STAT Ø3

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Stationary autoregressive processes

$$\begin{aligned} y_t &= \rho y_{t-1} + \varepsilon_t, t = 1, \ldots, T \\ y_t &= \rho^t y_0 + \varepsilon_t + \rho\varepsilon_{t-1} + \ldots + \rho^{t-1}\varepsilon_1 \end{aligned}$$

For $|\rho| < 1$ define

$$y_t^* = \sum_{i=0}^{\infty} \rho^i \varepsilon_{t-i} = \varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-2} + \ldots$$

Define y_t^* as $\lim_{N\to\infty} \sum_{i=0}^{N} \rho^i \varepsilon_{t-i} = \varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-2} + \ldots + \rho^N \varepsilon_{t-N}$

in the sense of

1. Probability

2. Distribution

3 Almost surely

4. L_2

THEOREM: If $|\rho| < 1$, the process y_t^* is a stationary linear process

Solution of the AR equations is stationary if we choose $y_0 = y_0^*$.

If y_0 is not chosen this way y_t is not stationary but $V(y_t - y_t^*) \to 0$.

MORE DIMENSIONS

$$\begin{aligned} Y_t &= \Pi_1 Y_{t-1} + \varepsilon_t, t = 1, \ldots, T \\ Y_t &= \Pi_1^t Y_0 + \varepsilon_t + \Pi_1 \varepsilon_{t-1} + \ldots + \Pi_1^{t-1} \varepsilon_1 \end{aligned}$$

THEOREM: If $|eig(\Pi_1)| < 1$, then $\Pi_1^t \to 0$ (exponentially)

"PROOF"

$$\Pi_1 = M \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_p \end{pmatrix} M^{-1}$$

$$\Pi_1^T = M \begin{pmatrix} \lambda_1^T & 0 & \ldots & 0 \\ 0 & \lambda_2^T & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_p^T \end{pmatrix} M^{-1} \to 0$$

MORE LAGS

$$\begin{aligned} X_t &= \Pi_1 X_{t-1} + \Pi_2 X_{t-2} + \varepsilon_t \\ X_{t-1} &= X_{t-1} \end{aligned}$$

Companion form

$$\begin{aligned} \begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} &= \begin{pmatrix} \Pi_1 & \Pi_2 \\ I & 0 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\ \tilde{X}_t &= \tilde{\Pi}_1 \tilde{X}_{t-1} + \tilde{\varepsilon}_t \end{aligned}$$

THEOREM: $\tilde{X}_t^* = \sum_{i=0}^{\infty} \tilde{\Pi}_1^i \tilde{\varepsilon}_{t-i}$ is stationary if $|eig(\tilde{\Pi}_1)| < 1$

What are the eigenvalues?

$$\begin{aligned} \begin{pmatrix} \Pi_1 & \Pi_2 \\ I & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \Pi_1 v_1 + \Pi_2 v_2 &= \lambda v_1 \\ v_1 &= \lambda v_2 \end{aligned}$$

implies

$$\begin{aligned} \lambda \Pi_1 v_2 + \Pi_2 v_2 &= \lambda^2 v_2 \\ (\lambda^2 I_p - \lambda \Pi_1 v_2 - \Pi_2) v_2 &= 0 \end{aligned}$$

Define

$$\Pi(z) = I_p - z\Pi_1 - z^2 \Pi_2$$

THEOREM: The eigenvalues of the companion form are the roots of the equation $|\Pi(z)| = 0$.

If $|\Pi(z)| = 0 \Longrightarrow |z| > 1$ the process $\tilde{X}_t^* = \sum_{i=0}^{\infty} \tilde{\Pi}_1^i \tilde{\varepsilon}_{t-i}$ is stationary and so is

$$X_t = \sum_{i=0}^{\infty} (I, 0) \tilde{\Pi}_1^i \begin{pmatrix} I \\ 0 \end{pmatrix} \varepsilon_{t-i} = \sum_{i=0}^{\infty} C_i \varepsilon_{t-i}$$

NOTE:

$$\begin{aligned}
\sum_{i=0}^{\infty} z^i (I_p, 0) \tilde{\Pi}_1^i \begin{pmatrix} I_p \\ 0 \end{pmatrix} &= (I_p, 0)[\sum_{i=0}^{\infty} z^i \tilde{\Pi}_1^i] \begin{pmatrix} I_p \\ 0 \end{pmatrix} \\
&= (I_p, 0)(I_{2p} - z\tilde{\Pi}_1)^{-1} \begin{pmatrix} I_p \\ 0 \end{pmatrix} \\
&= (I_p, 0) \begin{pmatrix} I_p - z\Pi_1 & -z\Pi_2 \\ -zI_p & I_p \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \\
&= (I_p - z\Pi_1 - (-zI_p) I_p^{-1} (-z\Pi_2))^{-1} \\
&= (I_p - z\Pi_1 - z^2\Pi_2)^{-1} = \Pi(z)^{-1}
\end{aligned}$$

The coefficients of the expansion of the solution are the coefficients in

Chapter 4

Assume throughout that

$$|\Pi(z)| = 0 \Longrightarrow |z| > 1 \text{ or } z = 1$$

ECM:

$$\begin{aligned} VAR &: X_t = \Pi_1 X_{t-1} + \Pi_2 X_{t-2} + \varepsilon_t \\ X_t - X_{t-1} &= (\Pi_1 - I_p) X_{t-1} + \Pi_2 X_{t-2} + \varepsilon_t \\ X_t - X_{t-1} &= (\Pi_1 - I_p + \Pi_2) X_{t-1} + \Pi_2 (X_{t-2} - X_{t-1}) + \varepsilon_t \\ ECM &: \Delta X_t = \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \varepsilon_t \end{aligned}$$

Note $\Pi(1) = -\Pi$ is singular and hence $\Pi = \alpha\beta'$ if there is unit root

Question: What is the solution of the VAR equations, when there is a unit root. One lag

$$\begin{aligned} \Delta X_t &= \alpha\beta' X_{t-1} + \varepsilon_t \\ \alpha'_\perp \Delta X_t &= \alpha'_\perp \varepsilon_t \\ \alpha'_\perp X_t &= \alpha'_\perp X_0 + \alpha'_\perp \sum_{i=1}^{t} \varepsilon_i : \text{Random walk} \end{aligned}$$

$$\begin{aligned} \beta' \Delta X_t &= \beta'\alpha\beta' X_{t-1} + \beta'\varepsilon_t \\ \beta' X_t &= (I_r + \beta'\alpha)\beta' X_{t-1} + \beta'\varepsilon_t \\ \beta' X_t &= \sum_{i=0}^{\infty} (I_r + \beta'\alpha)^i \beta' \varepsilon_{t-i} : \text{ IF } |eig(I_r + \beta'\alpha)| < 1 \end{aligned}$$

$$\begin{aligned} X_t &= \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp X_t + \alpha(\beta'\alpha)^{-1}\beta' X_t \\ &= \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i + \alpha(\beta'\alpha)^{-1} \sum_{i=0}^{\infty} (I_r + \beta'\alpha)^i \beta' \varepsilon_{t-i} \\ &\quad + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp X_0 \end{aligned}$$

$$\begin{aligned} GRT &: X_t = C \sum_{i=1}^{t} \varepsilon_i + Y_t + A \\ 1 &: \Delta X_t \text{ and } \beta' X_t \text{ stationary} \\ 2 &: X_t \text{ is non stationary} \\ 3 &: \text{Common trends } \alpha'_\perp \sum_{i=1}^{t} \varepsilon_i \end{aligned}$$

Equivalent conditions to $|eig(I_r + \beta'\alpha)| < 1$

$$\begin{aligned}
\Pi(z) &= (1-z)I_p - \alpha\beta' z \\
& \begin{pmatrix} \beta' \\ \alpha'_\perp \end{pmatrix} \Pi(z)(\alpha, \beta_\perp) \\
&= \begin{pmatrix} I_r(1-z)\beta'\alpha - (\beta'\alpha)^2 z & 0 \\ 0 & (1-z)\alpha'_\perp\beta_\perp \end{pmatrix} \\
& |\begin{pmatrix} \beta' \\ \alpha'_\perp \end{pmatrix}||\Pi(z)||(\alpha, \beta_\perp)| \\
&= |I_r(1-z)\beta'\alpha - (\beta'\alpha)^2 z|(1-z)^{p-r}|\alpha'_\perp\beta_\perp| \\
&= |\beta'\alpha||I_r - z(I_r + \beta'\alpha)|(1-z)^{p-r}|\alpha'_\perp\beta_\perp|
\end{aligned}$$

The condition $|eig(I_r + \beta'\alpha)| < 1$ implies $|\beta'\alpha| \neq 0$

$$\begin{aligned}
|\begin{pmatrix} \beta' \\ \beta'_\perp \end{pmatrix}||(\alpha, \beta_\perp)| &= |\begin{pmatrix} \beta'\alpha & 0 \\ \beta'_\perp\alpha & \beta'_\perp\beta_\perp \end{pmatrix}| = |\beta'\alpha||\beta'_\perp\beta_\perp| \\
|(\alpha, \beta_\perp)| &\neq 0 \Longleftrightarrow |\beta'\alpha| \neq 0
\end{aligned}$$

$$\begin{aligned}
|\begin{pmatrix} \alpha' \\ \alpha'_\perp \end{pmatrix}||(\alpha, \beta_\perp)| &= |\begin{pmatrix} \alpha'\alpha & \alpha'\beta_\perp \\ 0 & \alpha'_\perp\beta_\perp \end{pmatrix}| = |\alpha'\alpha||\alpha'_\perp\beta_\perp| \\
|(\alpha, \beta_\perp)| &\neq 0 \Longleftrightarrow |\beta'_\perp\alpha_\perp| \neq 0
\end{aligned}$$

$$|\Pi(z)| = |\beta'\alpha||I_r - z(I_r + \beta'\alpha)|(1-z)^{p-r}|\alpha'_\perp\beta_\perp|$$

Under condition $|eig(I_r + \beta'\alpha)| < 1$

$$\begin{aligned} |\Pi(z)| &= 0 \\ &\iff z = 1 \text{ or } |I_r - z(I_r + \beta'\alpha)| = 0 \end{aligned}$$

The $I(1)$ condition

$$\alpha'_\perp\beta_\perp \text{ full rank}$$

ensures that we only get $I(1)$ variables.

A simple model for $I(2)$

$$\begin{aligned}\Delta X_{1t} &= \varepsilon_{1t} \\ \Delta X_{2t} &= X_{1t-1} + \varepsilon_{2t}\end{aligned}$$

$$\begin{aligned}\Pi &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1,0) = \alpha\beta' \\ \alpha_\perp &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta_\perp = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \alpha'_\perp \beta_\perp = 0\end{aligned}$$

NOTE The matrix Π cannot be diagonalized:

$$\begin{aligned}\Pi &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ |\lambda I_2 - \Pi| &= \lambda^2 = 0 \Longrightarrow \text{The eigenvalue is } \lambda = 0 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 0 \Longrightarrow \text{The eigenvector is } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & \text{ONLY one eigenvector!}\end{aligned}$$

MORE LAGS

$$
\begin{aligned}
\Delta X_t &= \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \varepsilon_t \\
\begin{pmatrix} \Delta X_t \\ \Delta X_{t-1} \end{pmatrix} &= \begin{pmatrix} \alpha\beta' + \Gamma_1 & -\Gamma_1 \\ I_p & -I_p \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \Gamma_1 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \beta' & I_p \\ 0 & -I_p \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\
\Delta \tilde{X}_t &= \tilde{\alpha}\tilde{\beta}' \tilde{X}_{t-1} + \tilde{\varepsilon}_t
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\alpha}_\perp &= \begin{pmatrix} \alpha_\perp \\ -\Gamma_1' \alpha_\perp \end{pmatrix}, \tilde{\beta}_\perp = \begin{pmatrix} \beta_\perp \\ \beta_\perp \end{pmatrix} \\
\tilde{C} &= \tilde{\beta}_\perp (\tilde{\alpha}_\perp' \tilde{\beta}_\perp)^{-1} \tilde{\alpha}_\perp' = \begin{pmatrix} I_p \\ I_p \end{pmatrix} \beta_\perp [\alpha_\perp' (I_p - \Gamma_1) \beta_\perp]^{-1} \alpha_\perp' (I_p, -\Gamma_1) \\
&= \begin{pmatrix} I_p \\ I_p \end{pmatrix} C (I_p, -\Gamma_1)
\end{aligned}
$$

$$
\begin{aligned}
GRT &: \tilde{X}_t = \tilde{C} \sum_{i=1}^{t} \tilde{\varepsilon}_i + \tilde{Y}_t + \tilde{A} \\
X_t &= (I_p, 0)\tilde{X}_t = (I_p, 0)\tilde{C} \begin{pmatrix} I_p \\ 0 \end{pmatrix} \sum_{i=1}^{t} \varepsilon_i + Y_t + A \\
&= C \sum_{i=1}^{t} \varepsilon_i + Y_t + A
\end{aligned}
$$

The coefficients in the expansion of $\tilde{X}_t$ are $\tilde{\Pi}(z)^{-1}$

The coefficients in the expansion of X_t are $(I_p, 0)\tilde{\Pi}(z)^{-1}\begin{pmatrix} I_p \\ 0 \end{pmatrix}$

$$
\begin{aligned}
& (I_p, 0)\tilde{\Pi}(z)^{-1}\begin{pmatrix} I_p \\ 0 \end{pmatrix} \\
= \; & (I_p, 0)\left((1-z)I_{2p} - \tilde{\alpha}\tilde{\beta}' z\right)^{-1}\begin{pmatrix} I_p \\ 0 \end{pmatrix} \\
= \; & (I_p, 0)\left((1-z)I_{2p} - \begin{pmatrix} \alpha\beta' + \Gamma_1 & -\Gamma_1 \\ I_p & -I_p \end{pmatrix} z\right)^{-1}\begin{pmatrix} I_p \\ 0 \end{pmatrix} \\
= \; & (I_p, 0)\begin{pmatrix} (1-z)I_p - \alpha\beta' z - \Gamma_1 z & \Gamma_1 z \\ -I_p z & I_p \end{pmatrix}^{-1}\begin{pmatrix} I_p \\ 0 \end{pmatrix} \\
= \; & ((1-z)I_p - \alpha\beta' z - \Gamma_1 z - (-I_p z)(I_p)^{-1}(\Gamma_1 z))^{-1} \\
= \; & ((1-z)I_p - \alpha\beta' z - \Gamma_1 z(1-z))^{-1} = \Pi(z)^{-1}
\end{aligned}
$$