Exercises on Fourier Series

1. Calculate the Fourier series of the 2π-periodic functions given by

$$f(x) = \begin{cases} 0 & (-\pi < x \leq 0) \\ 1 & (0 < x \leq \pi) \end{cases}$$
$$g(x) = \pi - |x| \qquad (-\pi \leq x < \pi)$$

2. Use the binomial formula and Euler's identity $e^{ix} = \cos(x) + i\sin(x)$ to compute the Fourier series of the function f given by

$$f(x) = \cos^m(x) + \sin^n(x),$$

where $m, n \in \mathbf{N}$.

3. Let f be a 2π-periodic function given by $f(x) = x$ for $0 < x < 2\pi$ and $f(0) = \pi$.

(a) Show that

$$f(x) = \pi - 2\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \qquad (x \in \mathbf{R})$$

(b) Deduce Leibniz's series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

4. Let f_2 and f_4 be the 2π-periodic functions given by $f_n(x) = x^n$ for $0 < x < 2\pi$ and $f_n(0) = 2^{n-1}\pi^n$ for $n = 2, 4$.

(a) Show that

$$f_2(x) = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} - 4\pi\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

and

$$f_4(x) = \frac{16\pi^4}{5} + \sum_{k=1}^{\infty} \left(\frac{32\pi^2}{k^2} - \frac{48}{k^4}\right)\cos(kx) + \sum_{k=1}^{\infty} \left(\frac{48\pi}{k^3} - \frac{16\pi^3}{k}\right)\sin(kx)$$

(b) Deduce the identities

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$