GEOM2 TEST, 2012-13. SOLUTIONS

1 Solve Exercise 16 from Chapter 1. Let $\mathcal{S} = M \setminus \{P, Q\}$ and define $F : \mathcal{S} \to \mathcal{S}$ by $F(x, y, z) = (-x, y, -z)$. Why is F smooth?

Let $a = (0, 0, 1)$ and $b = (0, 0, -1)$. Determine the tangent spaces $T_a\mathcal{S}$ and $T_b\mathcal{S}$ and the differential dF_a.

Solution

P is isolated: If $y < 0$ then $x^2 = yz^2$ implies $x = z = 0$ and then $y = -1$ follows from $x^2 + y^2 + z^2 = 1$. Hence P is the only point in M from the open set $\{(x, y, z) \in \mathbb{R}^3 \mid y < 0\}$.

Let $Q = (0, 1, 0)$. Define $f(x, y, z) = (x^2 + y^2 + z^2, x^2 - yz^2)$, then M is the level set of the equation $f(x, y, z) = (1, 0)$ and

$$Df(x, y, z) = \begin{pmatrix} 2x & 2y & 2z \\ 2x & -z^2 & -2yz \end{pmatrix}.$$

By Thm. 1.6 we can establish that $\mathcal{S}$ is a manifold in $\mathbb{R}^3$ by showing that Df has rank 2 for all $(x, y, z) \in M \setminus \{P, Q\}$.

Let $(x, y, z) \in M \setminus \{P, Q\}$. If $z = 0$ then $x^2 = yz^2$ implies $x = 0$ and then $x^2 + y^2 + z^2 = 1$ implies $y = \pm 1$, which was excluded. It follows that $z \neq 0$. As $y \geq 0$ it follows from

$$\det \begin{pmatrix} 2x & 2y \\ 2x & -z^2 \end{pmatrix} = -2x(z^2 + 2y) \neq 0,$$

that Df has rank 2 if $x \neq 0$. On the other hand, if $x = 0$ then $x^2 = yz^2$ implies $y = 0$. In this case we also see that

$$Df(0, 0, z) = \begin{pmatrix} 0 & 0 & 2z \\ 0 & -z^2 & 0 \end{pmatrix}$$

has rank 2.

F is smooth: Note that $F(\mathcal{S}) \subset \mathcal{S}$. Now F is the restriction to $\mathcal{S}$ of the smooth map $(x, y, z) \mapsto (-x, y, -z)$ of $\mathbb{R}^3$ to itself. Hence F is smooth by Definition 2.6.1.

Tangent spaces: It is clear that $a, b \in \mathcal{S}$ and $F(a) = b$. We use Example 3.2.2. As

$$Df(a) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

it follows that $T_a\mathcal{S} = \mathbb{R}e_1$. Similarly we find $T_b\mathcal{S} = \mathbb{R}e_1$.

The differential dF_a is determined by Lemma 3.8.2: It is the restriction to $T_a\mathcal{S}$ of the linear map $\mathbb{R}^3 \to \mathbb{R}^3$ with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

hence $dF_a(v) = -v$ for all $v \in T_a\mathcal{S} = \mathbb{R}e_1$.

2 Let X be a topological space and let $Y \subset X$. Let U and V be subsets of Y. Show that if U is open in the topology of X and V is open in the relative topology of Y, then $U \cap V$ is open in the topology of X.

Solution

If V is relatively open in Y, then $V = Y \cap W$ where $W \subset X$ is open. Then

$$U \cap V = U \cap Y \cap W = U \cap W,$$

which is open when U is open.

3 Let $f : M \to N$ be a smooth map between differentiable manifolds. Show that if f is submersive at $p \in M$ then $f(p)$ is interior in $f(M)$. (Hint: use Theorem 4.3.2).

Solution

Let $\sigma : U \to M$ and $\tau : V \to N$ be as in Theorem 4.3.2, so that

$$f(\sigma(x)) = \tau(\pi(x)), \quad (x \in U).$$

Since U is open and contains 0, there exist open sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^{m-n}$ such that $0 \in A \times B \subset U$. Then $0 \in A = \pi(A \times B)$ and hence

$$f(p) \in f(\sigma(A \times B)) = \tau(\pi(A \times B)) = \tau(A).$$

Since τ is a homeomorphism, $\tau(A)$ is open in N, hence $f(\sigma(A \times B))$ is a neighborhood of $f(p)$ in $f(M)$.

4 Let M be an m-dimensional differentiable manifold and let $p_1, \dots, p_k$ be distinct points in M. Let $c_1, \dots, c_k \in \mathbb{R}$ be arbitrary numbers. Show that there exists $f \in C^\infty(M)$ with $f(p_j) = c_j$ for all $j = 1, \dots, k$.

Solution

Let $\Omega_i = M \setminus \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k\}$ for $i = 1, \dots, k$. Then each Ω_i is open since finite sets are closed (M being Hausdorff), and $M = \cup_{i=1}^k \Omega_i$. By Theorem 5.5, let $(f_i)_{i=1,\dots,k}$ be a partition of unity with $\operatorname{supp} f_i \subset \Omega_i$. Then $f_i(p_j) = 0$ for $i \neq j$, and since the functions sum to 1 it follows that $f_j(p_j) = 1$ for all j. Now $f = \sum_{i=1}^k c_i f_i$ is smooth and has the desired property.

5 Let $f : M \to N$ be a smooth map between differentiable manifolds. Show that f is constant on every component of M if and only if the differential $df_p : T_pM \to T_{f(p)}N$ is zero for all $p \in M$.

Solution

If f is constant on every component, then the function $f \circ \gamma$ is constant for every parametrized smooth curve $\gamma : I \to M$, because as $f(\gamma(I))$ is connected it has to lie in a component. It then follows from Theorem 3.8 together with the last statement in Theorem 3.3 that $df_p(v) = 0$ for every $v \in T_pM$.

Assume conversely that $df_p = 0$ for all $p \in M$, and consider a level set

$$L = \{p \in M \mid f(p) = c\} \subset M$$

for some $c \in \mathbb{R}$. Let $p \in L$ and let $\sigma : U \to M$ be a chart around p with $U \subset \mathbb{R}^m$ convex. Then, since σ is smooth, we see by applying σ to linear curves in U that for every $q \in \sigma(U)$ there exists a parametrized smooth curve $\gamma : [0,1] \to \sigma(U)$ with endpoints $\gamma(0) = p$ and $\gamma(1) = q$. Now Theorem 3.8 implies that $(f \circ \gamma)' = 0$, and hence $f \circ \gamma$ is constant, so that $f(q) = f(p) = c$. Hence $\sigma(U) \subset L$. It follows that the level set L is open, and as it is also closed (by continuity of f), its intersection with any component of M cannot be a non-empty proper subset of that component. Hence every component is contained in a level set, which exactly means that f is constant along the components.

6 (more challenging) Let X be a connected topological space, and let $E \subset X$ be a connected subset. Assume $X = A \cup B$ where A and B are open and where $A \cap B \subset E$. Show that $E \cup A$ and $E \cup B$ are connected.

Solution

By the symmetry between A and B it suffices to show that $E \cup A$ is connected. Assume that $E \cup A = A_1 \cup A_2$ is a disjoint union of relatively open sets in $E \cup A$. As E is not empty, one of the sets A_1 and A_2 must contain a point from E. Let us assume it is A_2. The aim is then to show that A_1 is empty.

We shall first use that E is connected to show that $A_1 \cap E$ is empty. It is clear that E is the disjoint union of its intersections with A_1 and A_2. As A_1 and A_2 are relatively open, we have

$$A_1 = (E \cup A) \cap W_1 \quad \text{and} \quad A_2 = (E \cup A) \cap W_2$$

for some open sets $W_1, W_2 \subset X$. Then $E \cap A_1 = E \cap W_1$ and $E \cap A_2 = E \cap W_2$, and hence these intersections with E are relatively open in E. Since E is connected, $E \cap A_1$ or $E \cap A_2$ must be empty. As we assumed $E \cap A_2 \neq \emptyset$, we conclude that $E \cap A_1 = \emptyset$ and $E \subset A_2$.

Next we want to employ that X is connected. As $X = A \cup B$ and $A \subset E \cup A = A_1 \cup A_2$ we find

$$X = A_1 \cup A_2 \cup B.$$

We claim that A_1 and $A_2 \cup B$ are open, and that $A_1 \cap (A_2 \cup B)$ is empty.

1) A_1 open: As $E \cap A_1 = \emptyset$, we have $A_1 \subset A$. Hence $A_1 \subset A \cap W_1$. On the other hand it is clear that $A \cap W_1 \subset A_1$. Hence $A_1 = A \cap W_1$ which is open.

2) $A_2 \cup B$ open: Since $X = A \cup B$ we find $A_2 \cup B = (A_2 \cap A) \cup B$. Now $A_2 \cap A = ((E \cup A) \cap W_2) \cap A = A \cap W_2$ is open, and hence so is $A_2 \cup B$.

3) Intersection empty: As A_1 and A_2 are disjoint we find $A_1 \cap (A_2 \cup B) = A_1 \cap B$. This set is empty because $A_1 \cap B \subset A \cap B \subset E$ but $A_1 \cap E = \emptyset$.

The claim follows. Since X is connected and $A_2 \cup B$ is not empty, we conclude that A_1 must be empty.