This note replaces the theory of p. 13-14 (Definition 1.12 and Theorem 1.13), which is quite technical.

Proposition 1. *Let $\mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^n$ and let $X : \mathcal{D} \to \mathbb{R}^n$ be continuous. Let (I, y) be a solution to*

$$\dot{y} = X(t, y) \tag{1}$$

with $I = (a, b)$ where $a < b$.

If there exists a compact set $K \subset \mathcal{D}$ such that $(t, y(t)) \in K$ for all $t \in I$, then y admits an extension $(\hat{I}, \hat{y})$ to a solution on the closed interval $\hat{I} = [a, b]$.

Proof. Since K is compact and X is continuous, there exists $M < \infty$ such that $\|X(t, x)\| \leq M$ for all $(t, x) \in K$. Here we use the standard norm

$$\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

on $\mathbb{R}^n$. In particular, it follows from (1) that $\|\dot{y}(t)\| \leq M$ for all $t \in I$. Hence each coordinate of $\dot{y}(t)$ is bounded by M for $t \in I$.

Let $t_1, t_2 \in I$. From the mean value theorem it follows that for each coordinate y_i of y there exists t between t_1 and t_2 such that

$$y_i(t_2) - y_i(t_1) = \dot{y}_i(t)(t_2 - t_1),$$

and hence $|y_i(t_2) - y_i(t_1)| \leq M|t_2 - t_1|$. We thus conclude

$$\|y(t_2) - y(t_1)\| \leq \sqrt{n}M|t_2 - t_1|.$$

Let s_k be an arbitrary sequence in I converging to the end-point a. Then s_k is a Cauchy sequence, and the inequality above then implies that so is its image $y(s_k)$ in $\mathbb{R}^n$. Hence $y(s_k) \to \eta$ as $k \to \infty$, for some $\eta \in \mathbb{R}^n$. It is easily seen that if s'_k is another sequence converging to a, then its image $y(s'_k)$ will have same limit η. For example, one can argue that the mixed sequence $y(s_1), y(s'_1), y(s_2), y(s'_2) \ldots$ is also Cauchy and contains both $y(s_k)$ and $y(s'_k)$ as subsequences. It follows that if we define $\hat{y}(a) = \eta$ and $\hat{y} = y$ on I, then $\hat{y}$ is continuous at a.

Next, we show that $\hat{y}$ solves the differential equation (1) also at a. We have to show that $\hat{y}$ is right-differentiable at a with derivative $X(a, \hat{y}(a))$, that is,

$$\lim_{t \to a_+} \frac{\hat{y}(t) - \hat{y}(a)}{t - a} = X(a, \hat{y}(a)).$$

Again we employ the mean value theorem to each coordinate. With $i \in \{1, \ldots, n\}$ fixed it follows that here exists, for each $t \in I$, an element $c_t \in (a, t)$ such that

$$\frac{\hat{y}_i(t) - \hat{y}_i(a)}{t - a} = y_i'(c_t)$$

(note that the mean value theorem requires continuity of $\hat{y}_i$ in $[a, t]$ but differentiability only in (a, t)). From (1) it follows that

$$y_i'(c_t) = X(c_t, y(c_t))_i$$

Now $t \to a_+$ implies $c_t \to a$ since $a < c_t < t$, and as $\hat{y}$ and X are continuous, it then follows that $X(c_t, y(c_t))_i \to X(a, \hat{y}(a))_i$, which implies what we wanted to show for each i.

Similarly, we obtain an extension of y at b. □

Example 3.10. The book treats this example without reference to Theorem 1.13. With Proposition 1 it reads as follows. The differential equation is

$$\dot{y} = y^{1/3} + \frac{1}{2}t^{1/2}, \qquad (t, y) \in \mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_+. \tag{2}$$

By an Ansatz, the solution $(\mathbb{R}_+, t^{2/3})$ was found in Example 1.4.

Claim: Let $(t_0, \eta) \in \mathcal{D}$ be an initial condition with $0 < \eta < t_0^{3/2}$. The corresponding maximal solution (I, y) with $y(t_0) = \eta$ satisfies $[t_0, \infty[\subseteq I$.

Proof. Suppose the right end-point b of I is finite. By Theorem 3.1 and Remark 3.2, I is open and the graph of y does not intersect with the graph of $t^{3/2}$. Thus $y(t) < t^{3/2} \leq B := b^{3/2}$ for all $t \in I$. Furthermore, from (2) we derive that $\dot{y} \geq 0$ and hence y is increasing. Hence $(t, y(t))$ belongs to the compact set

$$K = [t_0, b] \times [\eta, B]$$

for all $t \in (t_0, b)$. By Proposition 1, the solution $([t_0, b), y)$ admits an extension $([t_0, b], \hat{y})$. According to Lemma 1.17, we can glue this extension together with (I, y) and thus obtain an extension of the latter in the end-point b, which is a contradiction.

HS