P2. Class program for Tuesday Nov 23

Select among the following exercises.

1 Prove Lemma 1.4

2 Find an atlas for the cylinder (Example 1.5.2), consisting of a single chart.

3 Let $S = \{(x, y, z) \mid z = 0\}$ be the $xy$-plane, let $U = \{(u, v) \in \mathbb{R}^2 \mid u > v\}$, and let $\sigma : U \to \mathbb{R}^3$ be given by $\sigma(u, v) = (u + v, uv, 0)$. Verify that $\sigma$ is a chart on $S$. After that, replace the condition $u > v$ by $u \neq v$ in the definition of $U$, and prove that $\sigma$ is no longer a chart.

4 Give an example of a level set $S = \{(x, y, z) \mid f(x, y, z) = 0\}$ which is a surface in $\mathbb{R}^3$, and yet there exists a point in $S$ for which $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (0, 0, 0)$.

5 Let $S = \{(x, y, z) \mid xyz = c\}$ where $c \in \mathbb{R}$. Show that $S$ is a surface in $\mathbb{R}^3$ if $c \neq 0$. Determine $S$ explicitly when $c = 0$ and prove that it is not a surface.

6 Let $C \subset \mathbb{R}^2$ be a curve in $\mathbb{R}^2$, and put $S = C \times \mathbb{R} = \{(x, y, z) \mid (x, y) \in C\}$. Prove that $S$ is a surface in $\mathbb{R}^3$ (it is called a generalized cylinder).

7 Let $S \subset \mathbb{R}^3$. Show that $S$ is a surface if and only if the following holds. For each $p \in S$ there exists an open neighborhood $W \subset \mathbb{R}^3$ of $p$ and a smooth function $f : W \to \mathbb{R}$ such that
   
   (i) $S \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$
   
   (ii) $(f_x'(p), f_y'(p), f_z'(p)) \neq (0, 0, 0)$.

Propose (and prove) a similar statement for curves in $\mathbb{R}^2$.

8 Verify that $\{x \mid x_1^2 + x_2^2 - x_3^2 = 1\}$ is a 3-dimensional manifold in $\mathbb{R}^4$.

9 Let $M \subset \mathbb{R}^n$ be an $m$-dimensional manifold in $\mathbb{R}^n$, and let $N \subset M$ be a subset. Prove that $N$ is also an $m$-dimensional manifold in $\mathbb{R}^n$ if and only if it is relatively open in $M$ (see Definition 1.2.3)

10 Consider the set $M$ of points $(x, y, z)$ in $\mathbb{R}^3$ which satisfy both equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 = yz^2.$$ 

Show that $P = (0, -1, 0)$ is isolated in $M$ (that is, there is a neighborhood in $\mathbb{R}^3$, in which $P$ is the only point from $M$). Find another point $Q$ in $M$, such that $M \setminus \{P, Q\}$ is a manifold in $\mathbb{R}^3$.

11 Let $C$ be a curve in $\mathbb{R}^2$.

   a) Let $f : \Omega \to \mathbb{R}^2$ be a smooth map, where $\Omega \subset \mathbb{R}^3$ is open. Let $S = f^{-1}(C)$, and assume that $Df(p)$ has rank 2 for all $p \in S$.

   Prove that $S$ is a surface in $\mathbb{R}^3$ (hint: Apply Exercise 7 above in order to describe $C$ as a level set (locally). Now consider the composed map).

   b) Assume that $x > 0$ for all $(x, y) \in C$, and let

   $$S = \{(x, y, z) \mid (\sqrt{x^2 + y^2}, z) \in C\}.$$ 

Prove that this is a surface, and describe it in words.