Let G be a Lie group, and H a subgroup. According to Theorem 2.16 the following conditions are equivalent:

(1) H is closed.
(2) H is a submanifold.

If (1)–(2) hold, then H is a Lie subgroup of G (see Def 7.1). These are the *good* Lie subgroups, but unfortunately there exist other Lie subgroups. An example is given in Ex 7.5. This note describes the same example but simplified slightly. The reason that one defines the notion of a Lie subgroup so that it includes *bad* guys which are not submanifolds, is that otherwise the beautiful bijection in Thm 7.11 would fail to hold.

First we consider the simplest case where $G = (\mathbb{R}, +)$.

Lemma 1. *Let $H \subset \mathbb{R}$ be a subgroup $\neq \{0\}$. Then H is either discrete in $\mathbb{R}$ or dense in $\mathbb{R}$. If it is discrete then $H = \mathbb{Z}a = \{ma \mid m \in \mathbb{Z}\}$ for some $a > 0$.*

In the discrete case, H is a 0-dimensional closed Lie subgroup of $\mathbb{R}$. In the other case (which includes for example $H = \mathbb{Q}$), if $H \neq \mathbb{R}$ then H is not locally Euclidean when equipped with the topology from $\mathbb{R}$, and hence it is not a Lie subgroup. However, if equipped with the discrete topology, then H is a 0-dimensional non-closed Lie subgroup.

Proof. Since H is non-trivial it contains numbers > 0. Let $a = \inf\{x \in H \mid x > 0\}$.

Assume first $a > 0$. If $a \notin H$ then there exists a decreasing sequence in H converging to a. The differences between the elements of the sequence belong to H, hence are $> a$, which contradicts the convergence. Hence $a \in H$, and hence $\mathbb{Z}a \subset H$. Suppose $b \in H \setminus \mathbb{Z}a$ and choose $n \in \mathbb{Z}$ such that $na < b < (n+1)a$. Then $b - na \in H$ and $0 < b - na < a$, again a contradiction. Hence $H = \mathbb{Z}a$.

Assume next that $a = 0$. Let $c < d$ be arbitrary in $\mathbb{R}$. Then there exists $x \in H$ with $0 < x < d - c$. We choose $n \in \mathbb{Z}$ such that $nx \leq c < (n+1)x$, then $(n+1)x \in]c, d[$. Hence H meets every open interval, and it is dense. □

The non-closed Lie subgroups that we have found in $\mathbb{R}$ are all disconnected. To find an example which is connected, we must pass to higher dimension.

The example that we shall consider is based on the following lemma. For $a \in \mathbb{R}$ we denote

$$\mathbb{Z}a + \mathbb{Z} = \{ma + n \mid m, n \in \mathbb{Z}\} \subset \mathbb{R}.$$

It is clear that this is a proper subset of $\mathbb{R}$. In fact, if a is rational then $\mathbb{Z}a + \mathbb{Z}$ consists only of rational numbers, and if a is irrational (that is, $a \notin \mathbb{Q}$) then $\mathbb{Z}a + \mathbb{Z}$ does not include any element from $\mathbb{Q} \setminus \mathbb{Z}$.

Lemma 2. *Let $a \in \mathbb{R}$ be irrational. Then $\mathbb{Z}a + \mathbb{Z}$ is dense in $\mathbb{R}$.*

Proof. Otherwise there would exist $b \in \mathbb{R}$ such that $\mathbb{Z}a + \mathbb{Z} = \mathbb{Z}b$, according to Lemma 1. The inclusion $\mathbb{Z} \subset \mathbb{Z}b$ implies that b is rational, which then contradicts the inclusion $\mathbb{Z}a \subset \mathbb{Z}b$. □

Now let $G = \mathbb{T}^2 = S^1 \times S^1$, viewed as a submanifold of $\mathbb{R}^4 \simeq \mathbb{C}^2$. Let

$$\varphi\colon \mathbb{R} \to G, \qquad \varphi(t) = (e^{2\pi it}, e^{2\pi iat}),$$

for some fixed non-zero number $a \in \mathbb{R}$. Then φ is a smooth homomorphism.

The kernel of φ is $\{t \in \mathbb{R} \mid e^{2\pi it} = e^{2\pi iat} = 1\} = \mathbb{Z} \cap \mathbb{Z}a$, and hence φ is injective if and only if a is irrational. In that case, we give $H = \varphi(\mathbb{R})$ the manifold structure for which φ is a diffeomorphism, and then it is a 1-dimensional Lie subgroup of G.

Lemma 3. *Assume $a \notin \mathbb{Q}$. Then $H = \varphi(\mathbb{R})$ is a proper, dense, connected Lie subgroup of G. In particular, it is not closed.*

Proof. Choose $x \in \mathbb{R}$ outside of $\mathbb{Z}a + \mathbb{Z}$. Assume $(1, e^{2\pi ix}) \in H$. The there exists $t \in \mathbb{R}$ such that $1 = e^{2\pi it}$ and $e^{2\pi ix} = e^{2\pi iat}$. Hence $t \in \mathbb{Z}$ and $x = at$ modulo $\mathbb{Z}$, which contradicts the choice of x. Hence $(1, e^{2\pi ix}) \notin H$ and H is proper in G.

We claim that H is dense in G. To verify this, let $(e^{2\pi iu}, e^{2\pi iv}) \in G$ be arbitrary. We want to approximate this element by elements from H. By Lemma 2 there exist sequences m_k and n_k of integers such that $m_k a + n_k \to v - au \in \mathbb{R}$ as $k \to \infty$. Then

$$\varphi(m_k) = (e^{2\pi im_k}, e^{2\pi iam_k}) = (1, e^{2\pi i(am_k+n_k)}) \to (1, e^{2\pi i(v-au)})$$

and hence

$$\varphi(u + m_k) = \varphi(u)\varphi(m_k) \ \to \ \varphi(u)(1, e^{2\pi i(v-au)}) = (e^{2\pi iu}, e^{2\pi iv}),$$

which is the required approximation. Hence H is dense. Since it is proper, it cannot be closed. $\square$

Finally, let us call a Lie subgroup of G *ugly*, if it has uncountably many connected components. For example, in $G = \mathbb{R}$ with standard structure, the subgroup $H = G$ with the discrete topology is ugly.

Correction. *In Lemmas 7.7 and 7.10 one has to assume that H is not ugly.*

For the proof we need the following lemma. Recall that a topological space is called *second countable* if it has a countable base. Every subset of $\mathbb{R}^n$ (with the relative topology) is second countable.

Lemma 4. *The identity component G_e of a Lie group is second countable.*

Proof. Since $\mathfrak{g}$ is second countable, then so is its k-fold image $\exp(\mathfrak{g}) \cdots \exp(\mathfrak{g})$ in G for each $k \in \mathbb{N}$. Now G_e is the countable union over $k \in \mathbb{N}$ of these sets. $\square$

It follows that a Lie group is second countable if and only if it has at most countably many components.

For the proof of Lemma 7.7, one follows the notes until the paragraph which begins by "The map ...". From there one can replace by the following:

We may assume Ω, and hence U, is connected. It is easy to see that

$$\varphi(]-\epsilon, \epsilon[\times\Omega) \cap H = \cup_{t\in T} \exp(tX)U$$

where $T = \{t \in]-\epsilon, \epsilon[\mid \exp tX \in H\}$. Since φ is injective, this is a disjoint union of non-empty open connected sets in H. Since H is second countable this union must be countable, hence T is strictly smaller than $]-\epsilon, \epsilon[$. Thus $X \notin V$. $\square$

HS