GEOM2, 2010

P2. Class program for Tuesday Nov 23

Select among the following exercises.

1 Prove Lemma 1.4

2 Find an atlas for the cylinder (Example 1.5.2), consisting of a single chart.

3 Let $\mathcal{S} = \{(x, y, z) \mid z = 0\}$ be the xy-plane, let $U = \{(u, v) \in \mathbb{R}^2 \mid u > v\}$, and let $\sigma : U \to \mathbb{R}^3$ be given by $\sigma(u, v) = (u + v, uv, 0)$. Verify that σ is a chart on $\mathcal{S}$. After that, replace the condition $u > v$ by $u \neq v$ in the definition of U, and prove that σ is no longer a chart.

4 Give an example of a level set $\mathcal{S} = \{(x, y, z) \mid f(x, y, z) = 0\}$ which is a surface in $\mathbb{R}^3$, and yet there exists a point in $\mathcal{S}$ for which $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (0, 0, 0)$.

5 Let $\mathcal{S} = \{(x, y, z) \mid xyz = c\}$ where $c \in \mathbb{R}$. Show that $\mathcal{S}$ is a surface in $\mathbb{R}^3$ if $c \neq 0$. Determine $\mathcal{S}$ explicitly when $c = 0$ and prove that it is *not* a surface.

6 Let $\mathcal{C} \subset \mathbb{R}^2$ be a curve in $\mathbb{R}^2$, and put $\mathcal{S} = \mathcal{C} \times \mathbb{R} = \{(x, y, z) \mid (x, y) \in \mathcal{C}\}$. Prove that $\mathcal{S}$ is a surface in $\mathbb{R}^3$ (it is called a *generalized cylinder*).

7 Let $\mathcal{S} \subset \mathbb{R}^3$. Show that $\mathcal{S}$ is a surface if and only if the following holds. For each $p \in \mathcal{S}$ there exists an open neighborhood $W \subset \mathbb{R}^3$ of p and a smooth function $f\colon W \to \mathbb{R}$ such that

(i) $\mathcal{S} \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$

(ii) $(f'_x(p), f'_y(p), f'_z(p)) \neq (0, 0, 0)$.

Propose (and prove) a similar statement for curves in $\mathbb{R}^2$.

8 Verify that $\{x \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}$ is a 3-dimensional manifold in $\mathbb{R}^4$.

9 Let $M \subset \mathbb{R}^n$ be an m-dimensional manifold in $\mathbb{R}^n$, and let $N \subset M$ be a subset. Prove that N is also an m-dimensional manifold in $\mathbb{R}^n$ if and only if it is relatively open in M (see Definition 1.2.3)

10 Consider the set M of points (x, y, z) in $\mathbb{R}^3$ which satisfy both equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 = yz^2.$$

Show that $P = (0, -1, 0)$ is isolated in M (that is, there is a neighborhood in $\mathbb{R}^3$, in which P is the only point from M). Find another point Q in M, such that $M \setminus \{P, Q\}$ is a manifold in $\mathbb{R}^3$.

11 Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$.

a) Let $f\colon \Omega \to \mathbb{R}^2$ be a smooth map, where $\Omega \subset \mathbb{R}^3$ is open. Let $\mathcal{S} = f^{-1}(\mathcal{C})$, and assume that $Df(p)$ has rank 2 for all $p \in \mathcal{S}$.

Prove that $\mathcal{S}$ is a surface in $\mathbb{R}^3$ (hint: Apply Exercise 7 above in order to describe $\mathcal{C}$ as a level set (locally). Now consider the composed map).

b) Assume that $x > 0$ for all $(x, y) \in \mathcal{C}$, and let

$$\mathcal{S} = \{(x, y, z) \mid (\sqrt{x^2 + y^2}, z) \in \mathcal{C}\}.$$

Prove that this is a surface, and describe it in words.