GEOM2, 2010

P6. Class program for Tuesday Dec 7

Presentation of E2. Besides select from the following exercises

1 Let M be an abstract manifold, and M' an open subset (see Example 2.3.2). Show:

a) There is a natural linear isomorphism of T_xM' onto T_xM for each $x \in M'$, and it is the differential of the inclusion map $M' \to M$.

b) If $f\colon M \to N$ is a smooth map into an abstract manifold, then the restriction of f to M' is also smooth, and it has the same differential as f at every $x \in M'$.

c) If $f\colon N \to M$ is a map from an abstract manifold into M with $f(N) \subset M'$, then f is smooth as a map into M if and only if it is smooth into M', and the two maps have the same differential at each $y \in N$.

2 Let M, N be abstract manifolds.

a) Let $\mu\colon M \times N \to M$ and $\nu\colon M \times N \to N$ be the projection maps $(p, q) \mapsto p$ and $(p, q) \mapsto q$.

Verify that the isomorphism $T_{(p,q)}M \times N \to T_pM \times T_qN$ constructed in Exercise P5-3 is $Z \mapsto (d\mu_{(p,q)}(Z), d\nu_{(p,q)}(Z))$.

b) For a given element $q \in N$ let $\phi^q\colon M \to M \times N$ be defined by $\phi^q(x) = (x, q)$ for $x \in M$. Define $\psi^p\colon N \to M \times N$ similarly by $\psi^p(y) = (p, y)$ for $p \in M$.

Verify that these maps are smooth, and that the inverse of the above mentioned isomorphism is $(X, Y) \mapsto d\phi^q_p(X) + d\psi^p_q(Y)$

3 Let M, N be abstract manifolds, and let $f\colon M \to N$ be smooth. Prove the following *inverse function theorem for manifolds.*

a) Let $p \in M$ be given, and assume that the differential $df_p\colon T_pM \to T_{f(p)}N$ is an isomorphism of vector spaces (hence, in particular $\dim M = \dim N$). Then there exist open sets U and V around p and $f(p)$, respectively, such that f restricts to a diffeomorphism $U \to V$.

b) Assume that df_p is bijective for all $p \in M$. Then $f(M)$ is open in N, hence a manifold. If in addition f is injective, then it is a diffeomorphism of M onto this manifold $f(M)$.

4 Let $\Omega \subset S^2$ be an open hemisphere. Verify that the restriction to Ω of the projection $\pi\colon S^2 \to \mathbb{R}P^2$ is a diffeomorphism onto its image.

5 Let G be a group with neutral element e, which at the same time is an abstract manifold, such that the multiplication map $m\colon G \times G \to G$ is smooth.

a) Show that left multiplication, $l_g\colon x \mapsto gx$ and right multiplication $r_g\colon x \mapsto xg$ are diffeomorphisms of G to itself, for all $g \in G$.

b) Show that when $T_{(x,y)}G \times G$ is identified with $T_xG \times T_yG$ as above, then the differential $dm_{(e,e)}\colon T_{(e,e)}G \times G \to T_eG$ is the map $(X, Y) \mapsto X + Y$ (Hint: consider the cases $Y = 0$ and $X = 0$ separately, and apply linearity)

c) Show that the map $(x, y) \mapsto (xy, y)$ is smooth and bijective $G \times G \to G \times G$, and determine the inverse $\Phi\colon G \times G \to G \times G$.

d) Show that Φ is smooth in a neighborhood of (e, e), and conclude that the inversion $i\colon g \mapsto g^{-1}$ is smooth in a neighborhood of e.

e) Prove that the inversion i is everywhere smooth (hint: consider $r_g \circ i \circ l_g$).

Conclude: The assumption about x^{-1} is superfluous in the definition of a Lie group.