GEOM2, 2013-14. TEST EXERCISES

Hand in solutions at the exercise class on Tuesday Jan 7, 2014. The solutions will be graded PASSED or FAILED. A passed test is required for participation in the exam in the end of January. It is not necessary for passing that you have a complete and correct answer to every exercise, but your solution should demonstrate a reasonable understanding of the material.

RULES. You are allowed to collaborate, but your final answers must be worked out individually. You are free to consult other sources as long as you do not copy-paste. If you ask one of the teachers we will not give hints, but we will answer general questions also when they are related to the test.

There are 4 exercises.

1 Show that

$$\mathcal{S}_1 = \{x \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}$$

is a 3-dimensional manifold in $\mathbb{R}^4$, and determine an atlas for it (this is Exercise 1.14 of the notes).

Let

$$\mathcal{S}_c = \{x \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 = c\}$$

for $c \in \mathbb{R}$. Show that $\mathcal{S}_c$ is a manifold in $\mathbb{R}^4$ for each $c \neq 0$, and find a diffeomorphism $\mathcal{S}_1 \to \mathcal{S}_c$.

Is $\mathcal{S}_0$ a manifold in $\mathbb{R}^4$?

2 Let

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Show that for every $x \in S^3 \subset \mathbb{R}^4$ a basis for the tangent space $T_x S^3$ is given by (Fx, Gx, Hx).

Let $f : S^3 \to \mathbb{R}$ be given by

$$f(x) = x_1 x_2.$$

Determine the differential df_x for each $x \in S^3$, in terms of the basis above. Show that the level set $\{x \in S^3 \mid f(x) = c\}$ is a submanifold of S^3 for each constant $c \in \mathbb{R}$ with $0 < |c| < 1/2$.

3 Show that if M and N are differentiable manifolds, where M is compact and connected, and $f : M \to N$ is a smooth map which is submersive at every point, then $f(M)$ is a component of N.

4 Let k, l, r be positive integers. Let

$$U = \mathrm{GL}(r, \mathbb{R}) \times M_{r,k} \times M_{l,r} \subset M_{r,r} \times M_{r,k} \times M_{l,r} \simeq \mathbb{R}^{r(r+k+l)}$$

and consider the map $\sigma : U \to M_{r+l,r+k} \simeq \mathbb{R}^{(r+l)(r+k)}$, given by

$$(B, C, D) \mapsto \begin{pmatrix} B & C \\ D & DB^{-1}C \end{pmatrix}.$$

Show that σ is an embedded parametrized manifold in $\mathbb{R}^{(r+l)(r+k)}$.

Show that the image $\sigma(U)$ is the set of all matrices $A \in M_{r+l,r+k}$ of rank r, for which the $r \times r$-submatrix B in the upper left corner is invertible.

Let Π_n denote the set of $n \times n$ permutation matrices, that is, matrices P which have exactly one entry 1 in each row and column, and 0 everywhere else. Multiplying a matrix A by P from the left, resp. right, results in a permutation of the rows, resp. columns, of A.

Consider the set

$$\mathcal{A} := \{\sigma_{P,Q} \mid P \in \Pi_{r+l}, Q \in \Pi_{r+k}\}$$

of maps

$$\sigma_{P,Q} : (B, C, D) \mapsto P \begin{pmatrix} B & C \\ D & DB^{-1}C \end{pmatrix} Q,$$

where $(B, C, D) \in U$ as above. Show that $\mathcal{A}$ is an atlas on the set $M(r, r + l.r + k)$ of all matrices $A \in M_{r+l,r+k}$ of rank r, such that $M(r, r + l, r + k)$ becomes a manifold in $\mathbb{R}^{(r+l)(r+k)}$.

Show that transposition is a smooth map

$$M(r, r + l, r + k) \to M(r, r + k, r + l).$$

Consider the map $f : M(r, r+l, r+k) \to \mathrm{Gr}_{r,r+l}$ which maps a matrix A to the span of its columns, viewed as an element in the Grassmannian. Show that f is smooth. Show also the analogous statement for the span of the rows.