Fourier series

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1 Introduction

In Fourier analysis one attempts to express a periodic function of a real variable x as a combination of *pure oscillations*

$$\cos(\lambda x), \quad \sin(\lambda x), \quad \lambda \in \mathbb{R}$$

or more conveniently in complex notation

$$e^{i\lambda x}, \quad e^{-i\lambda x}, \quad \lambda \in \mathbb{R}.$$

Fourier analysis is a recurring theme in all of modern mathematical analysis, especially in functional analysis, description of infinite vector spaces, operator algebras, spectral theory, and not least it is an indispensable tool in the study of partial differential equations.

In these notes we will focus on real or complex valued functions, which are periodic with a given period. When we try to express such a function by pure oscillations, only the pure oscillations having the same period are relevant. There are countably many such oscillations. More precisely, we will try to express such periodic functions by an infinite series, where the terms are the appropriate pure oscillations. Such a series will be called a *trigonometric series* (see Definition 2.3) or a *Fourier series* (see Definition 2.10).

2 Fourier series

2.1 Periodic functions, trigonometric polynomials and trigonometric series

2.1 DEFINITION (Periodic function)**.** A function $f : \mathbb{R} \to \mathbb{C}$ is called periodic with period $p > 0$ (or p-periodic) if

$$f(x+p) = f(x)$$

for all $x \in \mathbb{R}$.

2.2 DEFINITION (Periodic extension)**.** For a function $g : [a, a+p] \to \mathbb{C}$ defined on a closed interval of length $p > 0$ and for which $g(a) = g(a+p)$, one defines the p-periodic extension $\widetilde{g} : \mathbb{R} \to \mathbb{C}$ by

$$\widetilde{g}(x+np) = g(x)$$

for all $x \in [a, a+p[$ and all $n \in \mathbb{Z}$. In the same manner one defines the p-periodic extension of a function $g : [a, a+p[\to \mathbb{C}$.

2.3 DEFINITION (Trigonometric polynomial and trigonometric series)**.** A trigonometric polynomial of period $p > 0$ is a real or complex function of the form

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{n} \left(a_k \cos\left(\frac{2\pi}{p}kx\right) + b_k \sin\left(\frac{2\pi}{p}kx\right) \right),$$

where $a_k, b_k \in \mathbb{C}$, $k = 1, \ldots n$. Equivalently, [1]

$$\sum_{k=-n}^{n} c_k e^{i\frac{2\pi}{p}kx} = c_0 + \sum_{k=1}^{n} \left(c_k e^{i\frac{2\pi}{p}kx} + c_{-k} e^{-i\frac{2\pi}{p}kx} \right),$$

where $c_k \in \mathbb{C}$, $k = -n, \ldots, n$. The transition between the two formulas is given by

$$\begin{aligned} c_0 &= \tfrac{1}{2}a_0, & c_k &= \tfrac{1}{2}(a_k - ib_k), & c_{-k} &= \tfrac{1}{2}(a_k + ib_k), \\ a_0 &= 2c_0, & a_k &= c_k + c_{-k}, & b_k &= i(c_k - c_{-k}), \end{aligned} \tag{1}$$

for all $k = 1, \ldots, n$.

A trigonometric series of period $p > 0$ is a series of form

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi}{p}kx\right) + b_k \sin\left(\frac{2\pi}{p}kx\right) \right), \quad a_k, b_k \in \mathbb{C},\ k \in \mathbb{N}$$

[1] using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

or equivalently

$$\sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi}{p}kx} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{i\frac{2\pi}{p}kx} + c_{-k}e^{-i\frac{2\pi}{p}kx}\right), \quad c_k \in \mathbb{C},\ k \in \mathbb{Z}.$$

The transition formulas are as in (1).

2.4 THEOREM (Absolute convergence implies uniform convergence)**.** *If the series* $\sum_{k=-\infty}^{\infty} |c_k|$ *converges, then the trigonometric series* $\sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi}{p}kx}$ *converges uniformly on* $\mathbb{R}$*.*

To ease notation we will primarily use the period $p = 2\pi$. This is no real limitation, as the general case can be transformed by considering $\frac{2\pi}{p}x$ instead of x. To further ease the notation we introduce the notation

$$e_k(x) = e^{ikx}, \quad k \in \mathbb{Z}$$

for the 2π-periodic pure oscillations. An important property of these functions is the following relationship.

2.5 LEMMA (Orthonormality relation)**.** *For all* $n, m \in \mathbb{Z}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}dx = \begin{cases} 0, & \text{for} \quad n \neq m \\ 1, & \text{for} \quad n = m \end{cases}.$$

These orthonormality relations will allow us to use methods from linear algebra to study trigonometric polynomials and series. We will consider functions e_k, $k \in \mathbb{Z}$ as an orthonormal family of vectors in an inner product vector space.

2.2 Inner product formalism

2.6 DEFINITION (Piecewise continuous function)**.** A function $f : [a, b] \to \mathbb{C}$ is called piecewise continuous if there exists a division of the interval $[a, b]$ by dividing points $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ and for each $j \in \{1, \ldots, k\}$ a continuous function $f_j : [x_{j-1}, x_j] \to \mathbb{C}$ which is identical with f on the open interval $]x_{j-1}, x_j[$. A function $f : I \to \mathbb{C}$ on an interval $I \subseteq \mathbb{R}$ is said to be piecewise continuous, if it is piecewise continuous, on any closed and bounded sub-interval of I.

For a piecewise continuous function, we define the limits from the right and left

$$f(x+0) = \lim_{h\downarrow 0}(x+h), \quad f(x-0) = \lim_{h\downarrow 0} f(x-h). \tag{2}$$

Points of continuity are characterized by $f(x+0) = f(x-0) = f(x)$.

2.7 DEFINITION (Normalized piecewise continuous function)**.** A piecewise continuous function $f : I \to \mathbb{C}$ is called normalized, if

$$f(x) = \tfrac{1}{2}(f(x+0) + f(x-0))$$

for all $x \in I$ which are not end points of I.

An arbitrary piecewise continuous function can be normalized by substituting for $f(x)$ the value $\frac{1}{2}(f(x+0) + f(x-0))$ at any point x of discontinuity.

Note that if $g : [a, a+p] \to \mathbb{C}$, with $g(a) = g(a+p)$, is a normalized piecewise continuous function, then the p-periodic extension $\widetilde{g}$ is normalized if and only if

$$\tfrac{1}{2}(g(a+0) + g(a+p-0)) = g(a).$$

2.8 LEMMA (Mean value of a periodic function)**.** *If $f : \mathbb{R} \to \mathbb{C}$ is piecewise continuous and periodic with period p, then*

$$\mathcal{M}_p(f) := \frac{1}{p}\int_a^{a+p} f(x)dx$$

is independent of $a \in \mathbb{R}$. The number $\mathcal{M}_p(f) \in \mathbb{C}$ is called the mean value of f.

The set $PC_{2\pi}$ of 2π-periodic normalized piecewise continuous functions is a complex vector space with the addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x),$$

for all $f, g \in PC_{2\pi}$ and $\lambda \in \mathbb{C}$. On this vector space we define an inner product by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx. \tag{3}$$

We see immediately from Lemma 2.8 that

$$\langle f, g\rangle = \frac{1}{2\pi}\int_a^{a+2\pi} f(x)\overline{g(x)}dx.$$

for all $a \in \mathbb{R}$.

We can now express the orthonormality relations in Lemma 2.5 by the family of vectors (functions) $(e_n)_{n\in\mathbb{Z}}$ being an orthonormal family in $PC_{2\pi}$, i.e.,

$$\langle e_n, e_m\rangle = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \tag{4}$$

for all $n, m \in \mathbb{Z}$. The trigonometric polynomials belonging to $PC_{2\pi}$ are exactly the functions in the subspace spanned by the vectors $(e_n)_{n\in\mathbb{Z}}$, i.e., $\operatorname{span}\left\{e_n \,\middle|\, n \in \mathbb{Z}\right\}$.

Note that a sum function for a trigonometric *series* does not necessarily belong to the linear span, as the span of a family of vectors is defined as *finite* linear combinations of vectors from the family.

An important consequence of orthonormality is that if $s = \sum_{k=-n}^{n} c_k e_k$, i.e.,

$$s(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

is a trigonometric polynomial belonging to the period 2π, then the coefficients will be given by

$$c_k = c_k \langle e_k, e_k \rangle = \sum_{m=-n}^{n} c_m \langle e_m, e_k \rangle = \langle s, e_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} s(x) e^{-ikx} dx, \tag{5}$$

where $|k| \le n$. Moreover, it follows that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |s(x)|^2 &= \langle s, s \rangle = \sum_{k=-n}^{n} \sum_{m=-n}^{n} \langle c_k e_k, c_m e_m \rangle = \sum_{k=-n}^{n} \sum_{m=-n}^{n} c_k \overline{c_m} \langle e_k, e_m \rangle \\ &= \sum_{k=-n}^{n} |c_k|^2. \end{aligned} \tag{6}$$

An interesting question is whether the results in (5) and (6) generalize to sum functions for trigonometric series. If we assume that the series is uniformly convergent, this is true.

2.9 THEOREM (uniformly convergent trigonometric series)**.** *If a trigonometric series* $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ *is uniformly convergent with sum function* $f(x)$*, then* $f : \mathbb{R} \to \mathbb{C}$ *is continuous and* 2π*-periodic. In addition, the coefficients are given by*

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \tag{7}$$

and we have the Parseval identity

$$\mathcal{M}_{2\pi}(|f|^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{8}$$

The theorem tells us that if a trigonometric series converges uniformly, then the series is uniquely determined from its sum function by the coefficients in (7). For any piecewise continuous 2π-periodic function f, we can calculate these coefficients and hence a uniquely determined trigonometric series. We emphasize that this series is not necessarily uniformly convergent and does not necessarily have f as sum function. We call this series the *Fourier series* for the function.

2.10 DEFINITION (Fourier series)**.** If $f : \mathbb{R} \to \mathbb{C}$ is a piecewise continuous 2π-periodic function, then the numbers

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx, \quad k \in \mathbb{Z} \tag{9}$$

are called the Fourier coefficients of f and the series

$$\sum_{k=-\infty}^{\infty} c_k(f)e^{ikx}$$

is called the *Fourier series* for f.

More generally, if f is p-periodic and piecewise continuous, the Fourier coefficients and the Fourier series are defined by

$$c_k(f) = \frac{1}{p} \int_{-p/2}^{p/2} f(x)e^{-i\frac{2\pi k}{p}x}dx, \quad \sum_{k=-\infty}^{\infty} c_k(f)e^{i\frac{2\pi k}{p}x}. \tag{10}$$

We will also define the Fourier series for functions defined on bounded intervals.

2.11 DEFINITION (Fourier series for functions on bounded intervals)**.** If $f : I \to \mathbb{C}$ is defined on an interval of length p and has a piecewise continuous extension to the closure $\overline{I}$, we define the Fourier coefficients and Fourier series for f by

$$c_k(f) = \frac{1}{p} \int_I f(x)e^{-i\frac{2\pi k}{p}x}dx, \quad \sum_{k=-\infty}^{\infty} c_k(f)e^{i\frac{2\pi k}{p}x}. \tag{11}$$

In this case, the Fourier series for f is identical to the Fourier series for every p-periodic function $\widetilde{f} : \mathbb{R} \to \mathbb{C}$ for which $\widetilde{f} = f$ in the interior of I.

Theorem 2.9 thus says that if a trigonometric series converges uniformly, then it is the Fourier series for its sum function. It is quite analogous to the result for power series, which says that a convergent power series is the Taylor series for its sum.

Using inner product notation the Fourier coefficients of a 2π-periodic normalized piecewise continuous function f can also be written

$$c_k(f) = \langle f, e_k \rangle. \tag{12}$$

Using the transition formulas (1) we see that the Fourier series of a 2π-periodic piecewise continuous function f can also be written

$$\tfrac{1}{2}a_0(f) + \sum_{k=1}^{n} \left(a_k(f)\cos(kx) + b_k(f)\sin(kx)\right),$$

where

$$\begin{aligned} a_k(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx,\ \ k \in \mathbb{N}\cup\{0\}, \\ b_k(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx,\ \ k \in \mathbb{N}. \end{aligned} \tag{13}$$

2.12 REMARK (Fourier series for even and odd functions). If $f : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic, piecewise continuous function, which is also *even*, i.e., $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)dx$$

for all $k \in \mathbb{N}\cup\{0\}$ and

$$b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = 0,$$

for all $k \in \mathbb{N}$, since $f(-x)\cos(-kx) = f(x)\cos(kx)$ and $f(-x)\sin(-kx) = -f(x)\sin(kx)$. Hence the Fourier series for f is a pure cosine series (the sine terms are missing). Conversely if f is *odd* i.e. $f(x) = -f(-x)$ for all $x \in \mathbb{R}$ then

$$a_k(f) = 0, \quad b_k(f) = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(kx)dx$$

for all k. Hence the Fourier series is a pure sine series (the constant term and the cosine terms are missing).

We saw in Theorem 2.4 a criterion which ensures uniform convergence of a trigonometrical series. Another interesting question which we will study in the next section is which properties of a function ensure convergence of its Fourier series and convergence towards the function itself. Since continuity is maintained by uniform limits, it is clear that if f is not continuous, the Fourier series can not converge uniformly to f. There remains the possibility that it converges pointwise and we will give a sufficient criterion for this in the next section.

Although Fourier series do not always converge uniformly, it can be shown that Parseval's identity (8) holds for the Fourier coefficients of all piecewise continuous functions f. we will not show this result here, but merely show an inequality.

2.13 THEOREM (Bessel's inequality). *If f is piecewise continuous and 2π-periodic, the Fourier coefficients $c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$ satisfy Bessel's inequality*

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

In particular, the series on the left is convergent.

Bessel's inequality gives that when f is piecewise continuous and 2π-periodic, the series $\sum_{k=-\infty}^{\infty} |c_k(f)|^2$ is convergent. It follows that the terms must tend to zero. This result is one of the most crucial steps in the study of Fourier series and is known as Riemann's Lemma .

2.14 LEMMA (Riemann's Lemma). *If $f : [-\pi, \pi] \to \mathbb{C}$ is piecewise continuous, then*

$$\int_{-\pi}^{\pi} f(x)e^{-inx}dx \to 0 \quad \text{when } |n| \to \infty.$$

In particular, if $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and piecewise continuous, then $c_n(f) \to 0$ when $|n| \to \infty$.

3 Pointwise convergence of Fourier series

We shall in this section provide sufficient criteria for when the Fourier series of a function converges pointwise.

3.1 THEOREM (Convergence Test for Fourier series). *Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous 2π-periodic function. If there exist $s \in \mathbb{C}$ such that the limit*

$$\lim_{t \downarrow 0} \frac{f(x+t) + f(x-t) - 2s}{t}$$

exists, then the Fourier series for f converges in the point x with sum s. That is

$$\sum_{k=-\infty}^{\infty} c_k(f)e^{ikx} = s.$$

Using the newly derived criteria, we will now show that there is a large class of functions for which the Fourier series converges pointwise to the function itself. It does not apply to all piecewise continuous functions. It requires a little more *regularity.* In full analogy with the piecewise continuous functions, we define piecewise differentiable functions.

3.2 DEFINITION (piecewise differentiable functions). A function $f : [a, b] \to \mathbb{C}$ is called *piecewise differentiable*, if there exists a division of the interval $[a, b]$ by dividing points $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ and for each $j \in \{1, \ldots, k\}$ a differentiable function [2] $f_j : [x_{j-1}, x_j] \to \mathbb{C}$ which is identical with f on the open interval $]x_{j-1}, x_j[$. A function $f : I \to \mathbb{C}$ on any interval $I \subseteq \mathbb{R}$ is said to be piecewise differentiable, if it is piecewise differentiable on any closed and bounded subinterval of I.

[2] This will specifically say that f_j has a derivative from left in the x_{j-1} and a derivative from the right in the x_j

Note that a piecewise differentiable function is piecewise continuous, but not necessarily continuous. We have easily the following equivalent characterization of a piecewise differentiable function.

3.3 LEMMA (Characterization of piecewise differentiable functions)**.** *A piecewise continuous function $f : [a,b] \to \mathbb{C}$ is piecewise differentiable if and only if the right and left derivative*

$$f'_+(x) = \lim_{t \downarrow 0} \frac{f(x+t) - f(x+0)}{t} \quad \text{and} \quad f'_-(x) = \lim_{t \downarrow 0} \frac{f(x-t) - f(x-0)}{-t} \tag{14}$$

exist at all points $x \in]a,b[$, are the same except for at most finally many points, and $f'_+(a)$ and $f'_-(b)$ exist.

Using the convergence criterion from Theorem 3.1 it is easy to show the following main result on pointwise convergence of Fourier series for piecewise differentiable functions.

3.4 THEOREM (Fourier series for piecewise differentiable functions)**.** *For every piecewise differentiable 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ the Fourier series is pointwise convergent at all points with sum function equal to the function obtained by normalizing f.*

3.5 EXAMPLE. Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function on $[-\pi, \pi[$ given by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [0, \pi[\\ 0 & \text{if} \quad x \in [-\pi, 0[\,. \end{cases}$$

The function f is piecewise differentiable and it is only discontinuous at $p\pi$, $p \in \mathbb{Z}$.

By explicit calculation we find the Fourier coefficients

$$c_n(f) = \frac{1}{i\pi n}\,, \quad \text{if } n \text{ is odd}\,,$$

while $c_0(f) = \frac{1}{2}$ and $c_n(f) = 0$, if n is even and $\neq 0$. This gives that the Fourier series for f is

$$\begin{aligned} &\frac{1}{2} + \frac{1}{\pi} \sum_{\substack{n=1 \\ n \text{ ulige}}}^{\infty} \frac{1}{in}(e^{inx} - e^{-inx}) \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}\,. \end{aligned}$$

¿From theorem 3.4 one obtains, since $\frac{1}{2}(f(p\pi + 0) + f(p\pi - 0)) = \frac{1}{2}$ for $p \in \mathbb{Z}$, that the Fourier series is pointwise convergent in $[-\pi, \pi[$ and that

$$\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1} = \begin{cases} 0 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \\ \frac{1}{2} & \text{if} \quad x = -\pi \text{ or } x = 0\,. \end{cases}$$

For $x = -\pi$ or $x = 0$ this is also verified immediately. As the sum function is not continuous, it is clear that the the Fourier series does not converge uniformly.

Inserting $x = \frac{\pi}{2}$ and using $\sin((2k+1)\frac{\pi}{2}) = (-1)^k$ one obtains

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots = \frac{\pi}{4} \,.$$

4 Uniform convergence of Fourier series

We saw in the previous section that a piecewise differentiable function has a pointwise convergent Fourier series. If the function f is discontinuous the Fourier series cannot be uniformly convergent. We will in this section provide a criterion for uniform convergence. It is found not to be sufficient to require continuity of f.

4.1 DEFINITION (Normalized derivative)**.** If $f : I \to \mathbb{C}$ is a piecewise differentiable and continuous function on an arbitrary interval I, we define the normalized derivative

$$f'(x) = \tfrac{1}{2}(f'_+(x) + f'_-(x)),$$

for all x in the interior of I, where we note that according to Lemma 3.3 the right and left derivative $f'_\pm(x)$ exist for all $x \in I$. If $f'_+(x) = f'_-(x)$ then f is differentiable at x and our definition of $f'(x)$ is consistent with the usual derivative. However, if $f'_+(x) \neq f'_-(x)$ then f is not differentiable at x, but in this case we shall still denote the normalized derivative by $f'(x)$. The function obtained by taking the normalized derivative at every x is *not* necessarily piecewise continuous but if it is, we say that f is piecewise C^1.

Note that we only speak of the normalized derivative function if f is continuous, although the definition in a way makes sense even when f is not continuous.

4.2 LEMMA (Fourier coefficients of a continuous piecewise C^1-function)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic, continuous and piecewise C^1. Then the normalized derivative f' is a 2π-periodic normalized piecewise continuous function. Its Fourier coefficients satisfy*

$$c_n(f') = inc_n(f), \tag{15}$$

for all $n \in \mathbb{Z}$.

4.3 THEOREM (Uniform convergence of Fourier series for C^1-functions)**.** *If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, continuous and piecewise C^1 with Fourier coefficients $c_n(f)$, $n \in \mathbb{Z}$, then the series $\sum_{n=-\infty}^{\infty} |c_n(f)|$ converges. In this case the Fourier series for f converges uniformly and absolutely towards f.*

Proof. We know from Theorem 3.4 that the Fourier series for f converges pointwise to f. The last assertion follows, therefore, if we can show that it also converges uniformly. That follows from Theorem 2.4 if we can show the first claim, that $\sum_{k=-\infty}^{\infty} |c_k(f)|$ is convergent. We will use Bessel's inequality to show this, but Bessel's inequality comments on the sum of absolute values of *squares* on Fourier coefficients and not the sum of absolute values of the coefficient itself. The idea is not to use the Bessel for the function f, but for f' with the following observation: For arbitrary positive numbers $a, b > 0$ one has

$$ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2 \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

Using Lemma 4.2 and this inequality we see that for $|k| \geq 1$

$$|c_k(f)| = \frac{1}{|k|}|c_k(f')| \leq \frac{1}{2|k|^2} + \frac{1}{2}|c_k(f')|^2$$

We know that the series $\sum_{k=1}^{\infty} k^{-2}$ is convergent and from Bessel's inequality it follows that the series $\sum_{k=-\infty}^{\infty} |c_k(f')|^2$ is also convergent. The comparison criterion provides, therefore, that $\sum_{k=-\infty}^{\infty} |c_k(f)|$ is convergent, which was what we wanted to show. □

5 Functions on bounded intervals

In this section we will look at functions which are only defined on bounded intervals. We will see that they can be written in several different ways as sum functions for trigonometric series.

If $f : I \to \mathbb{C}$ is a function defined on an interval of length p, and f has a piecewise continuous extension to $\overline{I}$, then we can consider the Fourier series of f given in Definition 2.11. There exists a unique normalized piecewise continuous p-periodic function $f^\sharp$ which equals f on I except at most in finitely many points. We will refer to this function as the normalized p-periodic extension of f . We see that $f^\sharp$ is piecewise differentiable if f can be extended to a piecewise differentiable function on $\overline{I}$. The Fourier series of f is the same as the Fourier series for $f^\sharp$, which together with the Theorems 3.4 and 4.3 immediately gives the following result.

5.1 THEOREM (Convergence of Fourier series on a bounded interval)**.** *If the function $f : I \to \mathbb{C}$ defined on an interval I of length p, can be extended to a piecewise differentiable function on $\overline{I}$, the Fourier series for f converges pointwise on $\mathbb{R}$ and hence especially on I to the normalized p-periodic extension $f^\sharp$. If $f^\sharp$ is continuous and piecewise C^1 the convergence is uniform.*

Note that we decided to extend the function f to a periodic function with a period equal to the length of defining the range of f. One could have chosen to extend f to a periodic function on other ways and could thus have obtained trigonometric series with other periods, also converging to f. We now give two examples of this.

If $f : [0, \pi] \to \mathbb{C}$ is a normalized piecewise differentiable function which is continuous at 0 and π, we can extend f to an even function f_E on $[-\pi, \pi]$ by

$$f_E(x) = \begin{cases} f(x), & \text{if } x \in [0, \pi] \\ f(-x), & \text{if } x \in [-\pi, 0[. \end{cases}$$

Note that f_E is an even, normalized piecewise differentiable function with $f_E(-\pi) = f_E(\pi)$. (Why is it important that f is continuous at the endpoints?). Therefore, if we find the 2π periodic extension $\widetilde{f_E}$ of f_E we get a 2π-periodic, even, normalized piecewise differentiable function. We therefore know from Remark 2.12 that the Fourier series for $\widetilde{f_E}$ is a pure cosine series. The coefficients of this series can be expressed in terms of f by at $a_k(\widetilde{f_E}) = \frac{2}{\pi} \int_0^\pi f(x) \cos(kx) dx$. So we have the pointwise convergent series

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos(kx), \quad \text{where } a_k = \frac{2}{\pi} \int_0^\pi f(x) \cos(kx) dx, \ k = 0, 1, \ldots$$

for all $x \in [0, \pi]$. If in addition f is continuous and piecewise C^1 then the series is uniformly convergent. Note that it is important that our interval is $[0, \pi]$ as a function of an arbitrary interval $[a, a + \pi]$ can not necessarily be written as a cosine series.

As the last example, we will look at the odd extension of a function. If $f : [0, \pi] \to \mathbb{C}$ is a normalized piecewise differentiable function, which takes the value 0 at points 0 and π, we can extend f for an odd function $f_O : [-\pi, \pi]$ by

$$f_O(x) = \begin{cases} f(x), & \text{if } x \in [0, \pi] \\ -f(-x), & \text{if } x \in [-\pi, 0[. \end{cases}$$

Then f_O is an odd normalized piecewise differentiable function with $f_O(-\pi) = f_O(\pi)$. It is important here that we started with a function f satisfying that $f(0) = f(\pi) = 0$ (why?). We can therefore find the 2π-periodic extension $\widetilde{f_O}$ of f_U, which is a 2π-periodic, odd, normalized piecewise differentiable function. We therefore know from Remark 2.12 that the Fourier series for $\widetilde{f_O}$ is a pure sine series . The coefficients of this series can be expressed in terms of f by as $b_k(\widetilde{f_E}) = \frac{2}{\pi} \int_0^\pi f(x) \sin(kx) dx$. So we have the pointwise convergent series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(kx), \quad \text{where } b_k = \frac{2}{\pi} \int_0^\pi f(x) \sin(kx) dx, \ k = 1, 2 \ldots$$

for all $x \in [0, \pi]$. If in addition f is continuous and piecewise C^1, then the series is uniformly convergent.

Let us conclude by noting that if we have a normalized piecewise differentiable function $f : [0, \pi] \to \mathbb{C}$ for which $f(0) = f(\pi) = 0$ and which is continuous in these two points, then we can write the function as sum function of both a pure cosine series and a pure pure sine series. Moreover, we can also write the function as sum function of a π-periodic Fourier series as described first in this section.