Let G be a Lie group with Lie algebra $\mathfrak{g}$. Recall from Geometry 2 that for a pair $X, Y \in \mathfrak{g}$ we define the Lie bracket $[X, Y] \in \mathfrak{g}$ by the commutator rule

$$[X,Y]f := X(Yf) - Y(Xf) \tag{1}$$

for all functions $f \in C^\infty(G)$.

In the notes by Erik van den Ban the element $[X, Y]$ is defined differently (see Definition 4.8). Recall that for $g \in G$ one defines

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$$

as the differential at e of the conjugation map $x \mapsto gxg^{-1}$ from G to G, and

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

as the differential at e of

$$\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}).$$

Finally one defines $[X, Y] = \mathrm{ad}(X)Y$.

It is the purpose of this note to show that the two definitions of $[X, Y]$ agree. Reserving the notation $[X, Y]$ for the element defined by (1), we want to prove

Theorem. *The relation* $[X, Y] = \mathrm{ad}(X)Y$ *holds for all* $X, Y \in \mathfrak{g}$.

Proof. The following lemma will be used several times.

Lemma. *Let X be a left invariant vector field on G. Then $X(g)$ is the derivative at $t = 0$ of the curve $t \mapsto g\exp(tX)$, that is,*

$$Xf(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp tX) \tag{2}$$

for $g \in G$ and $f \in C^\infty(G)$.

Proof. For $g = e$ this follows directly from Lemma 3.6(a). The general case is reduced to $g = e$ by means of the left invariance. □

By consecutive applications of (2) we derive

$$\begin{aligned}
&([X,Y]f)(g) \\
&= \frac{d}{dt}\Big|_{t=0} Yf(g\exp tX) - \frac{d}{ds}\Big|_{s=0} Xf(g\exp sY) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(g\exp tX \exp sY) - \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f(g\exp sY \exp tX).
\end{aligned}$$

We change the order of the differentiations in the first term and the sign of t in the second term. Then

$$([X,Y]f)(g) = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\big(f(g\exp tX\exp sY) + f(g\exp sY\exp -tX)\big)$$

Next (with s fixed) we use the rule

$$\frac{d}{dt}\Big|_{t=0}\big(F(t,0)+F(0,t)\big) = \frac{d}{dt}\Big|_{t=0}F(t,t)$$

on the C^∞-function $F(x,y) = f(g\exp xX\exp sY\exp -yX)$, and conclude

$$([X,Y]f)(g) = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}f(g\exp tX\exp sY\exp -tX).$$

By Lemma 4.3

$$\exp tX\exp sY\exp -tX = \exp(\mathrm{Ad}(\exp tX)sY)$$

and hence (after another change of the order of differentiations)

$$([X,Y]f)(g) = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}f(g\exp(s\,\mathrm{Ad}(\exp tX)Y)).$$

Using (2) again, we conclude

$$([X,Y]f)(g) = \frac{d}{dt}\Big|_{t=0}\Big\{\Big(\big(\mathrm{Ad}(\exp tX)Y\big)f\Big)(g)\Big\}.$$

The map $Z \mapsto Zf(g)$ is linear, and hence

$$\frac{d}{dt}\Big|_{t=0}\Big\{\Big(\big(\mathrm{Ad}(\exp tX)Y\big)f\Big)(g)\Big\} = \left(\Big(\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\exp tX)Y\Big)f\right)(g)$$

where the differentiation now takes place in $\mathfrak{g}$, before the application to f. By Definition 4.5 we have

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\exp tX)Y = \mathrm{ad}(X)Y$$

in $\mathfrak{g}$, and hence we finally infer

$$([X,Y]f)(g) = \big((\mathrm{ad}(X)Y)f\big)(g).$$

Since g and f were arbitrary, we conclude that $[X,Y] = \mathrm{ad}(X)Y$. $\square$

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