Let X be a locally compact Hausdorff space. A subset $A \subset X$ is called a *Borel set* if it belongs to the *Borel algebra* $B(X)$, which by definition is the smallest σ-algebra containing all open subsets of X. A *Borel measure* on X is a measure μ which is defined on $B(X)$ and satisfies

$$\forall K \subset X \text{ compact: } \mu(K) < \infty.$$

A Borel measure is said to be *regular* if

$$\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}$$

for all Borel sets E.

A *Radon measure* is by definition a regular Borel measure. It is a theorem that if X is second countable, then every Borel measure is regular. For example Lebesgue measure on $\mathbb{R}^n$ is easily seen to be a Borel measure, hence it is a Radon measure.

The main result in the theory of Radon measures is the following, which is called the *Riesz representation theorem.* We denote by $C_c(X)$ the vector space of continuous functions of compact support. A *positive linear functional* on $C_c(X)$ is a linear map $\ell : C_c(X) \to \mathbb{C}$ such that $\ell(f) \geq 0$ whenever $f \geq 0$.

Theorem. *Let ℓ be a positive linear functional on $C_c(X)$. Then there exists a unique Radon measure μ on X such that*

$$\ell(f) = \int_X f \, d\mu$$

for all $f \in C_c(X)$.

By combining this result with Lemma 19.2 we conclude that for every positive density ω on a differentiable manifold X, there exists a unique Radon measure μ such that

$$\int_X f \, d\mu = \int_X f\omega$$

for all $f \in C_c(X)$.

The present definitions are taken from J. Elstrodt, *Maß- und Integrationstheorie.* There are many other presentations of the Riesz representation theorem, for example in W. Rudin, *Real and Complex Analysis.*

HS