Power series exercises

1. Determine the radius of convergence for the power series

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{n} t^n$$

and find its sum (hint: differentiate the series).

2. Consider the initial value problem

$$x'' - x' - x = 0, \quad x(0) = 0,\ x'(0) = 1.$$

2a. Determine the recursion formula for the coefficients c_n of a power series solution $\sum c_n t^n$. What is the radius of convergence for the series?

2b. Rewrite the recursion formula in terms of the sequence defined by $F_n = c_n n!$

2c. Solve the initial value problem without use of power series

3. The equation

$$x'' - tx = 0$$

is called *Airy's equation*. It plays a role in the theory of optics.

3a. Determine a recursion formula for a power series solution. What is the radius of convergence? Some terms are always zero, which?

3b. Denote the solution with $x(0) = 1$, $x'(0) = 0$ by $\mathrm{A}(t)$, and the solution with $x(0) = 0$, $x'(0) = 1$ by $\mathrm{B}(t)$. Verify that

$$\mathrm{A}(t) = 1 + \sum_{k=1}^{\infty} \frac{t^{3k}}{(2 \cdot 3)(5 \cdot 6) \ldots ((3k-1) \cdot (3k))}$$

and write down the similar formula for $\mathrm{B}(t)$.

4. Consider the Legendre equation

$$(1 - t^2)x'' - 2tx' + l(l+1)x = 0$$

with initial values: (a) $x(0) = 1$, $x'(0) = 0$, or: (b) $x(0) = 0$, $x'(0) = 1$.

4a. Determine for every $l \in \mathbf{C}$ the radius of convergence for the power series solution $x(t) = \sum_{n=0}^{\infty} c_n t^n$, in both cases (a) and (b).

4b. Consider the case (b) and $l = 0$. Show that

$$x(t) = \sum_{k \geq 1, \text{odd}} \frac{1}{k} t^k$$

for $-1 < t < 1$. Prove that $x(t) \to \infty$ for $t \uparrow 1$ (hint: For example, you can show that $x(t) \geq \frac{1}{2} f(t)$ for $0 < t < 1$, where f is as in Exercise 1).

5. Consider the equation

$$x'' + e^t x = 0.$$

5a. Determine a recursion formula for a power series solution. What is the radius of convergence for the series?

5b. Determine the first five terms (up to t^4) of the series in case $x(0) = 1$, $x'(0) = 0$.

6. Investigate what happens when you try to solve the equation

$$tx' + x = 0$$

by a power series Ansatz $x(t) = \sum_{n=0}^{\infty} c_n t^n$.

Explain the result by solving the equation explicitly.

7. Verify that the series for the Bessel function $J_p(t)$ converges for all t.