GEOM2, 2010-2011

P12. Class program for Friday Jan 14

Presentation of E6. Besides select from the following exercises

1 Verify the Jacobi identity for the commutator bracket on $\mathfrak{gl}(n, \mathbb{R})$ (Example 6.6.3).

2 Define $[(x, y), (v, w)] = (0, xw - yv)$ for (x, y) and (v, w) in $\mathbb{R}^2$. Show that in this fashion $\mathbb{R}^2$ is a Lie algebra [Hint: Rather than verifying the Jacobi identity directly, you can consider 2×2 matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$].

Let $\mathfrak{g}$ be a 2-dimensional Lie algebra for which there are basis vectors e, f such that $[e, f] = f$. Prove that it is isomorphic with the mentioned Lie algebra on $\mathbb{R}^2$.

Prove finally that every 2-dimensional Lie algebra with a non-trivial bracket (that is, $[X, Y]$ is not always zero) has such a basis.

Conclude that up to isomorphism there are only two different Lie algebras of dimension 2.

3 Let G be a Lie group. For two subsets $A, B \subset G$ let

$$AB = \{xy \mid x \in A, y \in B\}.$$

Show that AB is open if A or B is open, and that it is compact if A and B are compact.

4 Let G be a Lie group, and let H be an open subgroup. Show that H is closed [Hint: prove that the complement H^c satisfies $H^c = HH^c$]. Conclude that a connected Lie group has no open subgroups (besides itself).

5 Let G be a connected Lie group. Choose a neighborhood W of e, and put

$$D_n = W^n = \{x_1 \cdots x_n \mid x_1, \dots, x_n \in W\}.$$

Show that $G = \cup_n D_n$.

Prove that G has a countable atlas [Hint: Choose W with compact closure. Now apply Theorem 5.2].

6 Let G be a Lie group, and let G_0 be the component of G (see Section 5.9) which contains the neutral element e. Show that G_0 is an open subgroup, and that it is contained in every open subgroup of G.