Note 1.

This note replaces the theory of p. 13-14 (Definition 1.12 and Theorem 1.13), which is quite technical.

**Proposition 1.** Let \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) and let \( X : D \to \mathbb{R}^n \) be continuous. Let \((I, y)\) be a solution to

\[
\dot{y} = X(t, y)
\]

with \( I = (a, b) \) where \( a < b \).

If there exists a compact set \( K \subset D \) such that \((t, y(t)) \in K \) for all \( t \in I \), then \( y \) admits an extension \((\hat{I}, \hat{y})\) to a solution on the closed interval \( \hat{I} = [a, b] \).

**Proof.** Since \( K \) is compact and \( X \) is continuous, there exists \( M < \infty \) such that \( \|X(t, x)\| \leq M \) for all \((t, x) \in K\). Here we use the standard norm

\[
\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}
\]
on \( \mathbb{R}^n \). In particular, it follows from (1) that \( \|\dot{y}(t)\| \leq M \) for all \( t \in I \). Hence each coordinate of \( \dot{y}(t) \) is bounded by \( M \) for \( t \in I \).

Let \( t_1, t_2 \in I \). From the mean value theorem it follows that for each coordinate \( y_i \) of \( y \) there exists \( t \) between \( t_1 \) and \( t_2 \) such that

\[
y_i(t_2) - y_i(t_1) = \dot{y}_i(t)(t_2 - t_1),
\]

and hence \( |y_i(t_2) - y_i(t_1)| \leq M|t_2 - t_1| \). We thus conclude

\[
\|y(t_2) - y(t_1)\| \leq \sqrt{n}M|t_2 - t_1|.
\]

Let \( s_k \) be an arbitrary sequence in \( I \) converging to the end-point \( a \). Then \( s_k \) is a Cauchy sequence, and the inequality above then implies that so is its image \( y(s_k) \) in \( \mathbb{R}^n \). Hence \( y(s_k) \to \eta \) as \( k \to \infty \), for some \( \eta \in \mathbb{R}^n \). It is easily seen that if \( s'_k \) is another sequence converging to \( a \), then its image \( y(s'_k) \) will have same limit \( \eta \). For example, one can argue that the mixed sequence \( y(s_1), y(s'_1), y(s_2), y(s'_2), \ldots \) is also Cauchy and contains both \( y(s_k) \) and \( y(s'_k) \) as subsequences. It follows that if we define \( \hat{y}(a) = \eta \) and \( \hat{y} = y \) on \( I \), then \( \hat{y} \) is continuous at \( a \).

Next, we show that \( \hat{y} \) solves the differential equation (1) also at \( a \). We have to show that \( \hat{y} \) is right-differentiable at \( a \) with derivative \( X(a, \hat{y}(a)) \), that is,

\[
\lim_{t \to a^+} \frac{\hat{y}(t) - \hat{y}(a)}{t - a} = X(a, \hat{y}(a)).
\]
Again we employ the mean value theorem to each coordinate. With \( i \in \{1, \ldots, n\} \) fixed it follows that here exists, for each \( t \in I \), an element \( c_t \in (a, t) \) such that

\[
\frac{\hat{y}_i(t) - \hat{y}_i(a)}{t - a} = y'_i(c_t)
\]

(note that the mean value theorem requires continuity of \( \hat{y}_i \) in \([a, t]\) but differentiability only in \((a, t)\)). From (1) it follows that

\[
y'_i(c_t) = X(c_t, y(c_t))_i
\]

Now \( t \to a_+ \) implies \( c_t \to a \) since \( a < c_t < t \), and as \( \hat{y} \) and \( X \) are continuous, it then follows that \( X(c_t, y(c_t))_i \to X(a, \hat{y}(a))_i \), which implies what we wanted to show for each \( i \).

Similarly, we obtain an extension of \( y \) at \( b \). \( \square \)

**Example 3.10.** The book treats this example without reference to Theorem 1.13. With Proposition 1 it reads as follows. The differential equation is

\[
(2) \quad \dot{y} = y^{1/3} + \frac{1}{2} t^{1/2}, \quad (t, y) \in D = \mathbb{R}_+ \times \mathbb{R}_+.
\]

By an Ansatz, the solution \((\mathbb{R}_+, t^{2/3})\) was found in Example 1.4.

**Claim:** Let \((t_0, \eta) \in D\) be an initial condition with \( 0 < \eta < t_0^{3/2} \). The corresponding maximal solution \((I, y)\) with \( y(t_0) = \eta \) satisfies \([t_0, \infty[ \subseteq I\).

**Proof.** Suppose the right end-point \( b \) of \( I \) is finite. By Theorem 3.1 and Remark 3.2, \( I \) is open and the graph of \( y \) does not intersect with the graph of \( t^{3/2} \). Thus \( y(t) < t^{3/2} \leq B := b^{3/2} \) for all \( t \in I \). Furthermore, from (2) we derive that \( \dot{y} \geq 0 \) and hence \( y \) is increasing. Hence \((t, y(t))\) belongs to the compact set

\[
K = [t_0, b] \times [\eta, B]
\]

for all \( t \in (t_0, b) \). By Proposition 1, the solution \(((t_0, b), y)\) admits an extension \(((t_0, b), \hat{y})\). According to Lemma 1.17, we can glue this extension together with \((I, y)\) and thus obtain an extension of the latter in the end-point \( b \), which is a contradiction.

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