GEOM2, 2010

P5. Class program for Friday Dec 4
Select from the following exercises

1 Let
$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } x^2 = yz^2\}$$
and put $\mathcal{C} = M \setminus \{(0, \pm 1, 0)\}$ (see also Exercise P2-10).

a) Determine $T_p\mathcal{C}$ where $p = (0, 0, 1)$.

b) Prove that $\mathcal{C} \cup \{(0, 1, 0)\}$ is not a curve in $\mathbb{R}^3$.

2 Let $\mathcal{S} = \{(x, y, z) \mid x^2 + y^2 = 1\}$, and let $p = (1, 0, 0) \in \mathcal{S}$. Let $X = e_2 = (0, 1, 1)$.

a) Verify that $X \in T_p\mathcal{S}$.

b) Let $f(x, y, z) = y - z$. Determine the directional derivative $L_X f$ at p.

c) Same as before, but now with $f(\cos\theta, \sin\theta, z) = z - \theta$.

d) Determine a smooth function f on $\mathcal{S}$ for which $L_X f = 1$.

3 Let M and N be abstract manifolds, and let $p \in M$, $q \in N$ be given. Establish a natural isomorphism between the vector spaces $T_pM \times T_qN$ and $T_{(p,q)}(M \times N)$. Verify that the isomorphism amounts to the identity map, when $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^l$ and $\mathbb{R}^k \times \mathbb{R}^l$ is identified with $\mathbb{R}^{k+l}$.

4 Let $\pi: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ be the projection map. Show that there is a natural isomorphism of vector spaces between T_pS^2 and $T_{\pi(p)}\mathbb{R}P^2$ for each $p \in S^2$.

5 Consider the following curves on $\mathbb{R}P^2$:
$$\begin{aligned}
\alpha(t) &= \pi(\cos t, \sin t, 0), \\
\beta(t) &= \pi(-\cos t, 0, \sin t), \\
\gamma(t) &= \pi(1, t, t).
\end{aligned}$$
At $t = 0$ they all pass through the same point p. Determine a non-trivial linear relation between their equivalence classes at this point (we know that such a relation exists, since the tangent space is 2-dimensional).