1. The flow of a vector field

Let M be a differentiable manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field on M. By definition an *integral curve* of X is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, such that

$$\gamma'(t) = X(\gamma(t)) \tag{1}$$

for all $t \in I$. If an initial value $t_0 \in I$ has been selected, then $\gamma(t_0)$ is said to be the *initial point* of the curve. One notices that if $\gamma : I \to M$ is an integral curve and $c \in \mathbb{R}$ an arbitrary constant, then the shifted curve $s \mapsto \gamma(s-c)$ with domain $I + c := \{t + c \mid t \in I\}$ is again an integral curve. For this reason one often assumes that $0 \in I$ and selects $t_0 = 0$ as the initial value.

In this note we shall discuss the existence and uniqueness of integral curves.

Theorem 1. *Let $X \in \mathfrak{X}(M)$ and $p \in M$ be given. There exists a unique open interval $I_p \subset \mathbb{R}$ containing* 0*, and a unique integral curve $\alpha : I_p \to M$ with initial point $\alpha(0) = p$, such that if $\nu : J \to M$ is an arbitrary integral curve with $0 \in J$ and initial point $\nu(0) = p$, then $J \subset I_p$ and $\nu = \alpha|_J$.*

The integral curve $\alpha = \alpha_p : I_p \to M$ is called the *maximal integral curve* with initial point p.

Proof. Let $\sigma: U \to M$ be a chart and $\gamma : I \to M$ a smooth curve, then we can express (1) in local coordinates as follows. We assume that $\gamma(I)$ (or a subset of it) is contained in $\sigma(U)$ and write the coordinate expression for γ

$$\gamma = \sigma \circ \mu$$

with a smooth map $\mu : I \to U$ (or $\gamma^{-1}(\sigma(U) \cap \gamma(I)) \to U$). Likewise we can write X in the local coordinates as

$$X(\sigma(u)) = \sum_{i=1}^{m} a_i(u) d\sigma_u(e_i)$$

with some coordinate functions $a_i \in C^\infty(U)$. Then (1) is equivalent with the system of ordinary differential equations

$$\frac{dx_i}{dt} = a_i(x), \quad (i = 1, \dots, m) \tag{2}$$

where x maps some neighborhood of t_0 into U. More precisely, γ satisfies (1) if and only if $x = \mu$ is a solution to (2). Furthermore, the assumption that a point $q = \sigma(u_0) \in \sigma(U)$ is the initial point $\gamma(t_0)$ of γ corresponds to the initial condition $x(t_0) = u_0$ for (2). Using the fundamental theorem of existence and uniqueness for systems of first order ordinary differential equations (see for example A. Knapp, Basic Real Analysis, Theorems 4.1-4.2), one can conclude the following:

Lemma. Let $t_0 \in \mathbb{R}$ and $q \in M$ be arbitrary. There exists an open interval $I \subset \mathbb{R}$ and an integral curve $\gamma : I \to M$ of X with $t_0 \in I$ and initial point $\gamma(t_0) = q$. If $\nu : J \to M$ is a second integral curve with $t_0 \in J$ and initial point $\nu(t_0) = q$, then $\nu = \gamma$ on $I \cap J$.

We can now prove that any two integral curves $\alpha_i : I_i \to M$, $i = 1, 2$, with a common initial point $\alpha_1(0) = \alpha_2(0)$, coincide on the intersection $I_1 \cap I_2$. By continuity of the curves the subset L of $I_1 \cap I_2$, where the curves coincide, is closed. On the other hand, this subset L is also open, since for any $t_0 \in L$, say with $\alpha_1(t_0) = \alpha_2(t_0) = q$, the lemma implies that on a neighborhood of t_0 both α_1 and α_2 are equal to a common integral curve γ with initial point $\gamma(t_0) = q$. Since $I_1 \cap I_2$ is connected, it follows that $L = I_1 \cap I_2$.

To prove the theorem we let $I_p \subset \mathbb{R}$ be the union of all the open intervals I which contain 0 and which are domains of an integral curve γ with the initial point $\gamma(0) = p$. The lemma ensures that this is not an empty union. Let $t \in I_p$. Then it follows that all the integral curves $\gamma : I \to M$ with $\gamma(0) = p$ and with $t \in I$, have the same value $\gamma(t)$. We define $\alpha_p(t)$ to be this common value. All the statements in the theorem now follow easily. $\square$

Let

$$\Omega = \{(t, p) \in \mathbb{R} \times M \mid t \in I_p\}$$

and define $\Phi : \Omega \to M$ by

$$\Phi(t, p) = \alpha_p(t)$$

for $t \in I_p$, then Φ is called the *flow* of the vector field. We have seen above that $t \mapsto \Phi(t, p)$ is smooth for each $p \in M$. In fact, $\Phi(t, p)$ also depends smoothly on the initial point p. This is a consequence of a refined version of the fundamental existence theorem (Knapp, Theorem 4.3) in which the dependence of the solution on the initial condition is accounted for. The conclusion is as follows:

Theorem 2. *The set $\Omega \subset \mathbb{R} \times M$ is open and the flow $\Phi : \Omega \to M$ is smooth.*

Finally, we shall need to consider vector fields X which depend on an extra parameter λ. Let Λ be a differentiable manifold, and assume that for each $\lambda \in \Lambda$, a smooth vector field $X_\lambda \in \mathfrak{X}(M)$ is given. Furthermore, we assume that $X_\lambda(p)$ depends smoothly on the pair $(p, \lambda) \in M \times \Lambda$. For each $\lambda \in \Lambda$ we denote by $\Phi_\lambda : \Omega_\lambda \to M$ the flow of X_λ. Then

$$\{(t, p, \lambda) \in \mathbb{R} \times M \times \Lambda \mid (t, p) \in \Omega_\lambda\}$$

is open and $\Phi_\lambda(t, p)$ depends smoothly on (t, p, λ) in this set. This can be seen by considering the trivially extended vector field $(p, \lambda) \mapsto (X_\lambda(p), 0)$ on $M \times \Lambda$. Its integral curves are constant along the last coordinate and hence the flow of this vector field is given by $\Phi(t, p, \lambda) = (\Phi_\lambda(t, p), \lambda)$. Theorem 2 implies that it depends smoothly also on the additional parameter, and hence so does the original flow $\Phi_\lambda(t, p)$.

HS