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This is not an Exam in StatØ3 2006

There are three exercises with 4, 3 and 3 questions respectively.

All 10 questions need to be answered and the grade will be based on an overall evaluation.

Johansen, S. (1996) *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models.* Oxford University Press, Oxford

and relevant additional literature may be used during the exam.

Pen and pencil may be used.

Exercise 1

The uncovered interest rate parity (UIP) states that the interest rates in two countries reflect the economic expectations to the change in exchange rate. Let e_t be the exchange rate and i_t^* and i_t be the domestic and foreign interest rate respectively. The UIP is formulated as

$$E_t^e \Delta e_{t+1} = i_t - i_t^*, \tag{1}$$

where $E_t^e \Delta e_{t+1}$ denotes the economic expectations, given available information at time t, of the future change of the exchange rate.

We shall now translate this into a statement about a cointegrated VAR model as follows: We define

$$X_t = (e_t, i_t, i_t^*)'$$

and assume that data can be described by

$$\Delta X_t = \alpha\beta' X_{t-1} + \Phi \Delta X_{t-1} + \varepsilon_t, \tag{2}$$

where the cointegration rank is $r = 1$, so that α and β are vectors. Let us normalize β as $\beta = (\gamma_1, 1, \gamma_2)'$, so that the parameters $(\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2)$ are identified.

Q.1.1. Find from (2) an expression for

$$E(\Delta e_t | X_{t-1}, X_{t-2}) \tag{3}$$

in terms of $(\alpha, \gamma_1, \gamma_2, \Phi)$.

(Solution:

$$E(\Delta e_t | X_{t-1}, X_{t-2}) = \alpha_1(\gamma_1 e_{t-1} + i_{t-1} + \gamma_2 i^*_{t-1}) + \Phi_{11}\Delta e_{t-1} + \Phi_{12}\Delta i_{t-1} + \Phi_{13}\Delta i^*_{t-1}$$

The idea of Muth (1960) was to equate the economic expectation (1) with the expectation based upon the statistical model (3).

Q.1.2. Describe the restrictions on the parameters $(\alpha, \gamma_1, \gamma_2, \Phi)$ which express the UIP.

Solution: If

$$E(\Delta e_t | X_{t-1}, X_{t-2}) = \alpha_1(\gamma_1 e_{t-1} + i_{t-1} + \gamma_2 i^*_{t-1}) + \Phi_{11}\Delta e_{t-1} + \Phi_{12}\Delta i_{t-1} + \Phi_{13}\Delta i^*_{t-1}$$

and

$$E^e_t \Delta e_t = i_{t-1} - i^*_{t-1}$$

should be the same relation we must have the restrictions

$$i_{t-1} - i^*_{t-1} = \alpha_1(\gamma_1 e_{t-1} + i_{t-1} + \gamma_2 i^*_{t-1}) + \Phi_{11}\Delta e_{t-1} + \Phi_{12}\Delta i_{t-1} + \Phi_{13}\Delta i^*_{t-1}$$

which implies that

$$\alpha_2 = 1, \gamma_1 = 0, \gamma_2 = -1, \Phi_{11} = \Phi_{12} = \Phi_{13} = 0.$$

Q.1.3. Write the three equations from (2) under the restrictions imposed by the UIP.

Solution

$$\begin{aligned}
\Delta e_t &= i_{t-1} - i^*_{t-1} + \varepsilon_{1t} \\
\Delta i^*_t &= \alpha_2(i_{t-1} - i^*_{t-1}) + \Phi_{21}\Delta e_{t-1} + \Phi_{22}\Delta i_{t-1} + \Phi_{23}\Delta i^*_{t-1} + \varepsilon_{2t} \\
\Delta i_t &= \alpha_3(i_{t-1} - i^*_{t-1}) + \Phi_{31}\Delta e_{t-1} + \Phi_{32}\Delta i_{t-1} + \Phi_{33}\Delta i^*_{t-1} + \varepsilon_{3t}
\end{aligned}$$

Q.1.4. Describe how you would estimate the model under the assumption of the UIP. How many parameters do you have in the restricted model?

Solution: Because the errors are correlated we find the marginal model for the first equation and the conditional model for the last two equations given the first and that gives

$$\begin{aligned}
\Delta e_t &= i_{t-1}^* - i_{t-1} + \varepsilon_{1t} \\
\Delta i_t^* &= \omega_2 \Delta e_t + (\alpha_2 - \omega_2)(i_{t-1}^* - i_{t-1}) + \Phi_{21}\Delta e_{t-1} + \Phi_{22}\Delta i_{t-1} + \Phi_{23}\Delta i_{t-1}^* + \varepsilon_{2.1t} \\
\Delta i_t &= \omega_3 \Delta e_t + (\alpha_2 - \omega_3)(i_{t-1}^* - i_{t-1}) + \Phi_{31}\Delta e_{t-1} + \Phi_{32}\Delta i_{t-1} + \Phi_{33}\Delta i_{t-1}^* + \varepsilon_{3.2t}
\end{aligned}$$

where

$$\begin{aligned}
\omega_2 &= \Omega_{21}\Omega_{11}^{-1}, \omega_3 = \Omega_{31}\Omega_{11}^{-1}, \\
\varepsilon_{2.1t} &= \varepsilon_{2t} - \omega_2\varepsilon_{1t}, \varepsilon_{3.1t} = \varepsilon_{3t} - \omega_3\varepsilon_{3t}
\end{aligned}$$

We then estimate Ω_{11} *by* $T^{-1}\sum_{t=1}^{T}\left(\Delta e_t - i_{t-1}^* + i_{t-1}\right)^2$ *and the remaining parameters by regression of*

$$(\Delta i_t^*, \Delta i_t) \text{ on } \Delta e_t, (i_{t-1}^* - i_{t-1}), \Delta e_{t-1}, \Delta i_{t-1}, \Delta i_{t-1}^*.$$

Exercise 2

Consider the equation

$$X_t + X_{t-1} = \varepsilon_t + \mu, t = 1, \ldots, T, \tag{4}$$

where we assume throughout that ε_t are i.i.d. $(0, \Omega)$, and the variables are of dimension p.

Q.2.1. Show that the solution of (4) is

$$X_t = (-1)^t \sum_{i=1}^{t} (-1)^i \varepsilon_i + \frac{1}{2}\mu((-1)^{t-1} + 1) + (-1)^t X_0 \tag{5}$$

Solution

$$\begin{aligned}
X_t &= (-1)^t \sum_{i=1}^{t} (-1)^i (\varepsilon_i + \mu) + (-1)^t X_0 \\
&= (-1)^t \sum_{i=1}^{t-1} (-1)^i (\varepsilon_i + \mu) + (-1)^t (-1)^t (\varepsilon_t + \mu) + (-1)^t X_0 \\
&= -(-1)^{t-1} \sum_{i=1}^{t-1} (-1)^i (\varepsilon_i + \mu) + (\varepsilon_t + \mu) + (-1)^t X_0 \\
&= -X_{t-1} + \varepsilon_t + \mu
\end{aligned}$$

and

$$(-1)^t \sum_{i=1}^{t} (-1)^i \mu = (-1)^t \mu (-1) \frac{1-(-1)^t}{1-(-1)} = \frac{1}{2}\mu(1+(-1)^{t-1})$$

Now consider the model

$$X_t + X_{t-1} = \alpha\beta' X_{t-1} + \varepsilon_t + \mu, t = 1, \ldots, T$$

where α and β have rank $r < p$.

Q.2.2 Find the equation for $\alpha'_\perp X_t$ and show that it has a solution of the form (5).

Solution

$$\alpha'_\perp X_t + \alpha'_\perp X_{t-1} = \alpha'_\perp \varepsilon_t + \alpha'_\perp \mu, t = 1, \ldots, T$$

with solution

$$\alpha'_\perp X_t = (-1)^t \sum_{i=1}^{t} (-1)^i \alpha'_\perp \varepsilon_i + \frac{1}{2}\alpha'_\perp \mu((-1)^t + 1) + \alpha'_\perp X_0$$

Q.2.3. Find an equation for $\beta' X_t$ and find the condition for $\beta' X_t$ to be stationary, expressed in terms of α and β, and find an expression for the expectation of $\beta' X_t$.

Solution

$$\beta' X_t = (-I_r + \beta'\alpha)\beta' X_{t-1} + \beta'\varepsilon_t + \beta'\mu,$$

which is stationary if

$$|eig(-I_r + \beta'\alpha)| < 1$$

the stationary process is

$$\sum_{i=0}^{\infty} (-I_r + \beta'\alpha)^i (\beta'\varepsilon_{t-1} + \beta'\mu)$$

and the expectation is

$$\sum_{i=0}^{\infty} (-I_r + \beta'\alpha)^i \beta'\mu = (I_r - (-I_r + \beta'\alpha))^{-1}\mu = (2I_r - \beta'\alpha)^{-1}\mu.$$

Q.2.4. Combine the results from **Q.2.2.** and **Q.2.3.** to find an expression for X_t.

Solution

$$
\begin{aligned}
X_t &= \alpha(\beta'\alpha)^{-1}\beta' X_t + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp X_t \\
&= \alpha(\beta'\alpha)^{-1}\beta' \sum_{i=0}^{\infty}(-I_r + \beta'\alpha)^i(\beta'\varepsilon_{t-1} + \beta'\mu) \\
&\quad + C(-1)^t \sum_{i=1}^{t}(-1)^i \alpha'_\perp \varepsilon_i + \frac{1}{2}C\mu((-1)^{t-1} + 1) + C(-1)^t X_0
\end{aligned}
$$

Exercise 3

In the model

$$\Delta X_t = \Pi X_{t-1} + \varepsilon_t, t = 1, \ldots, T$$

and ε_t are i.i.d. $N_p(0, \Omega)$, we want to test that the rank of Π is $r \leq 1$.

The likelihood ratio statistics is given by

$$Q_1 = -2\log LR\{\text{rank}(\Pi) \leq 1\} = -T\sum_{i=2}^{p}\log(1 - \hat{\lambda}_i),$$

where the $\hat{\lambda}_i$ are the eigenvalues of

$$|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}| = 0. \qquad (6)$$

In the book you find the proof that if the rank is $r = 1$, then

$$Q_1 \xrightarrow{w} tr\{\int_0^1 (dB)B'(\int_0^1 BB'du)^{-1}\int_0^1 B(dB)'\},$$

where B is a standard Brownian motion of dimension $p - 1$.

Q. 3.1. Find the limit distribution of the matrices in the expression (6), when the rank is zero, $r = 0$, and the eigenvalues are normalized as $\lambda = \rho T^{-1}$ where $T \to \infty$.

Solution. when $r = 0$, *then process is*

$$X_t = X_0 + \sum_{i=1}^{t}\varepsilon_i,$$

so that

$$\frac{\rho}{T} S_{11} - S_{10} S_{00}^{-1} S_{01} \xrightarrow{w}$$
$$\rho \int_0^1 WW' du - \int_0^1 W(dW)' \Omega^{-1} \int_0^1 (dW) W'$$

Q. 3.2. Use this result to describe the limit distribution of Q_1 when $r = 0$.

Solution When $\rho = 0$, the eigenvalues $\hat{\lambda}_i$ are of the order of T^{-1}, and the test statistic is approximately

$$-T \sum_{i=2}^{p} \log(1 - \hat{\lambda}_i) \approx \sum_{i=2}^{p} T \hat{\lambda}_i \xrightarrow{w} \sum_{i=2}^{p} \hat{\rho}_i,$$

where ρ solves

$$|\rho \int_0^1 WW' du - \int_0^1 W(dW)' \Omega^{-1} \int_0^1 (dW) W'| = 0.$$

Thus the limit distribution is the sum of the smallest $p-1$ eigenvalues of the matrix

$$\int_0^1 (dB) B' [\int_0^1 BB' du]^{-1} \int_0^1 B(dB)',$$

where $B = \Omega^{-1/2} W$ is standard BM.