P8. Class program for Tuesday Dec 14

Presentation of E3. Besides select from the following exercises

1 Show that $S^3$ can be given a structure as a Lie group (hint: Consider $SU(2) = \{ U \in GL(2, \mathbb{C}) \mid U^*U = I, \det U = 1 \} = \{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \}).$

2 Let $M$ be an abstract manifold, and let $p \in M$.
   a) Let $f$ be a diffeomorphism of $M$ to itself with $p$ as fixed point, $f(p) = p$. Show that if $M$ is oriented, then $f$ preserves the orientation at $p$ if and only if $\det df_p > 0$.
   b) Let $\sigma$ and $\tilde{\sigma}$ be charts on $M$ around $p = \sigma(x) = \tilde{\sigma}(\tilde{x})$. Show that $\sigma$ and $\tilde{\sigma}$ induce the same orientation at $p$ if and only if the Jacobian matrix $D(\tilde{\sigma}^{-1} \circ \sigma)(x)$ has positive determinant.

3 Verify that the unit sphere $S^n$ is orientable. Let $\alpha: S^n \to S^n$ be the antipodal map. Verify that it is a diffeomorphism, and determine for each $n$ whether it is orientation preserving or reversing.

4 Let $A, B \subset S^2$ denote, respectively, the cap $\{ (x, y, z) \mid z > \frac{1}{2} \}$ and the band $\{ (x, y, z) \mid |z| < \frac{1}{2} \}$, and put $D = \pi(A), E = \pi(B) \subset \mathbb{R}P^2$. Prove that $D$ and $E$ are domains with a common smooth boundary $C = \partial D = \partial E$, which is diffeomorphic to a circle, and that $\mathbb{R}P^2 = D \cup C \cup E$ (in this fashion, $\mathbb{R}P^2$ can be thought of as a disc and a Möbius band, sewn together along the edge of each).

5 Let $F: M \to N$ be a diffeomorphism. If $D \subset M$ is a domain with smooth boundary, verify that so is $F(D)$, and $\partial(F(D)) = F(\partial D)$. Furthermore, if $M$ and $N$ are oriented, and $F$ is orientation preserving/reversing, then so is its restriction $\partial D \to \partial(F(D))$.

   Prove finally that the rotation of $S^{n-1}$ by an element $A \in O(n)$ is orientation preserving if $\det A = 1$ and -reversing if $\det A = -1$.

6 Prove that if $M$ and $N$ are orientable manifolds, then so is their product $M \times N$. 