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This is not an Exam in StatØ3 2006
but the real one will have approximately this format

There are three exercises with 4, 4 and 2 questions respectively.

All 10 questions need to be answered and the grade will be based on an overall evaluation.

Johansen, S. (1996) *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models.* Oxford University Press, Oxford

and relevant additional literature may be used during the exam.

Pen and pencil may be used.

Exercise 1

The uncovered interest rate parity (UIP) states that the interest rates in two countries reflect the economic expectations to the change in exchange rate. Let e_t be the exchange rate and i_t^* and i_t be the domestic and foreign interest rate respectively. The UIP is formulated as

$$E_t^e \Delta e_{t+1} = i_t - i_t^*, \tag{1}$$

where $E_t^e \Delta e_{t+1}$ denotes the economic expectations, given available information at time t, of the future change of the exchange rate.

We shall now translate this into a statement about a cointegrated VAR model as follows: We define

$$X_t = (e_t, i_t, i_t^*)'$$

and assume that data can be described by

$$\Delta X_t = \alpha\beta' X_{t-1} + \Phi \Delta X_{t-1} + \varepsilon_t, \tag{2}$$

where the cointegration rank is $r = 1$, so that α and β are vectors. Let us normalize β as $\beta = (\gamma_1, 1, \gamma_2)'$, so that the parameters $(\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2)$ are identified.

Q.1.1. Find from (2) an expression for

$$E(\Delta e_{t+1} | X_{t-1}, X_{t-2}) \tag{3}$$

in terms of $(\alpha, \gamma_1, \gamma_2, \Phi)$.

The idea of Muth (1960) was to equate the economic expectation (1) with the expectation based upon the statistical model (3).

Q.1.2. Describe the restrictions on the parameters $(\alpha, \gamma_1, \gamma_2, \Phi)$ which express the UIP.

Q.1.3. Write the three equations from (2) under the restrictions imposed by the UIP.

Q.1.4. Describe how you would estimate the model under the assumption of the UIP. How many parameters do you have in the restricted model?

Exercise 2

Consider the equation

$$X_t + X_{t-1} = \varepsilon_t + \mu, t = 1, \ldots, T, \tag{4}$$

where we assume throughout that ε_t are i.i.d. $(0, \Omega)$, and the variables are of dimension p.

Q.2.1. Show that the solution of (4) is

$$X_t = (-1)^t \sum_{i=1}^{t} (-1)^i \varepsilon_i + \frac{1}{2}\mu((-1)^t + 1) + X_0 \tag{5}$$

Now consider the model

$$X_t + X_{t-1} = \alpha\beta' X_{t-1} + \varepsilon_t + \mu, t = 1, \ldots, T$$

where α and β have rank $r < p$.

Q.2.2. Find the equation for $\alpha'_\perp X_t$ and show that it has a solution of the form (5).

Q.2.3. Find an equation for $\beta' X_t$ and find the condition for $\beta' X_t$ to be stationary, expressed in terms of α and β, and find an expression for the expectation of $\beta' X_t$.

Q.2.4. Combine the results from **Q.2.2.** and **Q.2.3.** to find an expression for X_t.

Exercise 3

In the model

$$\Delta X_t = \Pi X_{t-1} + \varepsilon_t, t = 1, \ldots, T$$

and ε_t are i.i.d. $N_p(0, \Omega)$, we want to test that the rank of Π is $r \leq 1$.

The likelihood ratio statistics is given by

$$Q_1 = -2 \log LR\{\text{rank}(\Pi) \leq 1\} = -T \sum_{i=2}^{p} \log(1 - \hat{\lambda}_i),$$

where the $\hat{\lambda}_i$ are the eigenvalues of

$$|\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}| = 0. \tag{6}$$

In the book you find the proof that if the rank is $r = 1$, then

$$Q_1 \xrightarrow{w} tr\{\int_0^1 (dB)B'(\int_0^1 BB'du)^{-1} \int_0^1 B(dB)'\},$$

where B is a standard Brownian motion of dimension $p - 1$.

Q.3.1. Find the limit distribution of the matrices in the expression (6), when the rank is zero, $r = 0$, and the eigenvalues are normalized as $\lambda = \rho T^{-1}$ where $T \to \infty$.

Q.3.2. Use this result to describe the limit distribution of Q_1 when $r = 0$.