1 Exercise 4.7 updated

Let X_t be given by the vector autoregressive model

$$\Delta X_t = \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \Gamma_2 \Delta X_{t-2} + \varepsilon_t, \tag{1}$$

where ε_t are i.i.d. $N_p(0, \Omega)$ and α and β are $p \times r$ of rank r.

1. Show that

$$\Delta \begin{pmatrix} X_t \\ X_{t-1} \\ X_{t-2} \end{pmatrix} = \begin{pmatrix} \alpha\beta' + \Gamma_1 & -\Gamma_1 + \Gamma_2 & -\Gamma_2 \\ I_p & -I_p & 0 \\ 0 & I_p & -I_p \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ X_{t-3} \end{pmatrix} + \begin{pmatrix} I_p \\ 0 \\ 0 \end{pmatrix} \varepsilon_t$$

so that the stacked process $\tilde{X}_t = (X_t, X_{t-1}, X_{t-2})'$ satisfies the error correction model

$$\Delta \tilde{X}_t = \tilde{\Pi} \tilde{X}_{t-1} + B\varepsilon_t$$

where

$$\tilde{\Pi} = \begin{pmatrix} \alpha\beta' + \Gamma_1 & -\Gamma_1 + \Gamma_2 & -\Gamma_2 \\ I_p & -I_p & 0 \\ 0 & I_p & -I_p \end{pmatrix} \text{ and } B = \begin{pmatrix} I_p \\ 0 \\ 0 \end{pmatrix}.$$

2. Show that $\tilde{\Pi} = \tilde{\alpha}\tilde{\beta}'$ where

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \Gamma_1 & \Gamma_2 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix} \text{ and } \tilde{\beta}' = \begin{pmatrix} \beta' & 0 & 0 \\ I_p & -I_p & 0 \\ 0 & I_p & -I_p \end{pmatrix}$$

3. Show that

$$\tilde{\alpha}_\perp = \begin{pmatrix} \alpha_\perp \\ -\Gamma_1'\alpha_\perp \\ -\Gamma_2'\alpha_\perp \end{pmatrix}, \text{ and } \tilde{\beta}_\perp = \begin{pmatrix} \beta_\perp \\ \beta_\perp \\ \beta_\perp \end{pmatrix}$$

and that

$$\tilde{C} = \tilde{\beta}_\perp(\tilde{\alpha}_\perp'\tilde{\beta}_\perp)^{-1}\tilde{\alpha}_\perp' = \begin{pmatrix} \beta_\perp \\ \beta_\perp \\ \beta_\perp \end{pmatrix} \left(\alpha_\perp'(I - \Gamma_1 - \Gamma_2)\beta_\perp\right)^{-1} (\alpha_\perp', -\alpha_\perp'\Gamma_1, -\alpha_\perp'\Gamma_2)$$

Next define

$$\Gamma = I_p - \Gamma_1 - \Gamma_2 \text{ and } C = \beta_\perp \left(\alpha_\perp'\Gamma\beta_\perp\right)^{-1} \alpha_\perp',$$

and show that

$$\tilde{C} = \begin{pmatrix} I_p \\ I_p \\ I_p \end{pmatrix} C(I_p, -\Gamma_1, -\Gamma_2).$$

4. Finally derive the Granger Representation Theorem for (1) from the Granger Representation Theorem for the model with one lag.