GEOM2, 2010

P8. Class program for Tuesday Dec 14

Presentation of E3. Besides select from the following exercises

1 Show that S^3 can be given a structure as a Lie group (hint: Consider $\mathrm{SU}(2) = \{U \in \mathrm{GL}(2,\mathbb{C}) \mid U^*U = I, \det U = 1\} = \{\left(\begin{smallmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{smallmatrix}\right) | \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}$).

2 Let M be an abstract manifold, and let $p \in M$.

a) Let f be a diffeomorphism of M to itself with p as fixed point, $f(p) = p$. Show that if M is oriented, then f preserves the orientation at p if and only if $\det df_p > 0$.

b) Let σ and $\tilde{\sigma}$ be charts on M around $p = \sigma(x) = \tilde{\sigma}(\tilde{x})$. Show that σ and $\tilde{\sigma}$ induce the same orientation at p if and only if the Jacobian matrix $D(\tilde{\sigma}^{-1} \circ \sigma)(x)$ has positive determinant.

3 Verify that the unit sphere S^n is orientable. Let $\alpha: S^n \to S^n$ be the antipodal map. Verify that it is a diffeomorphism, and determine for each n whether it is orientation preserving or -reversing.

4 Let $A, B \subset S^2$ denote, respectively, the cap $\{(x,y,z) \mid z > \frac{1}{2}\}$ and the band $\{(x,y,z) \mid |z| < \frac{1}{2}\}$, and put $D = \pi(A), E = \pi(B) \subset \mathbb{R}P^2$. Prove that D and E are domains with a common smooth boundary $C = \partial D = \partial E$, which is diffeomorphic to a circle, and that $\mathbb{R}P^2 = D \cup C \cup E$ (in this fashion, $\mathbb{R}P^2$ can be thought of as a disc and a Möbius band, sewn together along the edge of each).

5 Let $F: M \to N$ be a diffeomorphism. If $D \subset M$ is a domain with smooth boundary, verify that so is $F(D)$, and $\partial(F(D)) = F(\partial D)$. Furthermore, if M and N are oriented, and F is orientation preserving/reversing, then so is its restriction $\partial D \to \partial(F(D))$.

Prove finally that the rotation of S^{n-1} by an element $A \in \mathrm{O}(n)$ is orientation preserving if $\det A = 1$ and -reversing if $\det A = -1$.

6 Prove that if M and N are orientable manifolds, then so is their product $M \times N$.