GEOM2, 2013-14. Extra Notes

Let $1 \leq k < n$. The *Grassmannian* $\mathrm{Gr}_{n,k}$ is the set of all k-dimensional linear subspaces in $\mathbb{R}^n$. It generalizes the projective space $\mathbb{R}P^{n-1} = \mathrm{Gr}_{n,1}$. We will show that $\mathrm{Gr}_{n,k}$ can be equipped with the structure of an $(n-k)k$-dimensional differentiable manifold.

We first need a way to identify the elements of $\mathrm{Gr}_{n,k}$. A subspace of dimension k in $\mathbb{R}^n$ can be identified by means of a basis consisting of k linearly independent vectors from the subspace. We view these vectors as columns and collect them in an $n \times k$ matrix. Let $M_{n,k}$ denote the vector space of all $n \times k$ matrices, and $M'_{n.k}$ the subset of matrices with independent columns. For $B \in M_{n,k}$ let $\mathrm{Sp}(B) \subset \mathbb{R}^n$ denote the linear space spanned by the columns of B. That is, it is the image

$$\mathrm{Sp}(B) = \{Bx \mid x \in \mathbb{R}^k\} \tag{1}$$

of the linear map $\mathbb{R}^k \to \mathbb{R}^n$ represented by B. We thus obtain a surjective map $\mathrm{Sp}\colon M'_{n,k} \to \mathrm{Gr}_{n,k}$.

The atlas with which we shall equip $\mathrm{Gr}_{k,n}$ consists charts which are all constructed by means of this map. They are all defined on $\mathbb{R}^{(n-k)k}$, which we shall identify with $M_{n-k,k}$. For $A \in M_{n-k,k}$ we let

$$\tilde{A} = \begin{pmatrix} I \\ A \end{pmatrix} \in M'_{n,k},$$

the $n \times k$ matrix with a $k \times k$ unit matrix on top of A. The unit matrix in the top ensures the linear independence of the columns. We define $\sigma(A) = \mathrm{Sp}(\tilde{A})$ and more generally

$$\sigma_g\colon M_{n-k,k} \to \mathrm{Gr}_{n,k}, \quad \sigma_g(A) = \mathrm{Sp}(g\tilde{A}).$$

for each $g \in \mathrm{GL}(n, \mathbb{R})$. Note that $g\tilde{A} \in M'_{n,k}$ as $\tilde{A} \in M'_{n-k,k}$ and g is invertible.

Theorem 1. *There exists on $\mathrm{Gr}_{k,n}$ a topology and a differential structure for which the collection of maps $\{\sigma_g \mid g \in \mathrm{GL}(n, \mathbb{R})\}$ is an atlas.*

Proof. We first equip $\mathrm{Gr}_{k,n}$ with a topology. Here we use the quotient topology with respect to the map $\mathrm{Sp}\colon M'_{n,k} \to \mathrm{Gr}_{n,k}$, that is, we define a subset of $\mathrm{Gr}_{n,k}$ to be open if and only if its preimage in $M'_{n,k}$ is open. The axioms of a topological space are easily verified. Note that $M'_{n,k}$ is an open subset of $M_{n,k}$ as we can represent it by

$$M'_{n,k} = \{B \in M_{n,k} \mid \det(B^t B) \neq 0\},$$

and $B \mapsto \det(B^t B)$ is a continuous map. The Hausdorff axiom for $\mathrm{Gr}_{n,k}$ will be verified later.

Let $G = \mathrm{GL}(n, \mathbb{R})$. In the proof we shall use the natural *action* of G on $\mathrm{Gr}_{n,k}$, defined as follows. For every matrix $g \in G$ and subspace $V \subset \mathbb{R}^n$ we define a subspace $g.V \subset \mathbb{R}^n$ by

$$g.V = \{gu \mid u \in V\}. \tag{2}$$

Since g is invertible, V and $g.V$ have the same dimension. The resulting map

$$G \times \mathrm{Gr}_{n,k} \to \mathrm{Gr}_{n,k}, \quad (g, V) \mapsto g.V$$

is the mentioned action. The relation $g_1.(g_2.V) = (g_1 g_2).V$ for $g_1, g_2 \in G$ is easily seen. Note also that (1) and (2) imply

$$g.\,\mathrm{Sp}(B) = \mathrm{Sp}(gB) \tag{3}$$

for $g \in G$ and $B \in M_{n,k}$. In particular, since multiplication by g is continuous $M_{n,k} \to M_{n,k}$ it follows from (3) and the definition of the topology that $V \mapsto g.V$ is continuous $\mathrm{Gr}_{n,k} \to \mathrm{Gr}_{n,k}$. In fact, then it is a homeomorphism since the inverse $V \mapsto g^{-1}.V$ is also continuous.

We now study the proposed charts σ_g. Let $g \in G = \mathrm{GL}(n, \mathbb{R})$ and note that $\sigma_g(A) = g.\sigma(A)$. The map σ is continuous, being composed by the continuous maps $A \mapsto \tilde{A}$ and Sp. Hence σ_g is continuous as well.

Our next aim is to show σ_g is injective. It suffices to show σ is injective, since multiplication by g is injective. We shall employ the following lemma.

Lemma 1. *Let $B, B' \in M'_{n,k}$. Then $\mathrm{Sp}(B) = \mathrm{Sp}(B')$ if and only if there exists $C \in \mathrm{GL}(k, \mathbb{R})$ such that $B' = BC$.*

Proof. The equation $B' = BC$ implies the image $\mathrm{Sp}(B')$ of B' is contained in the image $\mathrm{Sp}(B)$ of B. The opposite inclusion follows from the equation $B = B'C^{-1}$.

Conversely, if $\mathrm{Sp}(B') = \mathrm{Sp}(B)$ then each column of B' is in $\mathrm{Sp}(B)$, hence can be expressed as a linear combination of the columns of B. Inserting the coefficients of these linear combinations in a $k \times k$ matrix C we find $B' = BC$. If the rank of C was smaller than k then the rank of BC would also be smaller than k. Since B' has rank k, the rank of C must be k. Hence C is invertible. □ □

We return to the proof that σ is injective. Let $A, A' \in M_{n-k,k}$ and assume $\sigma(A') = \sigma(A)$, that is, $\mathrm{Sp}(\tilde{A}') = \mathrm{Sp}(\tilde{A})$. According to Lemma 1 there exists $C \in \mathrm{GL}(k, \mathbb{R})$ such that $\tilde{A}' = \tilde{A}C$. The first k rows of this matrix equation express that $I = C$ and the remaining rows then express that $A' = A$. Hence the map is injective.

Next we want to study the inverse of σ_g, which is defined on its image set $\sigma_g(M_{n-k,k})$. We want to show that this is an open subset of $\mathrm{Gr}_{k,n}$ and that σ_g^{-1} is continuous into $M_{n-k,k}$. Again it suffices to consider $g = I$, as the multiplication by g is a homeomorphism of $\mathrm{Gr}_{n,k}$. We claim

$$\sigma(M_{n-k,k}) = \big\{\, \mathrm{Sp}\begin{pmatrix} C \\ D \end{pmatrix} \,\big|\, C \in \mathrm{GL}(k, \mathbb{R}), D \in M_{n-k,k} \big\}, \tag{4}$$

and with C, D as in (4),

$$\mathrm{Sp}\begin{pmatrix} C \\ D \end{pmatrix} = \sigma(DC^{-1}). \tag{5}$$

The inclusion $\subset$ in (4) is obvious from the definition of σ. The equality (5) follow from Lemma 1, and it implies $\supset$ in (4).

With (4) and Lemma 1 we conclude

$$\mathrm{Sp}^{-1}(\sigma(M_{n-k,k})) = \big\{ \begin{pmatrix} C \\ D \end{pmatrix} \,\big|\, C \in \mathrm{GL}(k, \mathbb{R}), D \in M_{n-k,k} \big\},$$

which is open in $M_{n,k}$. From (5) we find

$$\sigma^{-1}\left(\operatorname{Sp}\begin{pmatrix} C \\ D \end{pmatrix}\right) = DC^{-1}, \tag{6}$$

which shows that $\sigma^{-1} \circ \operatorname{Sp}$ is continuous. This implies that σ^{-1} is continuous.

We have now shown that each of the proposed charts σ_g is a homeomorphism onto an open set. In order to see that every point of $\mathrm{Gr}_{n,k}$ lies in some chart, we shall use that the action of $G = \mathrm{GL}(n, \mathbb{R})$ is *transitive*, by which we understand that for any pair of elements $V, V' \in \mathrm{Gr}_{n,k}$ there exists $g \in G$ such that $V' = g.V$. This can be seen by choosing a basis $u_1 \dots, u_n$ for $\mathbb{R}^n$ such that $u_1, \dots, u_k$ belong to V, and likewise a basis $u'_1 \dots, u'_n$ such that $u'_1, \dots, u'_k$ belong to V'. The linear transformation of $\mathbb{R}^n$ to itself, which maps u_j to u'_j for all j, is invertible and maps V to V'. Hence the action is transitive. In particular, let

$$V_0 = \{x \in \mathbb{R}^n \mid x_{k+1} = \dots = x_n = 0\} = \sigma(0),$$

then there exists for every $V \in \mathrm{Gr}_{n,k}$ an element $g \in G$ such that $V = g.V_0$. It follows that $V = \sigma_g(0)$. Hence the proposed atlas covers all of $\mathrm{Gr}_{n,k}$.

Next we want to show the topology is Hausdorff. We shall employ the following two lemmas.

Lemma 2. *The image of σ is*

$$\sigma(M_{n-k,k}) = \{V \in \mathrm{Gr}_{k,n} \mid V \cap W_0 = \{0\}\}$$

where $W_0 \subset \mathbb{R}^n$ is the subspace $W_0 = \{x \in \mathbb{R}^n \mid x_1 = \dots = x_k = 0\}$.

Proof. Let $A \in M_{n-k,k}$, then every nontrivial linear combination of the columns of $\tilde{A}$ must be non-zero in at least one of its first k coordinates. Hence $\sigma(A) \cap W_0 = \{0\}$.

Conversely, let $V \in \mathrm{Gr}_{n-k,k}$ and assume $V \cap W_0 = \{0\}$. Let $V = \operatorname{Sp}(B)$ where $B \in M'_{n,k}$ and write

$$B = \begin{pmatrix} C \\ D \end{pmatrix}$$

with $C \in M_{k,k}$ and $D \in M_{n-k,k}$. Then (4) implies $V \in \sigma(M_{n-k,k})$ provided we show that C is invertible. If C was not invertible, there would exist a non-zero vector $x \in \mathbb{R}^k$ with $Cx = 0$. The vector Bx would be zero in its first k coordinates, that is, $Bx \in W_0$, and from (1) and $V \cap W_0 = \{0\}$ we would then deduce that $Bx = 0$. Since $x \neq 0$ this would contradict that B has independent columns. □

It follows that

$$\sigma_g(M_{n-k,k}) = \{V \in \mathrm{Gr}_{k,n} \mid V \cap g.W_0 = \{0\}\} \tag{7}$$

for all $g \in \mathrm{GL}(n, \mathbb{R})$, where W_0 is as above.

Lemma 3. *Let $V_1, V_2 \subset \mathbb{R}^n$ be two arbitrary k-dimensional subspaces. Then there exist a $n-k$-dimensional subspace $W \subset \mathbb{R}^n$ such that $V_1 \cap W = V_2 \cap W = \{0\}$*

Proof. We may assume $k < n$. We claim that there exists a vector v outside of $V_1 \cup V_2$. This is clear if $V_1 = V_2$. Otherwise, if $V_1 \neq V_2$ there exists a vector $u_1 \in$

$V_1 \setminus V_2$ and also a vector $u_2 \in V_2 \setminus V_1$. Now any linear combination $v = a_1 u_1 + b_2 u_2$, with a and b both $\neq 0$, lies outside both subspaces, and the claim is established.

We will describe a basis $v_1, \dots, v_{n-k}$ for W. The first vector v_1 is the vector v of the just proved claim. Next we apply the claim (for k one higher) to the two subspaces $V_1 + \mathbb{R}v$ and $V_2 + \mathbb{R}v$, and find a vector v_2 in neither space. After $n - k$ such steps we reach W. □

We return to the proof of the Hausdorff property. Let $V_1, V_2 \in \mathrm{Gr}_{n,k}$ be given, and let W be as in the preceding lemma. Since G acts transitively on $\mathrm{Gr}_{n,n-k}$ there exists $g \in G$ such that $g.W_0 = W$, and hence by (7) both V_1 and V_2 belong to the image of σ_g. Since σ_g is a homeomorphism its image is Hausdorff. If V_1 and V_2 are not equal, they can thus be separated by open sets inside $\sigma_g(M_{n-k,k})$, hence also in $\mathrm{Gr}_{n,k}$. This establishes the Hausdorff property for the Grassmannian.

Finally we show that $\mathrm{Gr}_{n,k}$ is a differentiable manifold. It only remains to show that $\sigma_{g_2}^{-1} \circ \sigma_{g_1}$ is smooth for all pairs of elements $g_1, g_2 \in G$. Note that $\sigma_{g_2}^{-1} \circ \sigma_{g_1} = \sigma^{-1} \circ \sigma_g$ where $g = g_2^{-1} g_1$. It follows from (5) that $\sigma^{-1} \circ \sigma_g(A) = DC^{-1}$ where C and D are determined from $g\tilde{A} = \begin{pmatrix} C \\ D \end{pmatrix}$. The entries of C and D depend smoothly (in fact, linearly) on the entries of A. The entries of C^{-1} depend smoothly on the entries of C because $\mathrm{GL}(k, \mathbb{R})$ is a Lie group. Finally the product of D and C^{-1} is smooth (again in fact linear) with respect to the entries of D and C^{-1}. Hence $\sigma^{-1} \circ \sigma_g$ is a smooth map. We conclude that the proposed charts comprise an atlas. □

Exercise: Prove that Sp is a smooth map from $M'_{n,k}$ to $\mathrm{Gr}_{n,k}$. Use this to show that the action $(g, V) \mapsto g.V$ is smooth $G \times \mathrm{Gr}_{n,k} \to \mathrm{Gr}_{n,k}$.