Extra Exercise 4

The following differential equation is encountered in the geometry of surfaces as the *geodesic equation*,

$$\ddot{u}_i = \sum_{j,k=1}^{2} \Gamma^i_{jk}(u_1, u_2)\dot{u}_j\dot{u}_k \quad (i = 1, 2).$$

The eight functions Γ^i_{jk} (called the *Christoffel symbols* for the given surface) are indexed by numbers $i, j, k \in \{1, 2\}$. They are defined on an open set $U \subset \mathbf{R}^2$ and they are smooth $U \to \mathbf{R}$.

The theorem of existence of geodesics asserts that for every $w \in \mathbf{R}^2$ there exists a solution $u(s) = (u_1(s), u_2(s))$, defined for s in some interval I around $s = 0$, with $u(0) = (0, 0)$ and $\dot{u}(0) = w$. It also asserts that this solution is unique, in the sense that if another solution satisfying the same condition is defined on an interval $J \ni 0$, then it will agree with u in the intersection $I \cap J$.

1. Reformulate to a first order equation and verify the theorem.

2. For the Euclidean plane $U = \mathbf{R}^2$ and all the Γ^i_{jk} are 0. Solve the equation in this case. Which curves $s \mapsto u(s)$ are obtained in this case?

3. In another case (the sphere with spherical coordinates) we have

$$U = (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$$

and

$$\Gamma^1_{11} = \Gamma^1_{22} = 0, \quad \Gamma^1_{12}(u_1, u_2) = -2\tan u_2$$

and

$$\Gamma^2_{12} = \Gamma^2_{22} = 0, \quad \Gamma^2_{11}(u_1, u_2) = \sin u_2 \cos u_2$$

Without solving the equations, show that $\dot{u}_1(s)\cos^2(u_2(s))$ is constant for every solution $u = (u_1, u_2)$, $s \in I$.