Boundary value problems for partial differential equations

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1 Introduction

This note contains a brief introduction to linear partial differential equations. Partial differential equations differ from ordinary differential equations by being equations for functions that can depend on more than one variable.

We shall only focus on one particular method of solution, based on the theory of Fourier series. It is a special case of a more general method called *separation of variables*. The idea of the general method is to look for solutions that are products of functions of one variable, with the hope that all other solutions are obtained from taking, in general, infinite sums of such product functions. In the special case that we shall consider, these infinite sums will be Fourier series.

As with ordinary differential equations, the partial differential equations are usually given with an additional constraint on the solution. In the theory of ordinary equations, where the solution depends only on one variable, the extra constraint consists of the initial conditions which impose some given values for the function (and possibly its derivatives up to some order), at some given initial time t_0. For partial differential equations, the typical additional constraint is a so-called boundary condition, in which specified values are imposed at points on the boundary of the domain where the solution is supposed to be defined. This explains the title *boundary value problems* of this note.

There are three main types of partial differential equations of which we shall see examples of boundary value problems - the wave equation, the heat equation and the Laplace equation. They are all defined in terms of the *Laplacian* Δ. By definition this is the differential operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

which acts on (two times differentiable) functions of $x \in \mathbb{R}^n$.

The *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta u,$$

and the *heat equation*

$$\frac{\partial u}{\partial t} = \Delta u,$$

both for functions $u(x,t)$ of $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and the *Laplace equation*

$$\Delta u(x) = 0$$

for functions $u(x)$ of $x \in \mathbb{R}^n$. In each case, the differential equation is considered for elements (x,t) in some domain Ω, in which the solution is required to be defined and two times differentiable, and it is accompagnied by some conditions for u on the boundary of Ω. The exact type of boundary conditions can vary from case to case.

In this note we shall just look at one of these equations, the wave equation, and we shall only consider the simplest boundary value problem for it.

2 The wave equation

Let us consider the wave equation in (space) dimension one,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

for (x, t) in the domain $[0, a] \times [0, \infty[$, where $a > 0$. A typical boundary condition consists of

$$u(0, t) = u(a, t) = 0$$

for all $t \geq 0$, and

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x)$$

for all $x \in [0, a]$, for some prescribed functions f and g on the interval $[0, a]$. Clearly, a solution satisfying these boundary conditions can only exist if we have $f(0) = f(a) = 0$. Similarly we need $g(0) = g(a) = 0$.

The physical interpretation is that $u(x, t)$ describes the displacement at location x and time t of a freely vibrating string of length a (and with a suitably normalized tension). The first condition above signifies that the endpoints of the string are fixed, and the second condition specifies an initial position together with an initial velocity of the string. For simplicity we assume in the following that the initial velocity is $g = 0$. For simplicity we assume in the following also that $a = \pi$. This can easily be achieved by a change of variables, in which x and t are both scaled by the common factor π/a.

The idea of our approach is to seek a solution in the form of a Fourier series

$$\sum_{n \in \mathbb{Z}} c_n(t) e^{inx} \tag{1}$$

with t-dependent coefficients. The series is assumed to constitute the Fourier series of the function $x \mapsto u(x, t)$ at all times t. In order to solve the equation, one then needs to determine the coefficient functions $c_n(t)$ such that the differential equation and the boundary conditions are satisfied by the sum of the series.

Recall from the Fourier theory that a function defined for $x \in [0, \pi]$ allows two natural 2π-periodic extensions, given by the even and the odd extension to $[-\pi, \pi]$. For the even extension, the Fourier series becomes a pure cosine series, and for the odd extension it becomes a pure sine series. In our case, where the boundary conditions prescribe that the solution $u(x, t)$ vanishes for $x = 0$ and $x = \pi$, it is natural to use the odd extension, since the sine functions $\sin(nx)$ already satisfy this condition for all n. For this reason it is advantageous to replace the Fourier series (1) by a Fourier sine series

$$\sum_{n=1}^{\infty} b_n(t) \sin(nx) \tag{2}$$

again with t-dependent coefficients.

Let us postpone the question of convergence of (2) for the moment, and analyze what other conditions are needed in order that the sum $u(x,t)$ solves our boundary value problem. Assuming that the convergence of the series is strong enough, we can perform the partial differentiations with respect to t and x termwise in the series. Inserting the series into the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

we thus obtain

$$\sum_{n=1}^{\infty} \frac{\partial^2 b_n(t)}{\partial t^2} \sin(nx) = \sum_{n=1}^{\infty} -b_n(t) n^2 \sin(nx)$$

for all (x,t) and by uniqueness of the coefficients in such a series, we conclude that

$$\frac{\partial^2 b_n}{\partial t^2} = -n^2 b_n$$

for all n. This is an ordinary differential equation for the function b_n, which we can solve. We find, for each n,

$$b_n(t) = C_n \cos(nt) + D_n \sin(nt) \tag{3}$$

with arbitrary constants C_n, D_n. These constants will now be determined from the boundary conditions.

The vanishing at $x = 0$ and $x = \pi$ has been taken care of by choosing a series (2) for which the individual terms already vanish at these points. To satisfy the boundary condition that $u(x,0) = f(x)$, we need that the series

$$\sum_{n=1}^{\infty} b_n(0) \sin(nx)$$

sums to $f(x)$ for all x. From the Fourier theory we recall the following.

2.1 Theorem. *Let $f \in C^1([0,\pi])$ with $f(0) = f(\pi) = 0$ and let $B_1, B_2, \dots$ be given by*

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx. \tag{4}$$

Then $\sum_{n=1}^{\infty} |B_n| < \infty$ and the Fourier sine series

$$\sum_{n=1}^{\infty} B_n \sin(nx), \quad (x \in [0,\pi]),$$

converges uniformly to $f(x)$.

Proof. Let $F : \mathbb{R} \to \mathbb{C}$ be the 2π-periodic extension of of the odd extension $f_O : [-\pi, \pi] \to \mathbb{C}$ of f. Thus

$$F(x + n2\pi) = \begin{cases} f(x) & \text{if } x \in [0, \pi] \\ -f(-x) & \text{if } x \in [-\pi, 0] \end{cases}$$

for $n \in \mathbb{Z}$. It is easily seen that F is continuous and a piecewise C^1-function. [1] Now Theorem 5.1 from the notes about Fourier series can be applied to F. □

Returning to the analysis of when the series (2) is a solution to our boundary value problem, we see that in order to satisfy $u(x, 0) = f(x)$ we must choose the coefficients $b_n(t)$ such that $b_n(0) = B_n$, where B_n is defined by (4). For each n this fixes the value of the first constant in (3) to $C_n = B_n$.

For the other boundary condition, that $\frac{\partial u}{\partial t}(x, 0) = g(x) = 0$ for all x, we consider the termwise differentiated series

$$\sum_{n=1}^{\infty} b_n'(t) \sin(nx),$$

Although we have not yet argued for the validity of termwise differentiation, it appears that we need this series to represent the zero function $g(x) = 0$. This will be the case if $b_n'(0) = 0$ for all n, and this is achieved if $D_n = 0$ in (3) for all n.

The conclusion of this analysis is thus that we expect that the Fourier sine series

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(nx) \cos(nt)$$

will produce a solution to the boundary value problem if we define the coefficients by (4). However, we have not yet considered the delicate questions of convergence of the series and the validity of termwise differentiation in it. Some extra regularity of f is needed, and the precise result is stated in the following theorem.

2.2 Theorem. *Assume $f \in C^3([0, \pi])$, and that*

$$f(0) = f(\pi) = 0, \qquad f''(0) = f''(\pi) = 0. \tag{5}$$

Let $B_1, B_2, \ldots$ be given by

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx. \tag{6}$$

[1] In fact, F is C^1: This is clear away from the multiples of π. At $x = 0$ the derivative from the left of $-f(-x)$ is equal to the derivative from the right of $f(x)$. Futhermore, the derivative at π from the left of $f(x)$ is equal to the derivative at $-\pi$ from the right of $-f(-x)$.

Then the sum

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(nt) \tag{7}$$

converges to a C^2*-function* $u(x,t)$ *on* $[0,\pi]\times[0,\infty[$*, which solves the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{8}$$

with the boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{9}$$

and

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0. \tag{10}$$

Conversely, if a C^2*-function of* (x,t) *solves this boundary value problem, then it is the unique function given by the sum* (7)*.*

Proof. Let F be the odd 2π-periodic extension of f, as defined in the proof of Theorem 2.1. It was seen in the footnote to that proof that F is C^1. The present assumption on the second derivatives of f at 0 and π implies that F'' is continuous at these points, hence F is C^2. It is in fact C^3, since continuity of the third derivative can be seen as in the footnote.

It follows from Theorem 2.1 that $\sum |B_n| < \infty$ and that the Fourier sine series of f converges uniformly to f. It follows from the convergence of $\sum |B_n|$ that the series (7) is convergent for all (x,t). It is then clear that (9) and the first boundary condition in (10) hold for the sum $u(x,t)$ of (7).

It remains to be verified that $u(x,t)$ is two times continuously differentiable and solves both the wave equation (8) and the second boundary condition in (10). Here we need to be able to differentiate $u(x,t)$ twice. We shall do this by termwise differentiation of the series. From the theorem about *termwise differentiation of a series* it is known that this is allowed, provided that both the series itself and the series of derivatives converge uniformly in a neighborhood of the given point. Note that each termwise differentiation of the series (7) (with respect to either t or x) worsens its convergence by multiplying a factor n to the n-th term. However, in our case, since $F \in C^3(\mathbb{R})$, we have $\sum n^2 |B_n| < \infty$ (apply Theorem 4.3 of the Fourier notes to the second derivative). This implies that after up to two termwise differentiations, the series (7) is still uniformly convergent. Hence $u(x,t)$ is C^2 and it suffices to verify the remaining conditions for each individual term $\sin(nx)\cos(nt)$. This is a straightforward verification.

To establish the final statement of uniqueness, we assume that $u(x,t)$ is some C^2-function which solves the boundary value problem. We have to show that it satisfies (7). We now define the functions $b_n(t)$ by

$$b_n(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\sin(nx)\,dx, \tag{11}$$

then we know from Theorem 2.1 that

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$$

for all (x,t). We have to show

$$b_n(t) = B_n \cos(nt)$$

where B_n are the coefficents in (6). All we have to show is that $b_n(t)$ solves the ordinary differential equation

$$\frac{\partial^2 b_n}{\partial t^2} = -n^2 b_n \tag{12}$$

with the initial conditions

$$b_n(0) = B_n, \quad b_n'(0) = 0. \tag{13}$$

The first condition in (13) follows immediately from the condition on u that $u(x,0) = f(x)$. We are going to determine the derivative b_n' of b_n from (11). Here we use the theorem from analysis about *differentiation under the integral.* According to this theorem, the derivative with respect to t of an integral $\int_a^b \varphi(x,t)\,dx$ of a function of two variables (x,t) can be determined as

$$\frac{d}{dt}\int_a^b \varphi(x,t)\,dx = \int_a^b \frac{\partial \varphi}{\partial t}(x,t)\,dx$$

provided φ is C^1. We apply this to (11) and obtain

$$b_n'(t) = \frac{2}{\pi}\int_0^{\pi} \frac{\partial u}{\partial t}(x,t)\sin(nx)\,dx.$$

In particular,

$$b_n'(0) = \frac{2}{\pi}\int_0^{\pi} \frac{\partial u}{\partial t}(x,0)\sin(nx)\,dx = 0$$

since $u(x,t)$ is assumed to satisfy the boundary condition. This shows the second condition in (13). Furthermore, again applying differentiation under the integral,

$$b_n''(t) = \frac{2}{\pi}\int_0^{\pi} \frac{\partial^2 u}{\partial t^2}(x,t)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{\partial^2 u}{\partial x^2}(x,t)\sin(nx)\,dx$$

since $u(x,t)$ is assumed to satisfy the wave equation. By two consequtive partial integrations, the latter integral can be rewritten as

$$b_n''(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\frac{d^2}{dx^2}\sin(nx)\,dx, \tag{14}$$

where in the first partial integration we use that $\sin(nx)$ vanishes in the endpoints 0 and π, and in the second we use that $u(x,t)$ vanishes in these points. Finally, from (14) we obtain

$$b_n''(t) = -n^2 \frac{2}{\pi} \int_0^\pi u(x,t) \sin(nx)\, dx = -n^2 b_n$$

as claimed in (12). □ □

2.3 Remark. The result in Theorem 2.2 can be improved slightly. In fact, it suffices to assume f is two times differentiable and satisfies (5). However, the argument for the improvement does not apply in general to other partial differential equations, and for this reason the proof given above for the weaker result is preferable.

The improvement is based on the observation that due to the trigonometric formula

$$\sin(\theta)\cos(\varphi) = \tfrac{1}{2}(\sin(\theta+\varphi) + \sin(\theta-\varphi)),$$

the series (7) can be rewritten as a sum of two absolutely convergent Fourier sine series

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\sum_{n=1}^{\infty} B_n \sin(n(x+t)) + \tfrac{1}{2}\sum_{n=1}^{\infty} B_n \sin(n(x-t)) \\ &= \tfrac{1}{2}F(x+t) + \tfrac{1}{2}F(x-t). \end{aligned}$$

That the last expression is a solution to the wave equation when F is two times differentiable, is an immediate consequence of the chain rule. It is also easy to verify the boundary conditions (the expression above is known as *d'Alembert's solution*).

The solution found in Theorem 2.2,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(nt),$$

has the following physical interpretation. It represents a superposition of oscillations with frequencies $\nu_n = \frac{n}{2\pi}$. The lowest frequency, $\nu_1 = \frac{1}{2\pi}$, is called the *fundamental frequency.* The higher frequencies $\nu_2, \nu_3, \ldots$ are called *overtones.* Each overtone is exactly one octave higher than the preceding. The corresponding displacements $\sin(nx)\cos(nt)$ of the string are called *pure oscillations.* The sound obtained from the string will thus be a superposition of pure oscillations with frequencies equal to the overtones, and the coefficients in this superposition are determined from the Fourier series of the initial displacement function $f(x)$.

3 Separation of variables

As described in the introduction the technique used in the previous chapter is an instance of a more general technique called *separation of variables*, which we will briefly describe.

The starting point for the investigation above was the Ansatz that a solution $u(x,t)$ could be obtained in the form of a Fourier series

$$\sum_{n\in\mathbb{Z}} c_n(t)e^{inx}$$

or a Fourier sine series

$$\sum_{n\in\mathbb{Z}} b_n(t)\sin(nx).$$

The method we are about to describe explains *why* we made this Ansatz, and it can be used to find reasonable analogues in some more general situations.

In the more general approach one first seeks a solution $u(x,t)$ of arbitrary product form. That is, we assume

$$u(x,t) = X(x)T(t)$$

for some one-variable functions X and T to be determined. With this Ansatz, and still aiming for solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

we obtain

$$X(x)T''(t) = X''(x)T(t)$$

for all x and t. Dividing by $X(x)T(t)$ (if allowed) we obtain

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

in which the variables have been separated. Since t and x are independent variables, the two sides can agree for all t and all x only if both sides are independent of both variables. In other words, $T''(t)/T(t)$ and $X''(x)/X(x)$ both have to be equal to a constant κ. We conclude that X and T have to satisfy the following ordinary differential equations

$$X'' - \kappa X = 0, \quad T'' - \kappa T = 0. \tag{15}$$

We keep in mind that we also want the boundary conditions

$$u(0,t) = u(a,t) = 0$$

for all $t \geq 0$, and

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0,$$

to be satisfied.

Let us concentrate on the function $X(x)$. If T is nontrivial the first boundary condition applied to $u(x,t) = X(x)T(t)$ implies

$$X(0) = X(a) = 0.$$

Suppose first that $\kappa > 0$. Then the differential equation $X'' - \kappa X$ implies that X is a linear combination

$$X(x) = c_1 e^{\sqrt{\kappa}x} + c_2 e^{-\sqrt{\kappa}x}$$

and the boundary condition implies

$$c_1 + c_2 = 0, \quad c_1 e^{\sqrt{\kappa}a} + c_2 e^{-\sqrt{\kappa}a} = 0.$$

Since $e^{\sqrt{\kappa}a} \neq e^{-\sqrt{\kappa}a}$, the only solution is $c_1 = c_2 = 0$ and hence $X = 0$. If $\kappa = 0$ we obtain similarly that X is a linear combination

$$X(x) = c_1 + c_2 x$$

which only satisfies $X(0) = X(a) = 0$ if $X = 0$. We conclude that the constant κ has to be negative.

We assume $\kappa < 0$ and let $\omega = \sqrt{-\kappa}$. Then the solution to $X'' - \kappa X$ reads

$$X(x) = c_1 e^{i\omega x} + c_2 e^{-i\omega x}$$

and the boundary condition now becomes

$$c_1 + c_2 = 0, \quad c_1 e^{i\omega a} + c_2 e^{-i\omega a} = 0.$$

Here nontrivial solutions are obtained precisely when $e^{i\omega a} = e^{-i\omega a}$, or equivalently, when $\omega a = n\pi$ for some $n \in \mathbb{N}$. In that case we obtain with $c_1 = -c_2$, that X is (a constant multiple of) the function

$$X(x) = \sin(\omega x) = \sin(\frac{n\pi x}{a}).$$

When $a = \pi$ as above, the analysis thus leads to the conclusion that it is reasonable to expect the product form of a solution only if $X(x)$ is a multiple of $\sin(nx)$ for some $n \in \mathbb{N}$.

Since the differential equation is linear, a more general solution is obtained by taking linear combinations of products $X(x)T(t)$ where $X(x) = \sin(nx)$ with different values of $n \in \mathbb{Z}$. The factor $T(t)$ is allowed to vary with n, and writing it as $T(t) = b_n(t)$ this leads exactly to the Ansatz

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$$

of the previous section. Notice that with $\kappa = -\omega^2 = -n^2$ the equation for $T(t)$ in (15) becomes

$$T'' = -n^2 T,$$

which is exactly the equation (12) which we solved in order to find b_n.

4 The wave equation, higher dimension

The vibrations of a circular drum are solutions of the wave equation in two space dimensions

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Here (x, y) describes a position on the membrane of the drum. We denote by $a > 0$ the radius of the membrane, then positions belong to

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}.$$

The variable t represents time, and the function $u(x, y, t)$ describes the displacement of the membrane in location (x, y) at time t. Assuming that the circular frame of the drum head is fixed we have the natural boundary condition

$$u(t, x, y) = 0, \quad \text{for } x^2 + y^2 = a^2.$$

In this example we shall discuss some solutions to this problem. Since the drum head is circular it is natural to transform (x, y) into polar coordinates

$$(x, y) = (r\cos\theta, r\sin\theta),$$

where $0 \leq r \leq a$ and $\theta \in \mathbb{R}$. In these coordinates the differential equation reads

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

and the boundary condition reads

$$u(t, a, \theta) = 0$$

for all t and θ.

The simplest vibrations are those for which the displacement $u(t, r, \theta)$ is independent of θ (called *radial vibrations*), and we shall confine ourselves to the consideration of these. The problem we want to solve is thus

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

with

$$u(t, a) = 0$$

for all t. Additional boundary conditions

$$u(0, r) = f(r), \quad \frac{\partial u}{\partial t}(0, r) = g(r)$$

describe the initial displacement and velocity at time $t = 0$ by some given functions f and g. We assume $f = 0$ in the following.

We follow the general strategy outlined in the preceding section and look for solutions in separated variables, $u(r, t) = R(r)T(t)$. Substituting in the equation above and dividing both sides by $R(r)T(t)$ yields

$$\frac{T''(t)}{T(t)} = \frac{R''(r) + \frac{1}{r}R'(r)}{R(r)}$$

for all t and all r. As before we conclude that both sides of this equation must be independent of both t and r, say equal to κ. We obtain the equations

$$T'' - \kappa T = 0$$

and

$$R'' + \frac{1}{r}R' - \kappa R = 0.$$

As before we separate in cases depending on the sign of κ, and motivated by the previous case we concentrate immediately on negative κ and put $\omega = \sqrt{-\kappa}$. We conclude that $R(r)$ must satisfy

$$R'' + \frac{1}{r}R' + \omega^2 R = 0$$

and $R(a) = 0$. Changing variables to $s = \omega r$ we obtain for the function $S(s) = R(s/\omega)$:

$$S'' + \frac{1}{s}S + S = 0$$

which we recognize as the Bessel equation of order $p = 0$. The equation was treated in the notes about power series solutions, where the solution $J_0(s)$ was found. There is a second independent solution (usually denoted $Y_0(s)$), but it behaves badly for $s \to 0$ and can be excluded for this reason. We conclude that

$$R(r) = J_0(\omega r)$$

and from the boundary condition $R(a) = 0$ we deduce that only the values of $\omega > 0$ for which $J_0(\omega a) = 0$ are relevant. Let $0 < \gamma_1 < \gamma_2 < \dots$ be the positive roots of $J_0(\gamma) = 0$, then $\omega = \gamma_n/a$ for some n.

Finally, we solve the equation $T'' = \kappa T = -\omega^2 T$, subject to the boundary condition (from the assumption $f = 0$) that $T(0) = 0$. We obtain

$$T(t) = \sin(\omega t) = \sin(\gamma_n t/a)$$

and conclude that the separated solutions are

$$u(t, r) = J_0(\gamma_n r/a) \sin(\gamma_n t/a).$$

A reasonable Ansatz for solving the problem is thus that

$$u(t, r) = \sum_{n=1}^{\infty} c_n J_0(\gamma_n r/a) \sin(\gamma_n t/a)$$

for some constants $c_1, c_2, \dots \in \mathbb{R}$.

The next step would be to determine the coefficients c_n in terms of the given function g, and prove that the infinite sum converges to a solution. The frequencies $\omega_n = \gamma_n/a$, which are determined from the zeros of the Bessel function and the radius of the drum, are the frequencies that one will hear when the drum is vibrating (provided it vibrates radially). Since these are not integral multiples of a fixed frequency, the series of sine functions above is not a series that can be treated within ordinary Fourier theory. One needs a more general theory, which is beyond the scope of this course.