1. The flow of a vector field

Let $M$ be a differentiable manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field on $M$. By definition an integral curve of $X$ is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, such that

$$\gamma'(t) = X(\gamma(t))$$

for all $t \in I$. If an initial value $t_0 \in I$ has been selected, then $\gamma(t_0)$ is said to be the initial point of the curve. One notices that if $\gamma : I \to M$ is an integral curve and $c \in \mathbb{R}$ an arbitrary constant, then the shifted curve $s \mapsto \gamma(s - c)$ with domain $I + c := \{t + c \mid t \in I\}$ is again an integral curve. For this reason one often assumes that $0 \in I$ and selects $t_0 = 0$ as the initial value.

In this note we shall discuss the existence and uniqueness of integral curves.

**Theorem 1.** Let $X \in \mathfrak{X}(M)$ and $p \in M$ be given. There exists a unique open interval $I_p \subset \mathbb{R}$ containing 0, and a unique integral curve $\alpha : I_p \to M$ with initial point $\alpha(0) = p$, such that if $\nu : J \to M$ is an arbitrary integral curve with $0 \in J$ and initial point $\nu(0) = p$, then $J \subset I_p$ and $\nu = \alpha|_J$.

The integral curve $\alpha = \alpha_p : I_p \to M$ is called the maximal integral curve with initial point $p$.

**Proof.** Let $\sigma : U \to M$ be a chart and $\gamma : I \to M$ a smooth curve, then we can express (1) in local coordinates as follows. We assume that $\gamma(I)$ (or a subset of it) is contained in $\sigma(U)$ and write the coordinate expression for $\gamma$

$$\gamma = \sigma \circ \mu$$

with a smooth map $\mu : I \to U$ (or $\gamma^{-1}(\sigma(U) \cap \gamma(I)) \to U$). Likewise we can write $X$ in the local coordinates as

$$X(\sigma(u)) = \sum_{i=1}^{m} a_i(u) d\sigma_u(e_i)$$

with some coordinate functions $a_i \in C^\infty(U)$. Then (1) is equivalent with the system of ordinary differential equations

$$\frac{dx_i}{dt} = a_i(x), \quad (i = 1, \ldots, m)$$

where $x$ maps some neighborhood of $t_0$ into $U$. More precisely, $\gamma$ satisfies (1) if and only if $x = \mu$ is a solution to (2). Furthermore, the assumption that a point $q = \sigma(u_0) \in \sigma(U)$ is the initial point $\gamma(t_0)$ of $\gamma$ corresponds to the initial condition $x(t_0) = u_0$ for (2). Using the fundamental theorem of existence and uniqueness for systems of first order ordinary differential equations (see for example A. Knapp, Basic Real Analysis, Theorems 4.1-4.2), one can conclude the following:
Lemma. Let \( t_0 \in \mathbb{R} \) and \( q \in M \) be arbitrary. There exists an open interval \( I \subset \mathbb{R} \) and an integral curve \( \gamma : I \to M \) of \( X \) with \( t_0 \in I \) and initial point \( \gamma(t_0) = q \). If \( \nu : J \to M \) is a second integral curve with \( t_0 \in J \) and initial point \( \nu(t_0) = q \), then \( \nu = \gamma \) on \( I \cap J \).

We can now prove that any two integral curves \( \alpha_i : I_i \to M \), \( i = 1, 2 \), with a common initial point \( \alpha_1(0) = \alpha_2(0) \), coincide on the intersection \( I_1 \cap I_2 \). By continuity of the curves the subset \( L \) of \( I_1 \cap I_2 \), where the curves coincide, is closed.

On the other hand, this subset \( L \) is also open, since for any \( t_0 \in L \), say with \( \alpha_1(t_0) = \alpha_2(t_0) = q \), the lemma implies that on a neighborhood of \( t_0 \) both \( \alpha_1 \) and \( \alpha_2 \) are equal to a common integral curve \( \gamma \) with initial point \( \gamma(t_0) = q \). Since \( I_1 \cap I_2 \) is connected, it follows that \( L = I_1 \cap I_2 \).

To prove the theorem we let \( I_p \subset \mathbb{R} \) be the union of all the open intervals \( I \) which contain 0 and which are domains of an integral curve \( \gamma \) with the initial point \( \gamma(0) = p \). The lemma ensures that this is not an empty union. Let \( t \in I_p \). Then it follows that all the integral curves \( \gamma : I \to M \) with \( \gamma(0) = p \) and with \( t \in I \), have the same value \( \gamma(t) \). We define \( \alpha_p(t) \) to be this common value. All the statements in the theorem now follow easily. □

Let

\[
\Omega = \{(t, p) \in \mathbb{R} \times M \mid t \in I_p\}
\]

and define \( \Phi : \Omega \to M \) by

\[
\Phi(t, p) = \alpha_p(t)
\]

for \( t \in I_p \), then \( \Phi \) is called the flow of the vector field. We have seen above that \( t \mapsto \Phi(t, p) \) is smooth for each \( p \in M \). In fact, \( \Phi(t, p) \) also depends smoothly on the initial point \( p \). This is a consequence of a refined version of the fundamental existence theorem (Knapp, Theorem 4.3) in which the dependence of the solution on the initial condition is accounted for. The conclusion is as follows:

**Theorem 2.** The set \( \Omega \subset \mathbb{R} \times M \) is open and the flow \( \Phi : \Omega \to M \) is smooth.

Finally, we shall need to consider vector fields \( X \) which depend on an extra parameter \( \lambda \). Let \( \Lambda \) be a differentiable manifold, and assume that for each \( \lambda \in \Lambda \), a smooth vector field \( X_\lambda \in \mathfrak{X}(M) \) is given. Furthermore, we assume that \( X_\lambda(p) \) depends smoothly on the pair \((p, \lambda) \in M \times \Lambda\). For each \( \lambda \in \Lambda \) we denote by \( \Phi_\lambda : \Omega_\lambda \to M \) the flow of \( X_\lambda \). Then

\[
\{(t, p, \lambda) \in \mathbb{R} \times M \times \Lambda \mid (t, p) \in \Omega_\lambda\}
\]

is open and \( \Phi_\lambda(t, p) \) depends smoothly on \((t, p, \lambda) \) in this set. This can be seen by considering the trivially extended vector field \((p, \lambda) \mapsto (X_\lambda(p), 0) \) on \( M \times \Lambda \). Its integral curves are constant along the last coordinate and hence the flow of this vector field is given by \( \Phi(t, p, \lambda) = (\Phi_\lambda(t, p), \lambda) \). Theorem 2 implies that it depends smoothly also on the additional parameter, and hence so does the original flow \( \Phi_\lambda(t, p) \).

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