathbb{Z}$. □

We thank Prof. George A. Elliott for informing us of [5, Proposition 4.4.1]. As a consequence we have the following Corollary.

Corollary 4.1 *If a k-graph Λ satisfies Assumption 4.1 above then $C^*(\Lambda)$ is not an AF-algebra.*

Proof. By Proposition 4.1 $C^*(\Lambda)$ has a corner sub-algebra isomorphic to a unital crossed product by the integers which cannot be AF by [5, Proposition 4.4.1]. Thus $C^*(\Lambda)$ is not AF. □

We close this section by listing some necessary conditions on a k-graph so that its C^*-algebra is an AF-algebra.

Corollary 4.2 *If Λ is a k-graph with $C^*(\Lambda)$ AF then*

(i) if $\lambda, \mu \in \Lambda\backslash\Lambda^0$ with $r(\lambda) = r(\mu),\ s(\lambda) = s(\mu)$ and $E^q_{\lambda,\mu} \neq \Lambda^{q-d(\lambda)}(s(\lambda))$ for some $q \in \mathbb{N}^k,\ q > d(\lambda),\ q > d(\mu)$ then $E^q_{\mu,\lambda} \neq \Lambda^{q-d(\mu)}(s(\mu))$,

(ii) Λ *does not contain an* n*-loop with* n*-exit for all* $n \in \mathbb{N}^k \backslash \{0\}$ *and*

(iii) Λ *does not contain an* e_i*-loop.*

Proof. Conditions (i) and (ii) follow from Lemma 4.3 and Lemma 4.6 respectively. To prove (iii) we shall prove its negation. Suppose that Λ contains an e_i-loop α. Then either α has an e_i-exit or it does not. If it does then $C^*(\Lambda)$ is not an AF-algebra by Lemma 4.4. If α does not have an e_i-exit then we may assume that every me_i-loop in Λ does not have an me_i-exit for all $m \in \mathbb{N} \backslash \{0\}$ thus Assumption 4.1 is satisfied and $C^*(\Lambda)$ is not an AF-algebra by Corollary 4.1. □

4.3 A necessary and sufficient condition

In this section we propose a single necessary and sufficient condition on a k-graph Λ so that $C^*(\Lambda)$ is an AF-algebra.

Definition 4.11 *An infinite path* $x \in \Lambda^\infty$ *is* **eventually injective** *if* $\sigma^p(x)$ *is injective for some* $p \in \mathbb{N}^k$ *(see Definition 2.5).*

If there are no eventually injective infinite paths in Λ then it is likely that there exists a finite number of elements in $C^*(\Lambda)$ that cannot be approximated by elements that lie in a common finite dimensional sub-C^*-algebra of $C^*(\Lambda)$.

For example, we conjectured in Section 4.1 that there exists a 2-graph $\mathcal{L}$ which does not have any infinite periodic paths. However it is clear that $\mathcal{L}$ must have an eventually injective infinite path (since e.g. $x((1,0),(1,0)) = x((0,1),(0,1))$ for all $x \in \mathcal{L}^\infty$).

Let $s_0 := s_\alpha^* s_a,\ s_1 := s_\alpha^* s_b,\ s_2 := s_\beta^* s_a,\ s_3 := s_\beta^* s_b$ where a, b are the blue edges with range 0 and α, β are the red edges with range 0 in Figure 4.3. Then it is conceivable that we may choose $\mathcal{L}$ such that the sequence, $\{w_n\}_{n\in\mathbb{N}}$, of words in $F := \{s_0, s_1, s_2, s_3\}$ defined by $w_0 := s_0,\ w_n := w_{n-1} s_{n \bmod 4}^{n+1}$ is a sequence of distinct, non-zero partial isometries in $C^*(\Lambda)$. Furthermore, it is easy to show that any finite sub-sequence of F is linearly independent, thus F cannot be contained in any finite dimensional sub-C^*-algebra of $C^*(\Lambda)$. Moreover, we believe that the elements in F cannot be approximated by elements that lie in a common finite dimensional sub-C^*-algebra of $C^*(\Lambda)$.

Conversely, if Λ has no eventually injective infinite paths then it is reasonable to believe that given a finite set $F := \{s_\lambda,\ s_\lambda^* \mid \lambda \in \Delta\}$ with Δ a finite subset of Λ, then the set of words in F is finite. Since otherwise we may recover an eventually injective infinite path from the infinite set of words. It then follows that $C^*(\Lambda)$ is an AF-algebra.

Hence we propose the following Conjecture.

Conjecture 4.3 *Let Λ be a k-graph. Then $C^*(\Lambda)$ is an AF-algebra if and only if every $x \in \Lambda^\infty$ is eventually injective.*

Chapter 5

K-theory of 2-graph C^*-algebras

Let Λ be a row finite k-graph with no sinks nor sources. We aim to calculate the K-groups of $C^*(\Lambda)$ solely in terms of Λ.

By [35, Theorem 5.5 and Theorem 5.7] (or Takesaki-Takai duality), $C^*(\Lambda)$ is stably isomorphic to a crossed product of an AF-algebra by $\mathbb{Z}^k$:

$$C^*(\Lambda) \otimes \mathbb{K} \cong (C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{Z}^k$$

For the rest of this chapter we let $\mathcal{G} := C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k$.

The K-theory [61, 2, 43] of a particular kind of 2-graph C^*-algebra, $\mathcal{A}$, is calculated in [53] by means of realising $\mathcal{A} \otimes \mathbb{K}$ as a crossed product of an AF-algebra, $\mathcal{F}$, by $\mathbb{Z}^2$ and considering the chain complex [62]:

$$0 \longleftarrow K_0(\mathcal{F}) \xleftarrow{d_1} K_0(\mathcal{F}) \oplus K_0(\mathcal{F}) \xleftarrow{d_2} K_0(\mathcal{F}) \longleftarrow 0$$

where the differential maps d_1, d_2 depend on the action of $\mathbb{Z}^2$ on $\mathcal{F}$. In fact the K-theory of $\mathcal{A}$ is given by the homology groups of the chain complex above.

We aim to take a similar approach. When $k = 2$, we can define an analogous chain complex:

$$0 \longleftarrow K_0(\mathcal{G}) \overset{\delta_1}{\longleftarrow} K_0(\mathcal{G}) \oplus K_0(\mathcal{G}) \overset{\delta_2}{\longleftarrow} K_0(\mathcal{G}) \longleftarrow 0 \tag{5.1}$$

where the differential maps δ_1, δ_2 depend on the action of $\mathbb{Z}^2$ on $\mathcal{G}$ (the dual action of α). Let $\mathcal{H}_j$, $(j = 0, 1, 2)$ denote the j^{th} homology group of the chain complex (5.1). If $\mathcal{H}_2$ if free abelian then, by [53, Proposition 4.1] , the K-theory of $C^*(\Lambda)$ (when $k = 2$) is as follows:

$$K_0(C^*(\Lambda)) = \mathcal{H}_0 \oplus \mathcal{H}_2$$
$$K_1(C^*(\Lambda)) = \mathcal{H}_1$$

We shall combine the methods found in [53] and [44] to calculate the K-theory of $C^*(\Lambda)$ in terms of the 2-graph Λ alone. Since it is conceivable that the following method will generalise to k-graphs ($k > 2$) we shall work with k-graphs until the restriction to $k = 2$ is necessary.

Note that the gauge action α is of the form α^d (see section 2.1), thus by Lemma 3.4, α has large spectral subspaces. Therefore by a result of Kishimoto and Takai [33, Theorem 2] it follows that $C^*(\Lambda)^\alpha \otimes \mathbb{K} \otimes \mathbb{K}(L^2(\mathbb{T}^k))$ is isomorphic to $(C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k) \otimes \mathbb{K}$. Thus there exists an isomorphism $\eta : K_0(C^*(\Lambda)^\alpha) \longrightarrow K_0(\mathcal{G})$.

The Kishimoto-Takai isomorphism $B^\beta \otimes \mathbb{K} \otimes \mathbb{K}(L^2(G)) \longrightarrow (B \otimes \mathbb{K}) \rtimes_{\beta\otimes\iota} G$ is implemented by unitaries $v_\chi \in M(B \otimes \mathbb{K})^{\bar{\beta}}$ ($\bar{\beta} := \beta \otimes \iota$), $\chi \in \hat{G}$ such that

$$(B \otimes \mathbb{K})^{\bar{\beta}}(\chi) = (B \otimes \mathbb{K})^{\bar{\beta}} v_\chi.$$

Here we will construct these unitaries explicitly for $B = C^*(\Lambda)$.

Let π be a faithful, non-degenerate representation of $C^*(\Lambda)$ on a Hilbert space K ($s_\lambda \mapsto S_\lambda$, $p_\lambda \mapsto P_\lambda$) and let H be an infinite dimensional separable Hilbert space. Let e_{ij}, $i, j \in \mathbb{N}$ be a complete system of matrix units for $\mathbb{K} = \mathbb{K}(H)$ the compact operators on H. For $N \in \mathbb{N}\backslash\{0\}$ and $\boldsymbol{i} = (i_1, i_2, \ldots, i_N)$, $\boldsymbol{j} = (j_1, j_2, \ldots, j_N) \in \mathbb{N}^N$ denote by $e_{\boldsymbol{ij}}$ the partial isometry $\bigotimes_{n=1}^N e_{i_n j_n} \in \bigotimes_{n=1}^N \mathbb{K}$. For $i, j \in \mathbb{N}$ and $n \in \{1, 2, \ldots, k\}$ let $e_{ij}^{(n)} := 1 \otimes \cdots \otimes 1 \otimes e_{ij} \otimes 1 \otimes \cdots \otimes 1 \in \bigotimes_{m=1}^k \mathbb{K}$ (e_{ij} in the n^{th} factor). Fix injective maps, $f_i : \Lambda^{e_i} \longrightarrow \mathbb{N}$, for $i = 1, 2, \ldots, k$.

Note that for each $\lambda \in \Lambda^{e_i}$ the operators $S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}$ are partial isometries in $\pi(C^*(\Lambda)) \otimes (\bigotimes_{j=1}^k \mathbb{K})$ with mutually orthogonal initial spaces and mutually orthogonal range spaces. Hence the (finite or infinite) sum $\sum_{\lambda \in \Lambda^{e_i}} S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}$, converges strongly to a partial isometry v_i.

We will apply the following lemma (taken from [61, Lemma 2.3.6]) to show that the partial isometries v_i ($i = 1, \ldots, k$) are in fact in $\mathcal{M}(\pi(C^*(\Lambda)) \otimes (\bigotimes_{j=1}^k \mathbb{K}))$.

Lemma 5.1 *Let A be a C^*-algebra and let $\{x_n\}_{n\in\mathbb{N}}$ be a norm bounded sequence in $\mathcal{M}(A)$. Let B be a total subset of A, i.e. a set with dense linear span. If $\{x_n b\}_{\mathbb{N}}$ and $\{b x_n\}_{\mathbb{N}}$ are norm convergent in A for all $b \in B$, then $\{x_n\}_{\mathbb{N}}$ is strictly convergent in $\mathcal{M}(A)$.*

Now $B_1 := \{S_\xi S_\eta^* \otimes e_{\boldsymbol{ij}} \,|\, \xi, \eta \in \Lambda,\ \boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^k\}$ is a total subset of $\pi(C^*(\Lambda)) \otimes (\bigotimes_{j=1}^k \mathbb{K})$. For each $n \in \mathbb{N}$ let $\Delta_n := f_i^{-1}(\{1, 2, \ldots, n\})$ and notice that:

- $\Delta_n \subset \Delta_{n+1}$, for all $n \in \mathbb{N}$
- $\bigcup_{n\in\mathbb{N}} \Delta_n = \Lambda^{e_i}$.

We wish to prove that $\{\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\}_n$ converges strictly in $\mathcal{M}(\pi(C^*(\Lambda))\otimes(\bigotimes_{j=1}^k \mathbb{K}))$.

Now, the sequence $\{\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\}_n$ is norm bounded since it is a sequence of partial isometries with norm 1. Let $b = S_\xi S_\eta^* \otimes e_{\boldsymbol{lm}}$ for some $\xi,\eta\in\Lambda$, $\boldsymbol{l},\boldsymbol{m}\in\mathbb{N}^k$. Then, for $\lambda\in\Lambda$,

$$S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\, b = \begin{cases} S_{\lambda\xi}S_\eta^* \otimes e_{\boldsymbol{l'm}} & \text{if } s(\lambda)=r(\xi),\ l_i = f_i(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\, b = \begin{cases} S_{\lambda\xi}S_\eta^* \otimes e_{\boldsymbol{l'm}} & \text{if } s(\lambda)=r(\xi),\ \lambda_0 := f_i^{-1}(l_i)\in\Delta_n \\ 0 & \text{otherwise} \end{cases} \quad (\dagger)$$

where $\boldsymbol{l'} = (l_1, l_2, \ldots, l_{i-1}, 1, l_{i+1}, \ldots, l_k)$. Clearly, $(\dagger)$ is equivalent to $s(\lambda) = r(\xi)$ and $n \geq l_i$ so for large enough n,

$$\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\, b = S_{\lambda\xi}S_\eta^* \otimes e_{\boldsymbol{l'm}}$$

if $s(\lambda_0) = r(\xi)$, where $\lambda_0 := f_i^{-1}(l_i)$. Thus $\{\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\}_n$ is eventually constant and therefore trivially norm convergent.

Now consider,

$$b\left(S_\lambda \otimes e^{(i)}_{1\,f_i(\lambda)}\right) = \begin{cases} S_\xi S_\eta^* S_\lambda \otimes e_{\boldsymbol{lm'}} & \text{if } m_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

But,

$$S_\xi S_\eta^* S_\lambda = \sum_{(\alpha,\beta)\in\mathcal{E}(\eta,\lambda)} S_{\xi\alpha} S_\beta^*$$

a finite sum. Also since Λ is row finite, there are only a finite number of edges $\lambda\in\Lambda^{e_i}$ such that $S_\eta^* S_\lambda$ is non-zero. In fact, $S_\eta^* S_\lambda = 0$ if $\lambda\in\Lambda^{e_i}\backslash\Lambda^{e_i}(r(\eta))$.

Consequently,

$$b\left(S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\right) = \begin{cases} S_\xi S_\eta^* S_\lambda \otimes e_{\boldsymbol{lm'}} & \text{if } m_i = 1 \text{ and } \lambda \in \Lambda^{e_i}\backslash\Lambda^{e_i}(r(\eta)) \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow b\left(\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\right) = \begin{cases} \displaystyle\sum_{\lambda\in\Delta_n\cap\Lambda^{e_i}(r(\eta))} S_\xi S_\eta^* S_\lambda \otimes e_{\boldsymbol{lm'}} & \text{if } m_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{m'} = (m_1, m_2, \ldots, m_{i-1}, f_i(\lambda), m_{i+1}, \ldots, m_k)$. Thus $\{b\left(\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\right)\}_n$ is eventually constant and therefore norm convergent. Hence, by Lemma 5.1, $\{\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\}_n$ is strictly convergent in $\mathcal{M}(\pi(C^*(\Lambda)) \otimes \bigotimes_{j=1}^k \mathbb{K})$.
Notice that as the sequences $\{b\left(\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\right)\}_n$ and $\{\left(\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\right) b\}_n$ are eventually constant for all $b \in B_1$, it follows that the strict limit of $\{\sum_{\lambda\in\Delta_n} S_\lambda \otimes e^{(i)}_{1\, f_i(\lambda)}\}_n$ does not depend on the choice of maps f_i.
Choose isometries $x_i : \bigotimes_{j=1}^{k+1} H \longrightarrow (e^{(i)}_{11} \otimes 1)\left(\bigotimes_{j=1}^{k+1} H\right)$ such that

$$x_i e_{(1,1,\ldots,1)\,(1,1,\ldots,1)} x_i^* = e_{(1,1,\ldots,1)\,(1,1,\ldots,1)}$$

Next for $\lambda \in \Lambda^{e_i}$ choose isometries

$$V_{i,\lambda} : \bigotimes_{j=1}^{k+1} H \longrightarrow \left(\sum_{\substack{\mu\in\Lambda^{e_i} \\ s(\mu)=r(\lambda)}} e^{(i)}_{f_i(\mu) f_i(\mu)} \otimes 1\right)\left(\bigotimes_{j=1}^{k+1} H\right)$$

where the (finite or infinite) sum is a strong operator limit of projections and so the sum converges in the strict topology in $\mathcal{M}(\bigotimes_{j=1}^{k+1}\mathbb{K})$. (Notice that we require Λ not to have a sink here to ensure that $V_{i,\lambda}$ is an isometry for all λ.) We will now check that the following strict limit (denoted by $\beta-\lim$) of partial isometries exists:

$$y_i := \sum_{\lambda\in\Lambda^{e_i}} (P_\lambda \otimes V_{i,\lambda}) := \beta-\lim_n \sum_{\lambda\in\Delta_n} (P_\lambda \otimes V_{i,\lambda}).$$

Notice that $\{P_\lambda \otimes V_{i,\lambda}\}_{\lambda\in\Lambda^{e_i}}$ is a family of partial isometries with mutually orthogonal initial spaces and mutually orthogonal range spaces. Therefore any finite sum of these partial isometries is itself a partial isometry. Thus $\{\sum_{\lambda\in\Delta_n} P_\lambda \otimes V_{i,\lambda}\}_n$ is norm bounded.

Let $B_2 := \{S_\xi S_\eta^* \otimes e_{\boldsymbol{lm}} \,|\, \xi,\eta \in \Lambda\; \boldsymbol{l},\boldsymbol{m} \in \mathbb{N}^{k+1}\}$. Now let $b = S_\xi S_\eta^* \otimes e_{\boldsymbol{lm}} \in B_2$, then

$$\begin{aligned}\left(\sum_{\lambda\in\Delta_n} P_\lambda \otimes V_{i,\lambda}\right) b &= \sum_{\lambda\in\Delta_n} P_\lambda S_\xi S_\eta^* \otimes V_{i,\lambda}\, e_{\boldsymbol{lm}} \\ &= \sum_{\Delta_n\cap\Lambda(r(\xi))} P_\lambda S_\xi S_\eta^* \otimes V_{i,\lambda}\, e_{\boldsymbol{lm}} \\ &= S_\xi S_\eta^* \otimes \sum_{\lambda\in\Lambda^{e_i}(r(\xi))} V_{i,\lambda}\, e_{\boldsymbol{lm}} \qquad \text{for large enough } n.\end{aligned}$$

Thus $\{\left(\sum_{\lambda\in\Delta_n} P_\lambda \otimes V_{i,\lambda}\right) b\}_n$ converges in norm for all $b \in B_2$.

Also,

$$\begin{aligned} b\left(\sum_{\lambda\in\Delta_n} P_\lambda \otimes V_{i,\lambda}\right) &= \sum_{\lambda\in\Delta_n} S_\xi S_\eta^* P_\lambda \otimes e_{\boldsymbol{lm}}\, V_{i,\lambda} \\ &= \sum_{\lambda\in\Delta_n\cap\Lambda^{e_i}(r(\eta))} S_\xi S_\eta^* P_\lambda \otimes e_{\boldsymbol{lm}}\, V_{i,\lambda} \\ &= S_\xi S_\eta^* \otimes e_{\boldsymbol{lm}} \sum_{\lambda\in\Lambda^{e_i}(r(\eta))} V_{i,\lambda} \qquad \text{for large enough } n.\end{aligned}$$

Thus $\{b\left(\sum_{\lambda\in\Delta_n} P_\lambda \otimes V_{i,\lambda}\right)\}_n$ converges in norm for all $b \in B_2$. Hence, $\beta-\lim_n \sum_{\lambda\in\Delta_n} P_\lambda \otimes V_{i,\lambda} \in \mathcal{M}(\pi(C^*(\Lambda)) \otimes \bigotimes_{j=1}^{k+1} \mathbb{K})$ exists and does not depend on the choice of $f_i,\ i = 1, 2, \ldots, k$.

We can now claim that $u_{e_i} := (1\otimes x_i^*)(v_i\otimes 1)y_i$ is an unitary in $\mathcal{M}(\pi(C^*(\Lambda))\otimes \bigotimes_{j=1}^{k+1} \mathbb{K})^{\overline{\alpha}}(e_i)$ (where $\overline{\alpha} := \alpha \otimes \iota \otimes \cdots \otimes \iota$ ($k+1$ ι's) and α also denotes the extension of α to the multiplier algebra) so that

$$(C^*(\Lambda)\otimes\bigotimes_{j=1}^{k+1}\mathbb{K})^{\overline{\alpha}}(e_i)=(C^*(\Lambda)\otimes\bigotimes_{j=1}^{k+1}\mathbb{K})^{\overline{\alpha}}\,u_{e_i}$$

for $i=1,\ldots,k$.

Clearly u_{e_i} is in $\mathcal{M}(C^*(\Lambda)\otimes\bigotimes_{j=1}^{k+1}\mathbb{K})$ for all $i=1,2,\ldots,k$. To see that u_{e_i} is unitary for all $i=1,2,\ldots,k$ consider the following.

$$\begin{aligned}
v_iv_i^*\otimes 1 &= \left(\sum_{\lambda,\mu\in\Lambda^{e_i}} S_\lambda S_\mu^*\otimes e^{(i)}_{1f_i(\lambda)}\,e^{(i)}_{f_i(\mu)1}\right)\otimes 1\\
&= \left[\left(\sum_{\lambda\in\Lambda^{e_i}} P_\lambda\right)\otimes e^{(i)}_{11}\right]\otimes 1\\
&= 1\otimes e^{(i)}_{11}\otimes 1 = 1\otimes x_ix_i^*
\end{aligned}$$

and

$$\begin{aligned}
y_iy_i^* &= \sum_{\lambda,\mu\in\Lambda^{e_i}} P_\lambda P_\mu\otimes V_{i,\lambda}V^*_{i,\mu}\\
&= \sum_{\lambda\in\Lambda^{e_i}} P_\lambda\otimes V_{i,\lambda}V^*_{i,\lambda}\\
&= \sum_{\lambda\in\Lambda^{e_i}} P_\lambda\otimes \sum_{\substack{\mu\in\Lambda^{e_i}\\ s(\mu)=r(\lambda)}} e^{(i)}_{f_i(\mu)f_i(\mu)}\otimes 1\\
&= \sum_{\mu\in\Lambda^{e_i}}\left(\sum_{\lambda\in\Lambda^{e_i}(s(\mu))} P_\lambda\right)\otimes e^{(i)}_{f_i(\mu)f_i(\mu)}\otimes 1\\
&= \sum_{\mu\in\Lambda^{e_i}} S_\mu^*S_\mu\otimes e^{(i)}_{f_i(\mu)f_i(\mu)}\otimes 1\\
&= \sum_{\mu,\lambda\in\Lambda^{e_i}} S_\mu^*S_\lambda\otimes e_{f_i(\mu)1}\,e_{1\,f_i(\lambda)}\otimes 1\\
&= v_i^*v_i\otimes 1
\end{aligned}$$

Thus,

$$\begin{aligned}
u_{e_i}^* u_{e_i} &= y_i^*(v_i^* \otimes 1)(1 \otimes x_i)(1 \otimes x_i^*)(v_i \otimes 1)y_i \\
&= y_i^*(v_i^* \otimes 1)(1 \otimes x_i x_i^*)(v_i \otimes 1)y_i \\
&= y_i^*(v_i^* v_i \otimes 1)y_i \\
&= y_i^* y_i \\
&= \sum_{\lambda,\mu \in \Lambda^{e_i}} P_\lambda P_\mu \otimes V_{i,\lambda}^* V_{i,\mu} \\
&= \sum_{\lambda \in \Lambda^{e_i}} P_\lambda \otimes 1 = 1
\end{aligned}$$

and

$$\begin{aligned}
u_{e_i} u_{e_i}^* &= (1 \otimes x_i^*)(v_i \otimes 1)y_i y_i^*(v_i^* \otimes 1)(1 \otimes x_i) \\
&= (1 \otimes x_i^*)(v_i \otimes 1)(v_i^* \otimes 1)(1 \otimes x_i) \\
&= 1 \otimes x_i^* x_i = 1.
\end{aligned}$$

Clearly, $\alpha_z(u_{e_i}) = z^{e_i} u_{e_i}$ for all $z \in \mathbb{T}^k$.

In order to express the K-groups of $C^*(\Lambda)$ in terms of Λ we will construct another chain complex whose homology groups will agree with those of chain complex (5.1).

Let ϕ_1 and ϕ_2 be chosen to make the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & K_0(\mathcal{G}) & \xleftarrow{(1-(\hat{\alpha}_{e_2})_*,(\hat{\alpha}_{e_1})_*-1)} & K_0(\mathcal{G}) \oplus K_0(\mathcal{G}) & \xleftarrow{\binom{1-(\hat{\alpha}_{e_1})_*}{1-(\hat{\alpha}_{e_2})_*}} & K_0(\mathcal{G}) & \longleftarrow & 0 \\
\Big\| & & \Big\uparrow \eta & & \Big\uparrow \eta\oplus\eta & & \Big\uparrow \eta & & \Big\| \\
0 & \longleftarrow & K_0(\mathcal{F}_\Lambda) & \xleftarrow{\phi_1} & K_0(\mathcal{F}_\Lambda) \oplus K_0(\mathcal{F}_\Lambda) & \xleftarrow{\phi_2} & K_0(\mathcal{F}_\Lambda) & \longleftarrow & 0
\end{array}$$

Thus,

$$K_0(C^*(\Lambda)) = \text{coker } \phi_1 \oplus \ker\, \phi_2$$

$$K_1(C^*(\Lambda)) = \ker\, \phi_1 / \text{im } \phi_2$$

To express the K-groups in terms of Λ we need to know what $K_0(\mathcal{F}_\Lambda)$ and ϕ_1, ϕ_2 are.

Definition 5.1 *Let $\tilde{\mathbb{Z}}^{\Lambda^0} := \bigoplus_{v\in\Lambda^0} \mathbb{Z}$ which is the group of all maps from Λ^0 into $\mathbb{Z}$ that have finite support with pointwise addition.*

Recall that M_i are the vertex matrices of Λ as defined in Definition 2.3.

Lemma 5.2 (c.f. [53] Lemma 4.5 and [44] Proposition 4.1.2) *Let $A_j = \tilde{\mathbb{Z}}^{\Lambda^0}$ for all $j \in \mathbb{N}^k$ then*

$$K_0(C^*(\Lambda)^\alpha) \cong \varinjlim(A_j, \jmath_{kj})$$

where $\jmath_{j+e_i,j} : A_j \longrightarrow A_{j+e_i}$ is given by M_i^t.

Proof. Recall from [35, Lemma 3.2 and Lemma 3.3] that $C^*(\Lambda)^\alpha = \mathcal{F}_\Lambda = \overline{\bigcup_{n\in\mathbb{N}^k} \mathcal{F}_n}$ where $\mathcal{F}_n = \bigoplus_{v\in\Lambda^0} \mathcal{F}_n(v)$ with $\mathcal{F}_n(v) := \overline{\operatorname{span}}\{s_\lambda s_\mu^* \,|\, \lambda, \mu \in \Lambda^n,\ s(\lambda) = s(\mu) = v\} \cong \mathbb{K}(\ell^2(\lambda \in \Lambda \,|\, s(\lambda) = v))$. By for example [61, Proposition 6.2.9], $K_0(C^*(\Lambda)^\alpha) \cong \lim_\rightarrow(K_0(\mathcal{F}_n), (i_{nm})_*)$, where $i_{nm} : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ is the inclusion map for $m, n \in \mathbb{N}^k,\ n \geq m$. For $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, $K_0(\mathcal{F}_n(v))$ is generated by any minimal projection in $\mathcal{F}_n(v)$. Therefore, to get a set of generators for $K_0(F_n)$ it suffices to write down a minimal projection from each $\mathcal{F}_n(v)$. So choose any $\mu(v) \in \Lambda^n$ such that $s(\mu(v)) = v$ and take $[p_{\mu(v)}]_n$ to be the generator of $K_0(\mathcal{F}_n(v))$, where $[\,\cdot\,]_n$ denotes the equivalence classes of $K_0(\mathcal{F}_n)$ for $n \in \mathbb{N}^k$. Thus the map $\theta : \tilde{\mathbb{Z}}^{\Lambda^0} \longrightarrow K_0(\mathcal{F}_n)$ given by

$$f \mapsto \sum_{u\in\Lambda^0} f(u)[p_{\mu(u)}]_n$$

is an isomorphism. We will now compute the maps $(i_{nm})_* : K_0(\mathcal{F}_m) \longrightarrow K_0(\mathcal{F}_n)$. The embedding $i_{nm} : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ is given by

$$i_{nm}(s_\lambda s_\mu^*) = \sum_{\alpha\in\Lambda^{n-m}(s(\lambda))} s_{\lambda\alpha} s_{\mu\alpha}^*$$

for all $\lambda, \mu \in \Lambda^m$. Therefore,

$$\begin{aligned}(i_{m+e_im})_*\left([p_{\mu(v)}]_m\right) &= \left[i_{m+e_im}\left(s_{\mu(v)}s^*_{\mu(v)}\right)\right]_{m+e_i} \\ &= \left[\sum_{\alpha\in\Lambda^{e_i}(v)} s_{\mu(v)\alpha}s^*_{\mu(v)\alpha}\right]_{m+e_i} \\ &= \sum_{u\in\Lambda^0} M_i(v,u)[p_\mu(u)]_{m+e_i}\end{aligned}$$

and

$$\sum_{v\in\Lambda^0} f(v)[p_{\mu(v)}]_m \mapsto \sum_{u\in\Lambda^0}\left(\sum_{v\in\Lambda^0} M_i(v,u)f(v)\right)[p_{\mu(u)}]_{m+e_i} = \sum_{u\in\Lambda^0}(M_i^t f)(u)[p_{\mu(u)}]_{m+e_i}.$$

Thus the following squares commute for all $i \in \{1, 2, \dots, k\}$.

$$\begin{array}{ccc} K_0(\mathcal{F}_n) & \xrightarrow{\left(i_{n+e_i,n}\right)_*} & K_0(\mathcal{F}_{n+e_i}) \\ \theta\big\uparrow & & \big\uparrow\theta \\ \tilde{\mathbb{Z}}^{\Lambda^0} & \xrightarrow{M_i^t} & \tilde{\mathbb{Z}}^{\Lambda^0} \end{array}$$

The result follows. □

Let $\psi_i := \eta^{-1}(\hat{\alpha}_{e_i})_*\eta$, $i = 1, \dots k$ and decompose the isomorphism $\eta : K_0(\mathcal{F}_\Lambda) \longrightarrow K_0(\mathcal{G})$ according to the diagram below.

$$\begin{array}{ccc} K_0(\mathcal{G}) & \xrightarrow{\left(\hat{\alpha}_{e_i}\right)_*} & K_0(\mathcal{G}) \\ \otimes e_*\big\downarrow & & \big\downarrow\otimes e_* \\ K_0(\mathcal{G}\otimes\mathbb{K}) & \xrightarrow{\left(\hat{\alpha}_{e_i}\otimes \mathrm{id}\right)_*} & K_0(\mathcal{G}\otimes\mathbb{K}) \\ \sigma_*\big\uparrow & & \big\uparrow\sigma_* \\ K_0((C^*(\Lambda)\otimes\mathbb{K})\rtimes_{\bar{\alpha}}\mathbb{T}^k) & \xrightarrow{\left(\widehat{\alpha}_{e_i}\right)_*} & K_0((C^*(\Lambda)\otimes\mathbb{K})\rtimes_{\bar{\alpha}}\mathbb{T}^k) \\ \kappa_*\big\uparrow & & \big\uparrow\kappa_* \\ K_0(\mathcal{F}_\Lambda\otimes\mathbb{K}\otimes\mathbb{K}(L^2(\mathbb{T}^k))) & & K_0(\mathcal{F}_\Lambda\otimes\mathbb{K}\otimes\mathbb{K}(L^2(\mathbb{T}^k))) \\ \big\uparrow\otimes^2 e_* & & \big\uparrow\otimes^2 e_* \\ K_0(\mathcal{F}_\Lambda) & \xrightarrow{\psi_i} & K_0(\mathcal{F}_\Lambda) \end{array}$$

The isomorphisms $\otimes e_*, \otimes^2 e_*$ are induced by the homomorphism $a \mapsto a \otimes e$ where e is any rank 1 projection. The isomorphism σ_* is induced by the isomorphism of $(C^*(\Lambda) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}} \mathbb{T}^k$ onto $\mathcal{G} \otimes \mathbb{K}$ and κ is the Kishimoto-Takai isomorphism. Following, for example [48, Proposition 5], the dual action of α is denoted by $\hat{\alpha}$ so that $\hat{\alpha}_{e_i}$ is an automorphism of $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k$ for all $i = 1, 2, \ldots, k$. We denote by $\bar{\alpha}$ the product action $\alpha \otimes \iota$ of $\mathbb{T}^k$ on $C^*(\Lambda) \otimes \mathbb{K}$ and $\widehat{\bar{\alpha}}$ its dual action, where ι is the trivial action of $\mathbb{T}^k$ on $\mathbb{K}$. Reading from top to bottom; the commutativity of the first square is clear, the second square commutes because of the functorality of K_0 and therefore it follows by the definition of ψ_i that the third square commutes.

Lemma 5.3 (c.f. [53] Lemma 4.10)

$$\varinjlim (K_0(\mathcal{F}_n) \xrightarrow{M_i^t} K_0(\mathcal{F}_n)) = K_0(\mathcal{F}_\Lambda) \xrightarrow{\psi_i} K_0(\mathcal{F}_\Lambda)$$

Proof. First we should check that the direct limit of maps is well defined by proving that

$$\begin{array}{ccc} K_0(\mathcal{F}_m) & \xrightarrow{(M_1,\ldots,M_k)^{n-m}} & K_0(\mathcal{F}_n) \\ {\scriptstyle M_i^t}\downarrow & & \downarrow{\scriptstyle M_i^t} \\ K_0(\mathcal{F}_m) & \xrightarrow{(M_1,\ldots,M_k)^{n-m}} & K_0(\mathcal{F}_n) \end{array}$$

is commutative for all $i = 1, 2, \ldots, k$ and $m, n \in \mathbb{N}^k$ such that $n \geq m$. In fact we need only prove the cases $n = m + e_j$ for all $j = 1, 2, \ldots, k$. But this is obvious since M_i and M_j commute for all $i, j = 1, 2, \ldots, k$ by [35]. Finally we must prove that

$$\begin{array}{ccc} K_0(\mathcal{F}_n) & \xrightarrow{(\iota_n)_*} & K_0(\mathcal{F}_\Lambda) \\ {\scriptstyle M_i^t}\downarrow & & \downarrow{\scriptstyle \psi_i} \\ K_0(\mathcal{F}_n) & \xrightarrow{(\iota_n)_*} & K_0(\mathcal{F}_\Lambda) \end{array}$$

commutes for all $i = 1, 2, \ldots, k$ and $n \in \mathbb{N}^k$. In fact, since $K_0(\mathcal{F}_n)$ is generated by $[p_{\mu(u)}]_n$, where for each $u \in \Lambda^0$ we fix a degree n path, $\mu(u)$ with source u, we need only apply our homomorphisms to these generators.

Now $(\iota_n)_*([p_{\mu(u)}]_n) = [p_{\mu(u)}]$. We may choose the rank 1 projection $e = e_{11} \otimes \lambda(\boldsymbol{z}^q) \in \mathbb{K} \otimes \mathbb{K}(L^2(\mathbb{T}^k))$ to induce the isomorphism $\otimes^2 e_*$, where $\boldsymbol{z}^q : \mathbb{T}^k \longrightarrow \mathbb{T}$ denotes the function $z \mapsto z^q$ for $q \in \mathbb{Z}^k$ and $\lambda(\chi) = \int_{\mathbb{T}^k} \chi(z)\lambda_z \, dz$, λ being the left regular representation of $\mathbb{T}^k$ on $L^2(\mathbb{T}^k)$. Thus we have

$$\otimes^2 e_*([p_{\mu(u)}]) = [p_\mu(u) \otimes e_{11} \otimes \lambda(\boldsymbol{z}^q)]$$

By the proof of [44, Theorem 3.2.2] the Kishimoto-Takai isomorphism κ has the following effect on the projections.

$$\kappa(p_{\mu(u)} \otimes e_{11} \otimes \lambda(\boldsymbol{z}^q)) = i_{C^*(\Lambda)\otimes\mathbb{K}}(u_q^*(p_{\mu(u)} \otimes e_{11})u_q) i_{\mathbb{T}^k}(\boldsymbol{z}^{-q})$$

where u_q are unitaries implementing the grading and we are using the notation of [48]. (Recall from [48] that

$$(C^*(\Lambda) \otimes \mathbb{K}) \rtimes_\alpha \mathbb{T}^k = \overline{\text{span}}\{i_{C^*(\Lambda)\otimes\mathbb{K}}(x) i_{\mathbb{T}^k}(\boldsymbol{z}) \,|\, x \in C^*(\Lambda) \otimes \mathbb{K}, \ \boldsymbol{z} \in C_c(\mathbb{T}^k)\}.)$$

Taking $q = 0$ with $u_0 = 1$ we have

$$\begin{aligned} \kappa_* \otimes^2 \ e_*([p_{\mu(u)}]) &= \kappa_*([p_{\mu(u)} \otimes e_{11} \otimes \lambda(\boldsymbol{z}^0)]) \\ &= [i_{C^*(\Lambda)\otimes\mathbb{K}}(p_{\mu(u)} \otimes e_{11}) i_{\mathbb{T}^k}(\boldsymbol{z}^{\mathbf{0}})]. \end{aligned}$$

Now by, for example [48, Proposition 5]

$$\begin{aligned} \widehat{\widehat{\alpha}}_{e_i}(i_{C^*(\Lambda)\otimes\mathbb{K}}(p_{\mu(u)} \otimes e_{11}) i_{\mathbb{T}^k}(\boldsymbol{z}^0)) &= i_{C^*(\Lambda)\otimes\mathbb{K}}(p_{\mu(u)} \otimes e_{11}) i_{\mathbb{T}^k}(\boldsymbol{z}^{e_i}\boldsymbol{z}^0) \\ &= i_{C^*(\Lambda)\otimes\mathbb{K}}(p_{\mu(u)} \otimes e_{11}) i_{\mathbb{T}^k}(\boldsymbol{z}^{e_i}) \end{aligned}$$

Notice that $u_{-e_i} := u_{e_i}^*$ implements the $-e_i$ grading of $C^*(\Lambda) \otimes \mathbb{K}$ (by identifying $\bigotimes_{j=1}^{k+1} \mathbb{K}$ with $\mathbb{K}$) therefore

$$\begin{aligned}
\kappa(u_{e_i}^*(p_{\mu(u)} \otimes e_{11})u_{e_i} \otimes \lambda(\boldsymbol{z}^{-e_i})) &= i_{C^*(\Lambda)\otimes\mathbb{K}}(u_{-e_i}^* u_{e_i}^*(p_{\mu(u)} \otimes e_{11})u_{e_i}u_{-e_i})i_{\mathbb{T}^k}(\boldsymbol{z}^{-e_i}) \\
&= i_{C^*(\Lambda)\otimes\mathbb{K}}(p_{\mu(u)} \otimes e_{11})i_{\mathbb{T}^k}(\boldsymbol{z}^{-e_i})
\end{aligned}$$

Consequently,

$$\kappa_*^{-1}(\hat{\alpha}_{e_i})_*\kappa_* \otimes^2 e_*([p_{\mu(u)}]) = [u_{e_i}^*(p_{\mu(u)} \otimes e_{11})u_{e_i} \otimes \lambda(\boldsymbol{z}^{e_i})]$$

In order to calculate $u_{e_i}^*(p_{\mu(u)} \otimes e_{11})u_{e_i}$ we must expand $\mathbb{K}$ to $\bigotimes_{j=1}^{k+1} \mathbb{K}$ by identifying e_{11} with $e_{\mathbf{11}} \otimes e_{11}$ where $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^k$. To ease the calculation, we make the assumption that $d(\mu(u)) \geq 2e_i$, which we are able to do because $K_0(\mathcal{F}_\Lambda) = \bigcup_{\substack{n\in\mathbb{N}^k \\ n\geq 2e_i}} (\iota_n)_*\mathcal{F}_n$.

Now

$$\begin{aligned}
u_{e_i}^*(p_{\mu(u)} \otimes e_{11} \otimes e_{\mathbf{11}})u_{e_i} &= y_i^*(v_i^* \otimes 1)(p_{\mu(u)} \otimes e_{\mathbf{11}} \otimes e_{11})(v_i \otimes 1)y_i \\
&\qquad (\text{since } x_i e_{\mathbf{11}} x_i^* = e_{\mathbf{11}}) \\
&= y_i^*(p_{\mu'(u)} \otimes e^{(i)}_{f_i(\mu_1)f_i(\mu_1)} \otimes e_{11})y_i \\
&\qquad (\text{where } \mu(u) = \mu_1\mu'(u) \text{ with } d(\mu_1) = e_i) \\
&= \sum_{\nu,\xi\in\Lambda^{e_i}} p_\nu p_{\mu'(u)} p_\xi \otimes V_{i,\nu}^*(e^{(i)}_{f_i(\mu_1)f_i(\mu_1)} \otimes e_{11})V_{i,\xi} \\
&= p_{\mu'(u)} \otimes V_{i,\mu_2}^*(e^{(i)}_{f_i(\mu_1)f_i(\mu_1)} \otimes e_{11})V_{i,\mu_2} \\
&\qquad (\text{where } \mu'(u) = \mu_2\mu''(u) \text{ with } d(\mu_2) = e_i)
\end{aligned}$$

Now $V_{i,\mu_2}^*(e^{(i)}_{f_i(\mu_1)f_i(\mu_1)} \otimes e_{11})V_{i,\mu_2}$ is a rank 1 projection in $\bigotimes_{j=1}^{k+1} \mathbb{K}$ which we will denote by f. Under the identification $\bigotimes_{j=1}^{k+1} \mathbb{K} \longrightarrow \mathbb{K}$ the rank 1 projection

f is sent to a rank 1 projection f' in $\mathbb{K}$. Thus

$$\begin{aligned}
[u_{e_i}^*(p_{\mu'(u)} \otimes e_{11})u_{e_i} \otimes \lambda(\boldsymbol{z}^{-e_i})] &= [p_{\mu'(u)} \otimes f' \otimes \lambda(\boldsymbol{z}^{-e_i})] \\
&= [p_{\mu'(u)} \otimes e_{11} \otimes \lambda(\boldsymbol{z}^{-e_i})] \\
&= [\sum_{\beta \in \Lambda^{e_i}} p_{\mu'(u)\beta} \otimes e_{11} \otimes \lambda(\boldsymbol{z}^{-e_i})] \\
&= \sum_{v \in \Lambda^0} M_i(u,v)[p_{\mu(v)} \otimes e_{11} \otimes \lambda(\boldsymbol{z}^{-e_i})]
\end{aligned}$$

since, if $\beta \in \Lambda^{e_i}$ with $s(\beta) = v$ then $s_{\mu'(u)\beta}s_{\mu(v)}^*$ is a partial isometry (in $\mathcal{F}_n$) implementing the equivalence of $p_{\mu'(u)\beta}$ and $p_{\mu(v)}$. Hence,

$$\psi_i((\iota_n)_*([p_{\mu(u)}]_n)) = \sum_{v \in \Lambda^0} M_i(u,v)[p_{\mu(v)}]$$

for all $i = 1, 2, \ldots, k$ since $\otimes^2 e_*$ is independent of the choice of rank 1 projection.

But

$$(\iota_n)_*(M_i^t[p_{\mu(u)}]_n)) = \sum_{v \in \Lambda^0} M_i^t(v,u)[p_{\mu(v)}] = \psi_i((\iota_n)_*([p_{\mu(u)}]_n))$$

hence, $\psi_i(\iota_n)_* = (\iota_n)_* M_i^t$ for all $i = 1, 2, \ldots, k,\ n \in \mathbb{N}^k$ such that $n \geq 2e_i$. □

Lemma 5.4 (c.f. [53, Lemma 4.9]) *For $i = 1, 2$, the map induced on the following complex by M_i^t acts as the identity on the homology groups.*

$$0 \longleftarrow \tilde{\mathbb{Z}}^{\Lambda^0} \xleftarrow{(1-M_2^t, M_1^t - 1)} \tilde{\mathbb{Z}}^{\Lambda^0} \oplus \tilde{\mathbb{Z}}^{\Lambda^0} \xleftarrow{\binom{1-M_1^t}{1-M_2^t}} \tilde{\mathbb{Z}}^{\Lambda^0} \longleftarrow 0$$

Proof. Virtually identical to the proof of [53, Lemma 4.9]. □

As in [53] it follows that

$$\mathrm{Hom}(0 \longleftarrow K_0(\mathcal{F}_\Lambda) \xleftarrow{\phi_1} K_0(\mathcal{F}_\Lambda) \oplus K_0(\mathcal{F}_\Lambda) \xleftarrow{\phi_2} K_0(\mathcal{F}_\Lambda) \longleftarrow 0)$$

$$= \mathrm{Hom}(0 \longleftarrow \tilde{\mathbb{Z}}^{\Lambda^0} \xleftarrow{(1-M_2^t, M_1^t-1)} \tilde{\mathbb{Z}}^{\Lambda^0} \oplus \tilde{\mathbb{Z}}^{\Lambda^0} \xleftarrow{\binom{1-M_1^t}{1-M_2^t}} \tilde{\mathbb{Z}}^{\Lambda^0} \longleftarrow 0)$$

From this we see that $\mathcal{H}_2$ is free abelian so we have proved the following Proposition.

Proposition 5.1 *If Λ is a row-finite 2-graph with no sinks nor sources, with vertex matrices M_1, M_2 then*

$$\begin{aligned} K_0(C^*(\Lambda)) &= \mathrm{coker}(1-M_2^t, M_1^t-1) \oplus \ker \begin{pmatrix} 1-M_1^t \\ 1-M_2^t \end{pmatrix} \\ K_1(C^*(\Lambda)) &= \ker(1-M_2^t, M_1^t-1)/\,\mathrm{im} \begin{pmatrix} 1-M_1^t \\ 1-M_2^t \end{pmatrix} \end{aligned}$$

where $(1-M_2^t, M_1^t-1) : \tilde{\mathbb{Z}}^{\Lambda^0} \oplus \tilde{\mathbb{Z}}^{\Lambda^0} \longrightarrow \tilde{\mathbb{Z}}^{\Lambda^0}$ *and* $\begin{pmatrix} 1-M_1^t \\ 1-M_2^t \end{pmatrix} : \tilde{\mathbb{Z}}^{\Lambda^0} \longrightarrow \tilde{\mathbb{Z}}^{\Lambda^0} \oplus \tilde{\mathbb{Z}}^{\Lambda^0}$ *(see Definition 5.1).*

Note that the K-theory of $C^*(\Lambda)$ depends soley on the vertex matrices of Λ so that we have the following Corollary.

Corollary 5.1 *If Λ and Δ are row-finite 2-graphs with no sinks nor sources and with the same skeleton then*

$$K_i(C^*(\Lambda)) = K_i(C^*(\Delta)) \quad \textit{for } i = 1, 2.$$

Definition 5.2 *A k-graph Λ is said to satisfy condition* **(P)** *if for every $v \in \Lambda^0$ there are $\lambda, \mu \in \Lambda$ with $d(\mu) \neq 0$ such that $r(\lambda) = v$ and $s(\lambda) = r(\mu) = s(\mu)$.*

Corollary 5.2 *Suppose that Λ and Δ are both row-finite, cofinal 2-graphs satisfying* **(A)**, *with no sinks nor sources and with the same skeleton. Furthermore suppose that Λ and Δ satisfy condition* **(P)**. *Then $C^*(\Lambda) \cong C^*(\Delta)$.*

Proof. It is clear that k-graph C^*-algebras are always separable. If the above hypotheses above hold then both $C^*(\Lambda)$ and $C^*(\Delta)$ are simple (by [35, Proposition 4.8]), purely infinite (by [35, Proposition 4.9]), nuclear and satisfies the Universal Coefficient Theorem ([46, Definition 4.2.3]) (by [35, Theorem 5.5] and [56, Theorem 1.17]). Moreover, since Λ and Δ have the same skeleton, there is a graded isomorphism $K_*(C^*(\Lambda)) \to K_*(C^*(\Delta))$ by Corollary 5.1. Therefore, by the Kirchberg-Phillips classification theorem (e.g. [46, Theorem 4.2.4]) $C^*(\Lambda) \cong C^*(\Delta)$. □

Remark 5.1 *If condition* **(P)** *(respectively cofinality) is satisfied by a k-graph Λ then condtion* **(P)** *(respectively cofinality) is satisfied by any other k-graph with the same skeleton as Λ.*

Appendix A

Alternative Cuntz-Pimsner *-representation

Here we give the details of an alternative *-representation of Λ in a Cuntz-Pimsner algebra. Unlike the construction in Section 2.2, we impose no restriction on the k-graph Λ apart from insisting the usual row-finiteness and no sources constraints. The construction was inspired by [59].

Let (Λ, d) be a k-graph and $\{s_\lambda \mid \lambda \in \Lambda\}$ be the canonical generating partial isometries of $C^*(\Lambda)$. Define:

$$\Sigma := \bigcup_{i=1}^{k} \Lambda^{e_i}, \quad A := C^*(s_\lambda^* s_\mu \mid \lambda, \mu \in \Sigma) \subset C^*(\Lambda)$$

$$X := \overline{\operatorname{span}}\{s_\alpha a \mid a \in A, \alpha \in \Sigma\} \subset C^*(\Lambda).$$

Then X is a A-bimodule with both actions given by multiplication in $C^*(\Lambda)$.

The right action is obviously well defined.

The left action is well defined because for $\lambda, \mu, \gamma \in \Sigma$

$$s_\lambda^* s_\mu s_\gamma = \sum_{\substack{\lambda\alpha=\mu\beta \\ d(\lambda\alpha)=e_i+e_j}} s_\alpha s_\beta^* s_\gamma \in X$$

as $\alpha, \beta, \gamma \in \Sigma,\ s_\beta^* s_\gamma \in A$

Therefore $a_0 x \in X$ for all $a_0 \in A_0, x \in X$ where A_0 is the dense *-subalgebra of A generated algebraically by $\{s_\lambda^* s_\mu \mid \lambda, \mu \in \Sigma\}$. Hence $ax \in X$ for all $a \in A$.

Furthermore, one can define a A valued inner product $\langle \cdot, \cdot \rangle$ on X by $\langle x, y \rangle = x^* y$ which makes X into a Hilbert A-bimodule.

This inner product is well defined since:

$$\langle s_\alpha a, s_\beta b \rangle = a^* s_\alpha^* s_\beta b \in A \quad \forall\, \alpha, \beta \in \Sigma,\ a, b \in A.$$

It is easy to check that X is full, i.e. $A = \overline{\text{span}}\{\langle x, y \rangle \mid x, y \in X\}$

Let $B_0 := \text{span}\{\langle x, y \rangle \mid x, y \in X\}$. Then given a monomial $w = \prod_{i=1}^n s_{\lambda_i}^* s_{\mu_i} \in A_0$ it is clear that $w = \langle s_{\lambda_1}, s_{\lambda_1} w \rangle$. It follows that $A_0 = B_0$.

The left action is clearly given by adjointable operators on X therefore the left action on X defines a homomorphism $\phi : A \longrightarrow \mathbb{B}(X)$ where $\mathbb{B}(X) :=$ the adjointable operators on X.

Actually ϕ maps into the compact adjointable operators on X, $\mathbb{K}(X) := \overline{\text{span}}\{\theta_{x,y} \mid x, y \in X\}$, where $\theta_{x,y}(z) = x\langle y, z \rangle$ for all $z \in X$.

For $\lambda, \mu \in \Sigma$ we have

$$\phi(s_\lambda^* s_\mu)x = s_\lambda^* s_\mu x = \sum s_\alpha s_\beta^* x = \sum s_\alpha \langle s_\beta, x \rangle = \sum \theta_{s_\alpha, s_\beta}(x) \in \mathbb{K}(X)$$

for all $x \in X$. So $\phi(s_\lambda^* s_\mu) = \sum \theta_{s_\alpha, s_\beta}$ where all the sums are over $\mathcal{E}(\lambda, \mu)$. Thus ϕ sends the generators of A into $\mathbb{K}(X)$ and it follows that $\phi(A) \subset \mathbb{K}(X)$.

We will now show that ϕ is injective.

Definition A.1 *For $V \subset \Lambda^0,\ v \in \Lambda^0$ and $m \in \mathbb{N}^k$ define*
$\Lambda^m(V) := \{\alpha \in \Lambda^m \mid r(\alpha) \in V\}$ *and* $\Lambda^m(V,v) := \{\alpha \in \Lambda^m(V) \mid s(\alpha) = v\}$*.*
Fix an enumeration of $\Lambda^0 = \{v_i\}_{i=1}^{\infty}$ and set $V_m^j := s\left(\Lambda^m(\{v_i\}_{i=1}^j)\right)$ where $j = 1, 2, \ldots$.

Define the following subset of A to be the set of homogeneous monomials of degree m

$$W_m^j := \{w \mid w = \prod_{i=1}^{n} s_{\lambda_i}^* s_{\mu_i} = \sum s_\alpha s_\beta^* \quad n = 1, 2, \ldots\ \lambda_i, \mu_i \in \Sigma,\ \alpha, \beta \in \Lambda^m(\{v_i\}_{i=1}^j)\}.$$

For $v \in V_m^j$, define

$$\mathcal{F}_m^j(v) := span\{s_\alpha s_\beta^* \mid \alpha, \beta \in \Lambda^m(\{v_i\}_{i=1}^j, v)\}$$

$$\mathcal{F}_m^j := span\{s_\alpha s_\beta^* \mid \alpha, \beta \in \Lambda^m(\{v_i\}_{i=1}^j)\}$$

$$\mathcal{G}_m^j := span\{W_m^j\} \quad \mathcal{G}_m := \overline{\bigcup_{j=1}^{\infty} \mathcal{G}_m^j} \quad \mathcal{G} := \overline{\bigcup_{m \in \mathbb{N}^k} \mathcal{G}_m}.$$

Remark A.1 *It is assumed that Λ is row finite therefore $\Lambda^m(\{v_i\}_{i=1}^j)$ is finite for all $m \in \mathbb{N}^k,\ j = 1, 2, \ldots$*

Lemma A.1 *For $m \in \mathbb{N}^k,\ j \in \mathbb{N}\backslash\{0\}$ and $v \in V_m^j$*

(1) $\mathcal{F}_m^j(v)$ *is isomorphic to a full matrix algebra. In fact*

$$\mathcal{F}_m^j(v) \cong M_{|\Lambda^m(\{v_i\}_{i=1}^j, v)|}$$

(2)

$$\mathcal{F}_m^j = \bigoplus_{v \in V_m^j} \mathcal{F}_m^j(v)$$

Proof (1) is a direct generalisation of [1, Lemma 2.1.7].

(2) if $\alpha, \beta \in \Lambda^m(\{v_i\}_{i=1}^j, v_1)$ and $\gamma, \delta \in \Lambda^m(\{v_i\}_{i=1}^j, v_2)$ for distinct $v_1, v_2 \in \Lambda^0$ then $\beta \neq \gamma$ and the matrix units $s_\alpha s_\beta^*,\ s_\gamma s_\delta^*$ are orthogonal.

Corollary A.1 *For $m \in \mathbb{N}^k,\ j \in \mathbb{N}\backslash\{0\},\ \mathcal{G}_m^j$ is a finite dimensional C^*-algebra. Hence $\mathcal{G}_m$ is an AF-algebra and so is $\mathcal{G}$.*

Proof Clearly W_m^j is closed under multiplication and $\mathcal{G}_m^j$ is a *-subalgebra of $\mathcal{F}_m^j$. Therefore as $\mathcal{G}_m$ is the closure of an increasing union of finite dimensional algebras, it is AF. For the last assertion it suffices to show that $\mathcal{G}_m \subset \mathcal{G}_n$ if $n \geq m$. This follows from

$$\mathcal{G}_m \ni \sum s_\alpha s_\beta = \sum_{\gamma \in \Lambda^(n-m)(s(\alpha)} s_{\alpha\gamma} s_{\beta\gamma}^* \in \mathcal{G}_n\,. \qquad \square$$

Let $\alpha : \mathbb{T}^k \longrightarrow \mathrm{Aut}(C^*(\Lambda))$ be the gauge action on $C^*(\Lambda)$ $(\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for all $\lambda \in \Lambda,\ z \in \mathbb{T}^k)$ and $\Phi : C^*(\Lambda) \longrightarrow C^*(\Lambda)^\alpha$ the faithful conditional expectation onto $C^*(\Lambda)^\alpha$ defined by

$$\Phi(a) = \int_{\mathbb{T}^k} \alpha_z(a)\, dz$$

where dz is the normalised Haar measure on $\mathbb{T}^k$. Also define

$$A^\alpha := \{a \in A \mid \alpha_z(a) = a\} = A \cap C^*(\Lambda)^\alpha$$

Then $\Phi(A) = A^\alpha$ (because $\alpha_z(A) = A$ for all $z \in \mathbb{T}^k$).

Lemma A.2 *We have the following identification of the fixed point algebra A^α with the AF-core of A.*

$$\begin{aligned}\mathcal{G} &= \overline{span}\{w = \prod_{i=1}^n s_{\lambda_i}^* s_{\mu_i} \mid n \in \mathbb{N}\backslash\{0\},\ \lambda_i, \mu_i \in \Sigma,\ \sum_{i=1}^n d(\lambda_i) = \sum_{i=1}^n d(\mu_i)\} \\ &= A^\alpha\end{aligned}$$

Proof Define $W := \{w \mid w = \prod_{i=1}^{l} s_{\lambda_i}^* s_{\mu_i} = \sum s_\alpha s_\beta^*\}$, $B_0 := \text{span}(W)$ and $B = \overline{B_0}$. Then a typical element b of B_0 is of the form $b = \sum_{r=1}^{l} b_r w_r$ where $b_r \in \mathbb{C}$ and $w_r \in W$ for $r = 1, 2, \ldots l$, l some posive integer. In particular $w_r = \sum_{(\alpha,\beta)\in S_r} s_{\alpha_r} s_{\beta_r}^*$ for some $S_r \subset \Lambda \times \Lambda$ with $d(\alpha_r) = d(\beta_r) =: d_r$ for all $(\alpha, \beta) \in S_r$, $r = 1, 2, \ldots, l$. Now choose $m \in \mathbb{N}^k$ and $j \in \mathbb{N}\backslash\{0\}$ so that $d_r \leq m$ for all $r = 1, 2, \ldots, l$ and $\{r(\alpha_r) \mid (\alpha_r, \beta_r) \in S_r,\ r = 1, 2, \ldots, l\} \subset V_m^j$. Then $b \in \mathcal{G}_m^j \subset \mathcal{G}$. It follows that $B \subset \mathcal{G}$.

Conversely, $\mathcal{G}_m^j \subset B$ for all $j \in \mathbb{N}\backslash\{0\}$, $m \in \mathbb{N}^k$ because $W_m^j \subset W$. It follows that $\mathcal{G} \subset B$. Hence $\mathcal{G} = B$.

To show that $B = A^\alpha$ pick a monomial $w \in W$, then $\alpha_z(w) = w$ because $w = \sum s_\alpha s_\beta^*$ for some $\alpha, \beta \in \Lambda^m$ some $m \in \mathbb{N}^k$. It follows from linearity and strong continuity of α that $B \subset A^\alpha$.

Conversely, note that for $b \in A^\alpha$, $b = \Phi(a)$ for some $a \in A$. Now given $\lambda_i, \mu_i \in \Sigma$ let $w := \prod_{i=1}^{n} s_{\lambda_i}^* s_{\mu_i} = \sum_{(\alpha,\beta)\in S} s_\alpha s_\beta^*$ some finite $S \subset \Lambda^n \times \Lambda^m$. Therefore $\Phi(w) = \delta_{nm} w$ and $\Phi(w) \in B$. Now as A is the closure of the span of monomials of the same form as w it follows by the continuity and linearity of Φ that $\Phi(A) \subset B$. Hence $A^\alpha = B$. $\square$

Lemma A.3 *For all $z \in \mathbb{T}^k$, we may define automorphisms, β_z, of $\phi(A)$ by $\beta_z \phi = \phi \alpha_z$ which in turn define an action, β, of $\mathbb{T}^k$ on $\phi(A)$.*

Proof First we must check that the maps β_z are well-defined for all $z \in \mathbb{T}^k$. To see this we note that,

$$\phi(a) = 0 \Rightarrow ax = 0\ \forall x \in X \Rightarrow \alpha_z(a)x = 0\ \forall x \in X \Rightarrow \phi(\alpha(a)) = 0.$$

Therefore, if $\phi(a) = \phi(b)$ then $\phi(a-b) = 0 \Rightarrow \phi(\alpha_z(a-b)) = 0 \Rightarrow \phi(\alpha_z(a)) = \phi(\alpha_z(b))$. Thus if $\phi(a) = \phi(b)$ for some $a, b \in A$ then $\beta_z(\phi(a)) = \beta_z(\phi(b))$ for all $z \in \mathbb{T}^k$.

Its clear that (since $\alpha_z(A) = A$ for all $z \in \mathbb{T}^k$) β_z is a surjective *-homomorphism for all $z \in \mathbb{T}^k$ and a routine check shows that $\beta : \mathbb{T}^k \longrightarrow \mathrm{End}(\phi(A))$ is a homomorphism. Therefore, it follows that β_z is also injective for all $z \in \mathbb{T}^k$. Thus β is indeed an action of $\mathbb{T}^k$ on $\phi(A)$ as claimed. $\square$

The injectivity of the left action can now be proved (c.f. [1]).

Lemma A.4 *The left action on X is faithful i.e. $\phi : A \longrightarrow \mathbb{B}(X)$ is injective.*

Proof First we claim that ϕ is faithful on A^α. Now by Lemma A.2 $A^\alpha = \overline{\bigcup_{m\in\mathbb{N}^k} \mathcal{G}_m}$ therefore by [43, Lemma 1.3]

$$\ker\phi_{|_{A^\alpha}} = \overline{\bigcup_{m\in\mathbb{N}^k} \ker\phi_{|_{A^\alpha}} \cap \mathcal{G}_m}.$$

Thus it will be enough to show that $\ker\phi_{|_{A^\alpha}} \cap \mathcal{G}_m = \{0\}$ for all $m \in \mathbb{N}^k$. Moreover $\mathcal{G}_m = \overline{\bigcup_{j=1}^\infty \mathcal{G}_m^j}$ so a similar argument to the above shows that it will be enough to show that

$$\ker\phi_{|_{\mathcal{G}_m^j}} = \ker\phi_{|_{\mathcal{G}_m}} \cap \mathcal{G}_m^j = \{0\}$$

for all $m \in \mathbb{N}^k,\ j = 1, 2, \ldots$.

Let $t = \sum_i c_i w_i \in \mathcal{G}_m^j$ where $c_i \in \mathbb{C},\ w_i \in W_m^j$. Then

$$\begin{aligned} t &= \sum_i \sum_{(\alpha_i,\beta_i)\in S_i} c_i\, s_{\alpha_i} s_{\beta_i}^* \quad \text{for some } S_i \subset \Lambda^m \times \Lambda^m \\ &= \sum_{(\alpha,\beta)\in S} \left(\sum_{i_{(\alpha,\beta)}} c_{i_{(\alpha,\beta)}} \right) s_\alpha s_\beta^* \quad \text{for some } S \subset \Lambda^m \times \Lambda^m \end{aligned}$$

For each $(\alpha_0, \beta_0) \in S$ write $\alpha_0 = \alpha_0^1 \alpha_0^2 \cdots \alpha_0^n$ and $\beta_0 = \beta_0^1 \beta_0^2 \cdots \beta_0^n$ where $\alpha_0^l, \beta_0^l \in \Sigma$ for $l = 1, 2, \ldots, n$ and $n = \sum_{i=1}^k m_i$ ($m = (m_1, m_2, \ldots, m_k)$).

Then

$$\begin{aligned}\left(\sum_{i_{(\alpha_0,\beta_0)}} c_{i_{(\alpha_0,\beta_0)}}\right) p_{s(\alpha)} &= s^*_{\alpha_0}\left[\sum_{(\alpha,\beta)\in S}\left(\sum_{i_{(\alpha,\beta)}} c_{i_{(\alpha,\beta)}}\right) s_\alpha s^*_\beta\right] s_{\beta_0} \\ &= \left\langle s_{\alpha_0^n}\left\langle s_{\alpha_0^{n-1}},\cdots,\left\langle s_{\alpha_0^1}, ts_{\beta_0^1}\right\rangle s_{\beta_0^2}\right\rangle\cdots\right\rangle s_{\beta_0^n}\right\rangle\end{aligned}$$

Therefore if $\phi(t) = 0$ then $tx = 0$ for all $x \in X$ so that $ts_{\beta_0^1} = 0$ in particular, and

$$\begin{aligned}\left(\sum_{i_{(\alpha_0,\beta_0)}} c_{i_{(\alpha_0,\beta_0)}}\right) p_{s(\alpha)} = 0 \quad &\Rightarrow \quad \sum_{i_{(\alpha_0,\beta_0)}} c_{i_{(\alpha_0,\beta_0)}} = 0 \quad \text{for all } (\alpha_0,\beta_0)\in S \\ &\Rightarrow \quad t = 0\end{aligned}$$

Hence $\ker\phi|_{\mathcal{G}_m^j} = \{0\}$.

Using the faithful conditional expectation Φ we can show now that ϕ is faithful. Choose a non-zero positive element $a \in A$. Then

$$0 \neq \phi(\Phi(a)) = \int_{\mathbb{T}^k} \phi\alpha_z(a)\,dz = \int_{\mathbb{T}^k} \beta_z\phi(a)\,dz$$

Thus $\phi(a) \neq 0$. Hence ϕ is faithful. □

Let $\mathcal{T}_X$ be the universal Toeplitz algebra associated to X generated by copies $\iota_X(X)$ of X and $\iota_A(A)$ of A subject to

$$\iota_X(x)\iota_A(a) = \iota_X(xa)$$

$$\iota_A(a)\iota_X(x) = \iota_X(\phi(a)x) = \iota_X(ax)$$

$$\iota_X(x)^*\iota_X(y) = \iota_A(\langle x,y\rangle)$$

where $\iota_X : X \longrightarrow \mathcal{T}_X$ is a right Hilbert module map, $\iota_A : A \longrightarrow \mathcal{T}_X$ is a $*$-monomorphism and $\kappa : \mathbb{K}(X) \longrightarrow \mathcal{T}_X$ is the *-homomorphism defined by

$$\kappa(\theta_{x,y}) = \iota_X(x)\iota_X(y)^* \quad \text{for all } x \in X\,.$$

Let $\mathcal{J}_X$ be the 2-sided closed ideal of $\mathcal{T}_X$ generated by $\{\iota_A(a) - \kappa(\phi(a)) \mid a \in A\}$. It follows that $\mathcal{O}_X = \mathcal{T}_X / \mathcal{J}_X$.

Definition A.2 *Define* $\tau_\lambda := \iota_X(s_\lambda) + \mathcal{J}_X$ *for all* $\lambda \in \Sigma$ *and* $\tau_{v,i} := \sum_{\xi \in \Lambda^{e_i}(v)} \tau_\xi \tau_\xi^*$ *for all* $v \in \Lambda^0,\ i = 1, 2, \ldots, k$.

We are nearly in a position to show that $C^*(\Lambda) \cong \mathcal{O}_X$ but first a couple of lemmas concerning the relations that $\tau_\lambda, \tau_{v,i}$ satisfy in $\mathcal{O}_X$.

Lemma A.5 *The following relations are satisfied in* $\mathcal{O}_X$.
For all $\alpha, \beta, \gamma, \delta \in \Sigma$

(1)

$$\tau_\alpha^* \tau_\beta = \sum_{\substack{\alpha\gamma = \beta\delta \\ d(\gamma) = d(\beta)}} \tau_\gamma \tau_\delta^*$$

(2)

$$\tau_\alpha^* \tau_\alpha = \tau_\beta^* \tau_\beta \quad \textit{if } s(\alpha) = s(\beta)$$

(3)

$$\sum_{\xi \in \Lambda^{e_j}(v)} \tau_\xi \tau_\xi^* \tau_\alpha = \tau_\alpha \quad \textit{if } \ r(\alpha) = v,\ j = 1, 2 \ldots, k$$

(4)

$$\tau_\alpha \tau_\gamma = \tau_\beta \tau_\delta \quad \textit{if } \ \alpha\gamma = \beta\delta$$

(5) $\tau_{v,i}$ *are projections and* $\tau_{v,i} = \tau_{v,j}$ *for all* $i, j = 1, 2, \ldots, k$.

Proof.

(1)

$$\begin{aligned}
\tau_\alpha^*\tau_\beta &= \iota_X(s_\alpha)^*\iota_X(s_\beta) + \mathcal{J}_X = \iota_A(s_\alpha^* s_\beta) + \mathcal{J}_X = \iota_A(\sum_{\substack{\alpha\gamma=\beta\delta \\ d(\gamma)=d(\beta)}} s_\gamma s_\delta^*) + \mathcal{J}_X \\
&= \kappa\phi(\sum_{\substack{\alpha\gamma=\beta\delta \\ d(\gamma)=d(\beta)}} s_\gamma s_\delta^*) + \mathcal{J}_X = \kappa(\sum_{\substack{\alpha\gamma=\beta\delta \\ d(\gamma)=d(\beta)}} \theta_{s_\gamma,s_\delta} + \mathcal{J}_X) \\
&= \sum_{\substack{\alpha\gamma=\beta\delta \\ d(\gamma)=d(\beta)}} \iota_X(s_\gamma)\iota_X(s_\delta) + \mathcal{J}_X = \sum_{\substack{\alpha\gamma=\beta\delta \\ d(\gamma)=d(\beta)}} \tau_\gamma\tau_\delta^*
\end{aligned}$$

(2)

$$\begin{aligned}
\tau_\alpha^*\tau_\alpha &= \iota_A(\langle s_\alpha, s_\alpha\rangle) + \mathcal{J}_X = \iota_A(p_s(\alpha)) + \mathcal{J}_X \\
&= \iota_A(\langle s_\beta, s_\beta\rangle) + \mathcal{J}_X = \tau_\beta^*\tau_\beta \quad \text{if } s(\alpha) = s(\beta)
\end{aligned}$$

(3)

$$\begin{aligned}
\sum_{\xi\in\Lambda^{e_j}(v)} \tau_\xi\tau_\xi^*\tau_\alpha &= \sum_{\xi\in\Lambda^{e_j}(v)} \iota_X(s_\xi)\iota_X(s_\xi)^* + \mathcal{J}_X = \sum_{\xi\in\Lambda^{e_j}(v)} \iota_X(\tau_\xi\tau_\xi^*\tau_\alpha) + \mathcal{J}_X \\
&= \iota_X(\sum_{\xi\in\Lambda^{e_j}(v)} \tau_\xi\tau_\xi^*\tau_\alpha) + \mathcal{J}_X = \iota_X(s_\alpha) + \mathcal{J}_X = \tau_\alpha
\end{aligned}$$

(4) If $\alpha\gamma = \beta\delta$ then

$$\begin{aligned}
\tau_\alpha\tau_\gamma &= \sum_{\xi\in\Lambda^{d(\gamma)}(r(\alpha))} \tau_\xi\tau_\xi^*\tau_\alpha\tau_\gamma = \sum_{\xi\in\Lambda^{d(\gamma)}(r(\alpha))} \tau_\xi \sum_{\substack{\xi\lambda=\alpha\mu \\ d(\lambda)=d(\alpha)}} \tau_\lambda\tau_\mu^*\tau_\gamma \\
&= \tau_\beta\tau_\delta\tau_\gamma^*\tau_\gamma = \tau_\beta\tau_\delta
\end{aligned}$$

(5) Clearly $\tau_{v,i}$ is self-adjoint. Now

$$\begin{aligned}
\tau_{v,i}^2 &= \sum_{\xi,\eta\in\Lambda^{e_i}(v)} \tau_\xi\tau_\xi^*\tau_\eta\tau_\eta^* \\
&= \sum_{\xi,\eta\in\Lambda^{e_i}(v)} \tau_\xi\big(\iota_A(\langle s_\xi, s_\eta\rangle) + \mathcal{J}_X\big)\tau_\eta^* \\
&= \sum_{\xi\in\Lambda^{e_i}(v)} \tau_\xi\tau_\xi^*\tau_\xi\tau_\xi^* = \sum_{\xi\in\Lambda^{e_i}(v)} \tau_\xi\tau_\xi^* \\
&= \tau_{v,i}
\end{aligned}$$

Hence $\tau_{v,i}$ is a projection for all $v \in \Lambda^0,\ i = 1, 2, \ldots, k$. Fix $v \in \Lambda^0$. To see that $\tau_{v,i} = \tau_{v,j}$ consider

$$\begin{aligned}
\tau_{v,i}\tau_{v,j} &= \sum_{\xi\in\Lambda^{e_i}(v)} \tau_\xi\tau_\xi^* \sum_{\eta\in\Lambda^{e_j}(v)} \tau_\eta\tau_\eta^* \\
&= \sum_{\eta\in\Lambda^{e_j}(v)} \sum_{\xi\in\Lambda^{e_i}(v)} \tau_\xi\tau_\xi^*\tau_\eta\tau_\eta^* \\
&= \sum_{\eta\in\Lambda^{e_j}(v)} \tau_\eta\tau_\eta^* = \tau_{v,j}
\end{aligned}$$

Therefore

$$\tau_{v,j} = \tau_{v,j}^* = \tau_{v,j}\tau_{v,i} = \tau_{v,i}\,.$$

□

Now we are able to make the following definition.

Definition A.3 *For $v \in \Lambda^0$ define $\tau_v = \tau_{v,i}$ for any $i = 1, 2, \ldots k$. For $\lambda \in \Lambda\backslash\Lambda^0$ define $\tau_\lambda := \tau_{\lambda_1}\tau_{\lambda_2}\cdots\tau_{\lambda_n}$ if $\lambda = \lambda_1\lambda_2\cdots\lambda_n$ for some $n \geq 1$ with $\lambda_j \in \Sigma,\ j = 1, 2, \ldots, n$.*

Lemma A.6 *The family $\{\tau_\lambda \mid \lambda \in \Lambda\}$ is a *-representation of Λ. i.e.*

(i) $\{\tau_v\}_{v\in\Lambda^0}$ are mutually orthogonal projections

(ii) $\tau_{\lambda\mu} = \tau_\lambda \tau_\mu$ *for all* $\lambda, \mu \in \Lambda$

(iii) $\tau_\lambda^* \tau_\lambda = \tau_{s(\lambda)}$ *for all* $\lambda \in \Lambda$

(iv) $\tau_v = \sum_{\xi \in \Lambda^{e_i}(v)} \tau_\xi \tau_\xi^*$ *for all* $i = 1, 2, \ldots, k$

Proof.

(i) For any $u, v \in \Lambda^0$ we have

$$\begin{aligned}
\tau_u \tau_v &= \sum_{\xi \in \Lambda^{e_i}(u)} \tau_\xi \tau_\xi^* \sum_{\eta \in \Lambda^{e_i}(v)} \tau_\eta \tau_\eta^* \\
&= \sum_{\substack{\xi \in \Lambda^{e_i}(u) \\ \eta \in \Lambda^{e_i}(v)}} \tau_\xi (\iota_A(\langle s_\xi, s_\eta \rangle) + \mathcal{J}_X) \tau_\eta^* \\
&= \begin{cases} \tau_u & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}
\end{aligned}$$

(ii) Obvious.

(iii) For any $\lambda \in \Lambda$ and $i = 1, 2, .., k$ we have

$$\begin{aligned}
\tau_\lambda^* \tau_\lambda &= \iota_A(\langle s_\lambda, s_\lambda \rangle) + \mathcal{J}_X \\
&= \iota_A(\sum_{\xi \in \Lambda^{e_i}(s(\lambda))} s_\xi s_\xi^*) + \mathcal{J}_X \\
&= \kappa\phi(\sum_{\xi \in \Lambda^{e_i}(s(\lambda))} s_\xi s_\xi^*) + \mathcal{J}_X \\
&= \kappa(\sum_{\xi \in \Lambda^{e_i}(s(\lambda))} \theta_{s_\xi, s_\xi}) + \mathcal{J}_X \\
&= \sum_{\xi \in \Lambda^{e_i}(s(\lambda))} \iota_X(s_\xi) \iota_X(s_\xi)^* + \mathcal{J}_X \\
&= \sum_{\xi \in \Lambda^{e_i}(s(\lambda))} \tau_\xi \tau_\xi^* = \tau_v
\end{aligned}$$

(iv) By definition. ∎

Theorem A.1 *k-graph C^*-algebras are isomorphic to Cuntz-Pimsner algebras.*

Proof The natural inclusions of X and A into $C^*(\Lambda)$ give a Toeplitz representation of X. Therefore there exists a homomorphism $\tilde{f} : \mathcal{T}_X \longrightarrow C^*(\Lambda)$ such that

$$\tilde{f}(\iota_X(x)) = x \quad \text{and} \quad \tilde{f}(\iota_A(a)) = a \quad \text{for all } x \in X,\ a \in A.$$

Now $\iota_A(a) - \kappa\phi(a) \in \ker \tilde{f}$ for all $a \in A$. Therefore there exists a *-homomorphism $f : \mathcal{O}_X \longrightarrow C^*(\Lambda)$ such that

$$f(\iota_X(x) + \mathcal{J}_X) = x \quad \text{and} \quad f(\iota_A(a) + \mathcal{J}_X) = a \quad \text{for all } x \in X,\ a \in A.$$

By the previous lemma $\{\tau_\lambda \mid \lambda \in \Lambda\}$ is a *-representation of Λ therefore there exists a *-homomorphism $g : C^*(\Lambda) \longrightarrow \mathcal{O}_X$ such that $g(s_\lambda) = \tau_\lambda$ for all $\lambda \in \Lambda$. f, g are inverses of each other, hence $C^*(\Lambda) \cong \mathcal{O}_X$. $\square$

Appendix B

K-theory calculation

It is conjectured that an example which shows that a k-graph Λ does not necessarily have to satisfy condition **(S)** in order for $C^*(\Lambda)$ to be an AF-algebra is the 2-graph $\mathcal{P}_{\theta_1}$.

To support the claim that $C^*(\mathcal{P}_{\theta_1})$ is an AF-algebra we will show that $K_1(C^*(\mathcal{P}_{\theta_1}))$ is the trivial group. Now

$$K_1(C^*(\mathcal{P}_{\theta_1})) = \ker(B_2, -B_1)/\operatorname{im}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

where $B_i := 1 - M_i^t$ and M_i are the vertex matrices of $\mathcal{P}_{\theta_1}$ for $i = 1, 2$. For $u \in \mathcal{P}_{\theta_1}^0$ let $\delta_u \in \tilde{\mathbb{Z}}^{\mathcal{P}_{\theta_1}^0}$ be defined by $\delta_u(v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$ for all $v \in \mathcal{P}_{\theta_1}^0$. Then for $i = 1, 2$, $M_i(u, v) = (\delta_{u+2e_i} + \delta_{u+(1,1)})(v)$ for all $u, v \in \mathcal{P}_{\theta_1}^0$. Therefore $B_i(u, v) = (\delta_u - \delta_{u-2e_i} - \delta_{u-(1,1)})(v)$ for all $u, v \in \mathcal{P}_{\theta_1}^0$. If $\binom{x}{y} \in \ker(B_2, -B_1)$ then there exists some $N \in \mathbb{N}$ such that $x(u) = y(u) = 0$ for all $u \in \mathcal{P}_{\theta_1}^0 \backslash [(-2N, -2N), (2N, 2N)]$ and $B_2 x = B_1 y$. Define a map

$z : \mathcal{P}^0_{\theta_1} \longrightarrow \mathbb{Z}$ by

$$z(u) = \begin{cases} 0 & \text{if } u \ngeq (-2N, -2N) \\ x(u) - z(u - 2e_1) - z(u - (1,1)) & \text{if } u \geq (-2N, -2N) \end{cases}.$$

We claim that

(i) $B_1 z = x$

(ii) $B_2 z = y$

(iii) z has finite support

Thus $\ker(B_2, -B_1) = \operatorname{im} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ and so $K_1(C^*(\mathcal{P}_{\theta_1})) = 0$.

It is clear that (i) is true by definition of z. To prove (ii) we need the following result.

Lemma B.1 *If* $\binom{x}{y} \in \ker(B_2, -B_1)$ *then*

$$y(-2N + 2m, -2N + 2n) = \sum_{l_0=0}^{m} \{a^{(n)}_{l_0} + \sum_{l_1=0}^{l_0-1} \{b^{(n-1)}_{l_1} + \sum_{l_2=0}^{l_1} \{a^{(n-2)}_{l_2} + \cdots + \sum_{l_{2n}}^{l_{2n-1}} a^{(0)}_{l_{2n-1}}\} \cdots \}$$

and

$$y(-2N + 1 + 2m, -2N + 1 + 2n) = \sum_{l_0=0}^{m} \{b^{(n)}_{l_0} + \sum_{l_1=0}^{l_0} \{a^{(n)}_{l_1} + \sum_{l_2=0}^{l_1-1} \{b^{(n-1)}_{l_2} + \cdots + \sum_{l_{2n+1}=0}^{l_{2n}} a^{(0)}_{l_{2n+1}}\} \cdots \}$$

for all $m, n \in \mathbb{N}$ *where* N *is chosen so that* $x(u) = y(u) = 0$ *for all* $u \in \mathcal{P}^0_{\theta_1} \backslash [(-2N, -2N), (2N, 2N)]$ *and*

$$\begin{aligned} a^{(n)}_l &:= x(-2N + 2l, -2N + 2n) - x(-2N + 2l, -2N + 2n - 2) \\ &- x(-2N + 2l - 1, -2N + 2n - 1) \\ b^{(n)}_l &:= x(-2N + 1 + 2l, -2N + 1 + 2n) - x(-2N + 1 + 2l, -2N - 1 + 2n) \\ &- x(-2N + 2l, -2N + 2n). \end{aligned}$$

Proof. By induction since

$$y(-2N+1+2m,-2N+1+2n) = \sum_{l=0}^{m}\{b_l^{(n)} + y(-2N+2l,-2N+2n)\}$$

and $y(-2N,-2N) = x(-2N,-2N)$ etc.

Proof of Claim (ii). By induction

$$z(-2N+2m,-2N+2n) \;=\; \sum_{l_0=0}^{m}\{\tilde{a}_{l_0}^{(n)} + \sum_{l_1=0}^{l_0-1}\{\tilde{b}_{l_1}^{(n-1)} + \cdots + \sum_{l_{2n}=0}^{l_{2m-1}} \tilde{a}_{l_{2n}}^{(0)}\}\cdots\}$$

and

$$\begin{aligned} z(-2N+1+2m,-2N+1+2n) \;=\; & \sum_{l_0=0}^{m}\{\tilde{b}_{l_0}^{(n)} + \sum_{l_1=0}^{l_0}\{\tilde{a}_{l_1}^{(n)} + \cdots + \sum_{l_{2n}=0}^{l_{2n-1}-1}\{\tilde{b}_{l_{2n}}^{(0)} \\ & + \sum_{l_{2n+1}=0}^{l_{2n}} \tilde{a}_{l_{2n+1}}^{(0)}\}\cdots\} \end{aligned}$$

for all $m,n \in \mathbb{N}$ where $\tilde{a}_l^{(n)} := x(-2N+2l,-2N+2n)$ and $\tilde{b}_l^{(n)} := x(-2N+1+2l_0,-2N+1+2n)$. Now

$$\begin{aligned} B_2 z(-2N+2m,-2N+2n) \;=\; & z(-2N+2m,-2N+2n) \\ - \; & z(-2N+2m,-2N+2(n-1)) \\ - \; & z(-2N+1+2(m-1),-2N+1+2(n-1)) \end{aligned}$$

and

$$\begin{aligned} a_l^{(n)} &= \tilde{a}_l^{(n)} - \tilde{a}_l^{(n-1)} - \tilde{b}_{l-1}^{(n-1)} \\ b_l^{(n)} &= \tilde{b}_l^{(n)} - \tilde{b}_l^{(n-1)} - \tilde{a}_{l-1}^{(n-1)} \end{aligned}$$

Therefore by substituting the above in this equation and using Lemma B.1 we see that $B_2 z(-2N+2m,-2N+2n) = y(-2N+2m,-2N+2n)$. Similarly $B_2 z(-2N+1+2m,-2N+1+2n) = y(-2N+1+2m,-2N+1+2n)$. It is clear that $B_2 z(u) = y(u)$ if $u \ngeq (-2N,-2N)$, hence $B_2 z = y$ as claimed.

It is easy to prove that (i),(ii) $\Rightarrow$ (iii).

Appendix C

Outlook

We discuss some possible future developments of the thesis.

- The concept of a graph trace as defined in [28] might be of use in proving that, for example, $C^*(\mathcal{L})$ and $C^*(\mathcal{P}_\theta)$ (see Chapter 4) are finite, for all factorisation rules θ (a C^*-algebra A is said to be finite if there are no infinite projections in A i.e. a projection p for which there is a partial isometry s such that $s^*s = p$ and $ss^* < p$). Note that all AF-algebras are finite. Therefore $C^*(\mathcal{L})$ may be an example of a simple k-graph C^*-algebra that is neither AF nor purely infinite (in the sense that every heredetary subalgebra contains an infinite projection) in contrast to the 1-graph case (see [37, Corollary 3.10]).

  Another such example might be $\mathcal{P}_{\theta_2}$ where $\theta_2 : A^1 * B^1 \longrightarrow B^1 * A^1$ is defined by

$$(a(u), \alpha(u+(1,1))) \mapsto (\alpha(u), a(u+(1,1)))$$
$$(b(u), \alpha(u+(2,0))) \mapsto (\alpha(u), b(u+(2,0)))$$
$$(a(u), \beta(u+(1,1))) \mapsto (\beta(u), a(u+(0,2)))$$
$$(b(u), \beta(u+(2,0))) \mapsto (\beta(u), b(u+(0,2)))$$

for all $u \in V$. We strongly suspect that $\mathcal{P}_{\theta_2}$ is not an AF-algebra, primarily because $\mathcal{P}_{\theta_2}$ contains infinite periodic paths. By the above argument $C^*(\mathcal{P}_{\theta_2})$ may not be purely infinite. It is certainly true that $C^*(\mathcal{P}_{\theta_2})$ is simple.

If $C^*(\mathcal{P}_{\theta_1})$ is indeed AF we have the curious situation of a skeleton giving rise to two 2-graphs $\mathcal{P}_{\theta_1}$, $\mathcal{P}_{\theta_2}$; one with an associated AF algebra, the other with an associated non AF algebra. Note also that $K_*(C^*(\mathcal{P}_{\theta_1})) = K_*(C^*(\mathcal{P}_{\theta_2}))$ by Corollary 5.1.

- Consider the 2-graph $O_n *_\theta O_m$ ($n, m \in \mathbb{N} \setminus \{0\}$) constructed from two 1-graphs O_n, O_m and a factorisation rule θ in the usual way (see Section 2.1) where $O_n^0 = O_m^0 = \{*\}$ and $O_n^1 := \{a_i \,|\, i = 1, \ldots, n\}$, $O_m^1 := \{\alpha_i \,|\, i = 1, \ldots, m\}$ (see Figure C.1).

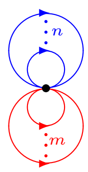

Figure C.1: The skeleton of $O_n *_\theta O_m$.

Note that $C^*(O_n)$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$, ($n \geq 2$), thus we call $C^*(O_n *_\theta O_m)$ a coloured Cuntz algebra. Also note that

$O_n *_\theta O_m$ is cofinal and satisfies the hypothesis of [35, Proposition 4.9] if $O_n *_\theta O_m$ satisfies the aperiodicity condition **(A)** [35, Definition 4.3] for $n, m \in \mathbb{N}\backslash\{0\}$ such that $\max\{n, m\} \geq 2$.

It would be interesting to know which factorisation rules gives rise to $O_n *_\theta O_m$ satisfying **(A)**, for then we may use Proposition 5.1 with the classification theory of Kirchberg and Phillips [32, 46] to determine which $C^*(O_n *_\theta O_m)$ are isomorphic to $\mathcal{O}_n \otimes \mathcal{O}_m$.

- Is there a way of embedding a 2-graph Δ with sinks into a 2-graph Λ without sinks such that $C^*(\Delta)$ is isomorphic to a full corner of $C^*(\Lambda)$? The answer is yes for 1-graphs however it is unclear whether a method will work in general for 2-graphs. There are obvious ad hoc methods for some 2-graphs.

  If such a construction is possible then we may us Proposition 5.1 and [9, Theorem 2.8] to calculate the K-theory of Δ since stably isomorphic C^*-algebras have the same K-theory.

- The study of AF k-graph C^*-algebras may be useful for the study of AF C^*-systems [7, 26, 27].

- A natural problem would be to calculate the K-theory of k-graph C^*-algebras for $k > 2$.

Bibliography

[1] T.G. Bates, *The Cuntz-Krieger algebras of row-finite graphs*, PhD Thesis, University of Newcastle, 1999.

[2] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249-275.

[3] B. Blackadar, *K-theory for C^*-algebras, Second Edition*, MSRI Publications 5, Cambridge University Press, Cambridge, 1998.

[4] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Transactions of the American Mathematical Society, **171** (1972), 195-234.

[5] O. Bratteli, G.A. Elliott, R.H. Herman, *On the possible temperatures of a dynamical system*, Commun. math. Phys. **74**, (1980) 281-295.

[6] O. Bratteli, P.E.T. Jorgensen, *Wavelets Through a Looking Glass. The world of the spectrum.* Applied and Numerical Harmonic Analysis. Birkhuser Boston, Inc., Boston, MA, 2002.

[7] O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum-Statistical Mechanics. II. Equilibrium States. Models in Quantum-Statistical Mechanics.* Texts and Monographs in Physics. Springer-Verlag, New York-Berlin, 1981.

[8] K. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[9] L.G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific Journal of Mathematics, **71**, No. 2, (1977), 335-348.

[10] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173-185.

[11] J. Cuntz, *A class of C^*-algebras and topological markov chains II: Reducible chains and the Ext-functor for C^*-algebras*, Invent. Math., **63** (1981), 25-40.

[12] J. Cuntz, W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math., **56** (1980), 251-268.

[13] D. Drinen, *Viewing AF-algebras as graph algebras*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 1991-2000.

[14] D. Drinen, M. Tomforde, *The C^*-algebras of arbitrary graphs*, Rocky Mountain J. Math. to appear.

[15] E.G. Effros, *Dimensions and C^*-Algebras*, CBMS Lectures, **46** (1981), American Mathematical Society.

[16] G.A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, journal of Algebra, **38**, 29-44.

[17] D.E. Evans, *On $\mathcal{O}_n$*, Publ. Res. Inst. Math. Sci. **16** (1980), no. 3, 915-927.

[18] D.E. Evans, *Gauge actions on $\mathcal{O}_A$*, J. Operator Theory **7** (1982), 79-100.

[19] D.E. Evans, *The C^*-algebras of topological Markov chains*, Lecture Notes, Tokyo Metropolitan University (1983).

[20] D.E. Evans, Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[21] R. Exel, M. Laca, *Cuntz-Krieger algebras for infinite matrices*, J. Reine Angew. Math. **512** (1999), 119-172.

[22] R. Exel, M. Laca, *The K-theory of Cuntz-Krieger algebras for infinite matrices*, K-Theory 19 (2000), no. 3, 251-268.

[23] R. Exel, M. Laca, *Partial dynamical systems and the KMS condition*, Preprint.

[24] N.J. Fowler, A. Sims, *Product systems over right-angled Artin semigroups*, Trans. Amer. Math. Soc. **354** (2002), no. 4, 1487-1509.

[25] N.J. Fowler, M. Laca, I. Raeburn, *The C^*-algebras of infinite graphs*, Proc. Amer. Math. Soc. **8** (2000), 2319-2327.

[26] V.Y. Golodets, S.V. Neshveyev, *Gibbs states for AF algebras*, Journal of Mathematical Physics, **39**, No. 12 (1998), 6329-6344.

[27] F. Hiai, D. Petz, *Quantum mechanics in AF C^*-systems*, Reviews in Mathematical Physics, Vol. 8, No.6 (1996), 819-859.

[28] J. Hjelmborg, *Purely infinite and stable C^*-algebras of graphs and dynamical systems*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1789-1808.

[29] A. an Huef, I. Raeburn, *The ideal structure of Cuntz-Krieger algebras*, Ergod. Th. & Dynam. Sys. **17** (1997), 611-624.

[30] M. Izumi, *Subalgebras of infinite C^*-algebras with finite Watatani indices, I, Cuntz algebras*, Commun. Math. Phys. **155**, no. 1, (1993) 157-182.

[31] M. Izumi, *Subalgebras of infinite C^*-algebras with finite Watatani indices, II, Cuntz-Krieger algebras*, Duke Math. J. **91** (1998), no. 3, 409-461.

[32] E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, preprint.

[33] A. Kishimoto, H. Takai, *Some remarks on C^*-dynamical systems with a compact abelian group*, Publ. RIMS Kyoto, **14** (1978) 383-397.

[34] A. Kumjian, *Notes on C^*-algebras of graphs*, in Operator Algebras and Operator Theory, Contemp. Math. **228**, Amer. Math. Soc., Providence (1998).

[35] A. Kumjian, D. Pask, *Higher rank graph C^*-algebras*, New York J. Math **6** (2000) 1-20.

[36] A. Kumjian, D. Pask, *Actions of $\mathbb{Z}^k$ associated to higher rank graphs* Preprint, 2001.

[37] A. Kumjian, D. Pask, I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific. J. Math. **184** (1998), 161-174.

[38] A. Kumjian, D. Pask, I. Raeburn, J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras,* J. Funct. Anal., **144** (1997), 505-541.

[39] E.C. Lance, *Hilbert C^*-Modules. A toolkit for operator algebraists*, LMS Lecture Note Series 210, Cambridge University Press, Cambridge 1995.

[40] C.-W.H. Lee, S.G. Rajeev, *Symmetry algebras of large-N matrix models for open strings*, Nuclear Phys. B **529** (1998), no. 3, 656-688.

[41] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Gttingen-Heidelberg 1963.

[42] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, no. 5, Springer-Verlag, Berlin, 1971.

[43] M. Nilsen, S. Adji, M. Laca, I. Raeburn, *Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups*, Proc. Amer. Soc. **122** (1994), 1133-1141.

[44] D. Pask, I. Raeburn, *On the K-theory of Cuntz-Krieger algebras* Publ. Res. Inst. Math. Sci. **32** (1996), no. 3, 415-443.

[45] G.K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.

[46] N.C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. 5 (2000), 49-114.

[47] M.V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Fields Inst. Comm. **12** (1997), 189-212.

[48] I. Raeburn, *On crossed products and Takai duality*, Proc. Edinburgh Math. Soc. **31** (1988), 321-330.

[49] I. Raeburn, A. Sims, T. Yeend, *Higher-rank graphs and their C^*-algebras* Preprint (2001).

[50] I. Raeburn, D.P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*. Mathematical Surveys and Monographs, **60**, American Mathematical Society, Providence, RI, 1998.

[51] G. Robertson, T. Steger, *C^*-algebras arising from group actions on the boundary of a triangle building*, Proc. London Math. Soc. **72** (1996), 613-637.

[52] G. Robertson, T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. Reine Angew. Math. **513** (1999), 115–144.

[53] G. Robertson, T. Steger, *Asymptotic K-Theory for groups acting on $\tilde{A}_2$ Buildings* Canad. J. Math. **53** (2001), no. 4, 809-833.

[54] M. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, Vol. 7, Academic Press, New York, 1989.

[55] M. Rørdam, F. Larsen, N. Laustsen, *An Introduction to K-Theory for C^*-Algebras*, London Mathematical Society Student Texts, **49**, Cambridge University Press, Cambridge, 2000.

[56] J. Rosenberg, C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431-474.

[57] J. Spielberg, *Free product groups, Cuntz-Krieger algebras, and covariant maps*, International J. Math. **2** (1991), 457-476.

[58] J. Spielberg, *A functorial approach to the C^*-algebras of a graph*, Internat. J. Math. 13 (2002), no. 3, 245-277.

[59] W. Szymański, *Bimodules for Cuntz-Krieger algebras of infinite matrices*, Bull. Austral. Math. Soc. **62** (2000), 87-94.

[60] M. Tomforde, *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs*, J. Operator Theory, to appear.

[61] N.E. Wegge-Olsen, *K-theory and C^*-Algebras: A Friendly Approach*, Oxford University Press, Oxford, 1993.

[62] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, Vol. 38, Cambridge University Press, Cambridge, 1994.