Extra Exercise 5

Let $G(t) = \int_0^t e^{-\frac{1}{2}s^2}\, ds$ for $t \in \mathbf{R}$ (which, up to normalization, is the error function).

Consider the homogeneous differential equation

$$\ddot{x} - t\dot{x} - x = 0. \tag{1}$$

1. Make the change of variables by the spatial transformation $x(t) = e^{\frac{1}{2}t^2}u(t)$
2. Solve the obtained equation for u, and determine the solution x to (1).
3. Find a particular solution to the inhomogeneous equation

   $$\ddot{x} - t\dot{x} - x = f(t) \tag{2}$$

   with $f(t) = t^2$, by making the Ansatz that x is a second order polynomial, and give the complete solution to this equation.
4. Find the corresponding first order equation $\dot{y} = A(t)y$ and determine a fundamental matrix $\Phi(t)$.
5. Write down the general formula from Theorem 8.19 for the solution to (2) with $f \in C^0(I)$ arbitrary.