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Exercise

This exercise is a guided tour through asymptotic inference for $\hat{\alpha}(\beta)$, in the cointegration model, when β is known. You should apply the Law of Large Numbers for an ergodic and stationary process Z_t, say, which states that if $E(|Z_1|^2) < \infty$, then

$$T^{-1}\sum_{t=1}^{T} Z_t \xrightarrow{P} E(Z_1). \tag{1}$$

You should also check the conditions and apply Theorem B.3 page 241.

Consider therefore the cointegration model

$$\Delta X_t = \alpha\beta' X_{t-1} + \varepsilon_t, t = 1, \ldots, T,$$

where ε_t are i.i.d. $N_p(0, \Omega)$. In the following we assume that β is known and want to make inference in α and Ω. We assume that $r = 1$, so that α and β are vectors, and hence that $\Sigma_{\beta\beta} = E(\beta' X_{t-1} X'_{t-1}\beta)$ is a real number, not a matrix, to simplify the notation. In the probability analysis below we assume that the $I(1)$ condition: $\alpha'_\perp\beta_\perp$ full rank is satisfied Hence the process X_t is $I(1)$ and $(\Delta X_t, \beta' X_{t-1}, \varepsilon_t)$ is stationary and ergodic.

Construction of the estimators

1. Show that for fixed β, the maximum likelihood estimators are

$$\begin{aligned}\hat{\alpha}(\beta) &= S_{01}\beta(\beta' S_{11}\beta)^{-1}\\ \hat{\Omega}(\beta) &= S_{00} - S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}\end{aligned}$$

so that

$$\hat{\alpha}(\beta) - \alpha = S_{\varepsilon 1}\beta(\beta' S_{11}\beta)^{-1}.$$

We have used the notation

$$\begin{aligned}S_{\varepsilon 1}\beta &= T^{-1}\sum_{t=1}^{T} \varepsilon_t X'_{t-1}\beta\\ S_{01}\beta &= T^{-1}\sum_{t=1}^{T} \Delta X_t X'_{t-1}\beta\\ \beta' S_{11}\beta &= T^{-1}\sum_{t=1}^{T} \beta' X_{t-1} X'_{t-1}\beta\end{aligned}$$

Properties of Moments

2. Show that

$$\begin{aligned} E(\varepsilon_t X_{t-1}'\beta|\varepsilon_1,\ldots\varepsilon_{t-1},X_0) &= 0 \\ Cov(\varepsilon_t X_{t-1}'\beta,\varepsilon_s X_{s-1}'\beta|\varepsilon_1,\ldots\varepsilon_{t-1},X_0) &= 0, s>t \\ V(\varepsilon_t X_{t-1}'\beta|\varepsilon_1,\ldots\varepsilon_{t-1},X_0) &= \Omega\beta' X_{t-1}X_{t-1}'\beta \end{aligned}$$

so that

$$\begin{aligned} E(\varepsilon_t X_{t-1}'\beta) &= 0 \\ E(\beta' X_{s-1}\varepsilon_s'\varepsilon_t X_{t-1}'\beta) &= 0, s>t \\ V(\varepsilon_t X_{t-1}'\beta) &= E(\varepsilon_t X_{t-1}'\beta\beta' X_{t-1}\varepsilon_t') = \Omega\Sigma_{\beta\beta} \end{aligned}$$

Consistency of $\hat{\alpha}(\beta)$

3. Show that it follows that

$$\begin{aligned} E(S_{\varepsilon 1}\beta) &= 0 \\ E(S_{\varepsilon 1}\beta\beta' S_{1\varepsilon}) &= T^{-1}\Sigma_{\beta\beta}\Omega \to 0 \end{aligned}$$

and hence that

$$S_{\varepsilon 1}\beta \xrightarrow{P} 0.$$

This is also a consequence of (1).

4. Show that it follows from (1) that

$$\beta' S_{11}\beta \xrightarrow{P} E(\beta' S_{11}\beta) = \Sigma_{\beta\beta},$$

so that $\hat{\alpha}(\beta)$ is consistent:

$$\hat{\alpha}(\beta) \xrightarrow{P} \alpha.$$

Consistency of $\hat{\Omega}(\beta)$

5. Show that

$$S_{00} = \alpha\beta' S_{11}\beta\alpha' + \alpha\beta' S_{1\varepsilon} + S_{\varepsilon 1}\beta\alpha' + S_{\varepsilon\varepsilon}$$

6. Find the probability limit of S_{00} and $S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}$ and use those to show that $\hat{\Omega}(\beta)$ is consistent.

Asymptotic distribution of $\hat{\alpha}(\beta)$

We have from above that

$$\sqrt{T}(\hat{\alpha}(\beta)-\alpha) = \sqrt{T}S_{\varepsilon 1}\beta(\beta' S_{11}\beta)^{-1}.$$

Define the process (see page 240)

$$X_{T,t} = \frac{1}{\sqrt{T}}\Omega^{-1/2}\varepsilon_t\frac{X_{t-1}'\beta}{\sqrt{\Sigma_{\beta\beta}}}, t=1,\ldots,T$$

and the $\sigma-$field

$$F_{T,t} = \sigma(X_0, \varepsilon_1, \ldots, \varepsilon_t)$$

7. Show that

$$E(X_{T,t}|F_{T,t-1}) = 0$$

8. Show that

$$V(X_{T,t}|F_{T,t-1}) = T^{-1}\frac{\beta' X_{t-1}X_{t-1}'\beta}{\Sigma_{\beta\beta}}E(\Omega^{-1/2}\varepsilon_t\varepsilon_t'\Omega^{-1/2}) = T^{-1}\frac{\beta' X_{t-1}X_{t-1}'\beta}{\Sigma_{\beta\beta}}I_p$$

9. Apply these results to show that Conditions B.1 and B.3 are satisfied.

Next we turn to Condition B.2. We define the squared norm by

$$|X_{T,t}|^2 = X_{T,t}'X_{T,t} = \varepsilon_t'\Omega^{-1}\varepsilon_t\frac{\beta' X_{t-1}X_{t-1}'\beta}{T\Sigma_{\beta\beta}}.$$

10. Use Chebychev's inequality to show that

$$\begin{aligned} E(|X_{T,t}|^2 I\{|X_{T,t}| &> \delta\}|F_{T,t-1}) \leq \frac{E(|X_{T,t}|^4|F_{T,t-1})}{\delta^2} \\ &= \frac{(\beta' X_{t-1}X_{t-1}'\beta)^2}{\delta^2 T^2\Sigma_{\beta\beta}^2}E(\varepsilon_t'\Omega^{-1}\varepsilon_t)^2 \end{aligned}$$

and use that to show that Condition B.3 is satisfied.

There is a simpler argument. Can you see that?

Applying Theorem B.3 we then get that

$$\frac{1}{\sqrt{T}}\Omega^{-1/2}S_{\varepsilon 1}\beta\frac{1}{\sqrt{\Sigma_{\beta\beta}}} = \sum_{t=1}^{T} X_{T,t} \xrightarrow{w} N_p(0, I_p),$$

and hence that

$$T^{-1/2}S_{\varepsilon 1}\beta \xrightarrow{w} N_p(0, \Omega\Sigma_{\beta\beta})$$

and finally

$$T^{1/2}(\hat{\alpha}(\beta) - \alpha) \xrightarrow{w} N_p(0, \Omega\Sigma_{\beta\beta}^{-1}).$$

For your information the same result holds even if $r > 1$ and β is estimated, see Theorem 13.3 last line. Note that then $\hat{\alpha}(\beta)$ is a matrix and the expression for the variance involves the Kronecker product, that is,

$$T^{1/2}(\hat{\alpha}(\beta) - \alpha) \xrightarrow{w} N_p(0, \Sigma_{\beta\beta}^{-1} \otimes \Omega).$$

This notation means that for $\lambda \in R^p$ and $\eta \in R^r$ we have the limits

$$T^{1/2}\lambda'(\hat{\alpha}(\beta) - \alpha)\eta \xrightarrow{w} N_1(0, \eta'\Sigma_{\beta\beta}\eta\lambda'\Omega\lambda).$$

Please note that in the book I use the notation

$$T^{1/2}(\hat{\alpha}(\beta) - \alpha) \xrightarrow{w} N_p(0, \Omega \otimes \Sigma_{\beta\beta}^{-1}),$$

for the same result.