GEOM2, 2010

P3. Class program for Friday Nov 26

Select from the following exercises

1 Prove Lemma 2.1.1

2 Let X be a finite set. Prove that it is a Hausdorff topological space in one and only one way. If X has 2 elements, in how many ways is it a topological space?

3 Equip $\mathbb{R}$ with the family of all intervals $]a, \infty[$ where $a \in \mathbb{R}$, including also $\emptyset$ and $\mathbb{R}$. Verify that it is a topological space, but not Hausdorff.

4 Let X, Y be topological spaces, and let $f, g \colon X \to Y$ be continuous maps. Prove that if Y is Hausdorff, then $\{x \in X \mid f(x) = g(x)\}$ is closed.

Give an example which shows the conclusion can fail if Y is not Hausdorff (for example, Y could be a two element set with the trivial topology)

5 Verify that compatibility of atlasses is an equivalence relation (see Defn 2.2.3). Prove that the union of all the atlasses in a given equivalence class is an atlas (called the *maximal atlas* for the structure).

6 Prove that if $A \subset S^2$ is open, then $\pi(A)$ is open in $\mathbb{R}P^2$.

7 The natural injection of S^1 into the equator of S^2 induces a map $f \colon \mathbb{R}P^1 \to \mathbb{R}P^2$. Prove that the image is closed in $\mathbb{R}P^2$, and that f is a homeomorphism onto it.

8 a. Let X be a topological space with an equivalence relation $\sim$, and let $M = X/\sim$ denote the set of equivalence classes, with $\pi \colon X \to M$ the natural map. A set $A \subset M$ is said to be open if and only if the preimage $\pi^{-1}(A)$ is open in X. Verify that M is a topological space (called the *quotient space*). – Unfortunately, it need not be Hausdorff, even if X is.

b. Define a relation on $X = \mathbb{R}^2$ by $(s, t) \sim (s', t')$ if and only if $s' - s$ is an integer and $t' = (-1)^{s'-s} t$. Verify it is an equivalence relation. Let M be the quotient space. Prove that it is Hausdorff.

c. For each $a \in \mathbb{R}$ let $U_a =]a, a+1[\times \mathbb{R} \subset \mathbb{R}^2$ and let $\sigma_a \colon U_a \to M$ be the restriction of π to this set. Prove that the collection of all these maps is an atlas on M.

9 Let $\mathbb{C}P^1$ denote the set of 1-dimensional (complex) subspaces of the complex vector space $\mathbb{C}^2$. Turn it into an abstract 2-dimensional manifold (you are entitled to skip the verification of the Hausdorff axiom for the relevant quotient space).

10 Verify that if M is a manifold in $\mathbb{R}^k$ and N is a manifold in $\mathbb{R}^l$, then $M \times N$ is a manifold in $\mathbb{R}^{k+l}$, and its smooth structure is identical to that of the product manifold.