Extra Exercise 3

Let $m : \mathbf{R} \to \mathrm{Mat}(n, \mathbf{R})$ be continuous. Consider the differential equation

$$\dot{y} = m\, y \tag{1}$$

(a) Assume $n = 1$. Show that

$$\mathbf{R} \ni t \mapsto c e^{\int_0^t m(s)\, ds} \qquad (c \in \mathbf{R}) \tag{2}$$

is a maximal solution to (1) and that all maximal solutions are of this form.

In the remainder of this exercise we will show that the naive generalization of (2) is in general not a solution to (1). For a continuous function $f : \mathbf{R} \to \mathrm{Mat}(n, \mathbf{R})$, we define $\int_a^b f(s)\, ds$ to be the matrix with coefficients

$$\left(\int_a^b f(s)\, ds \right)_{i,j} = \int_a^b f_{i,j}(s)\, ds.$$

(b) Assume $n = 2$. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $m : \mathbf{R} \to \mathrm{Mat}(2, \mathbf{R})$ be given by $m(t) = A + tB$ for $t \in \mathbf{R}$. Show that

$$e^{\int_0^t m(s)\, ds} = \begin{pmatrix} e^t & \frac{t}{2} \sinh(t) \\ 0 & e^{-t} \end{pmatrix}.$$

Prove that there exist $\eta \in \mathbf{R}^2$ such that $\mathbf{R} \ni t \mapsto e^{\int_0^t m(s)\, ds} \eta$ is not a solution of (1).

(c) Let $n > 1$. Use (b) to construct continuous functions $m : \mathbf{R} \to \mathrm{Mat}(n, \mathbf{R})$ with the property that there exist $\eta \in \mathbf{R}^n$ such that $\mathbf{R} \ni t \mapsto e^{\int_0^t m(s)\, ds} \eta$ is not a solution.

The reason that the functions considered in (b) and (c) are not solutions to (1) is that $n \times n$-matrices do not always commute for $n > 1$.

(d) Let A and B be commuting matrices and let $m : \mathbf{R} \to \mathrm{Mat}(n, \mathbf{R})$ be given by $m(t) = A + tB$ for $t \in \mathbf{R}$. Show that $\mathbf{R} \ni t \mapsto e^{\int_0^t m(s)\, ds} \eta$ is a solution of (1) for all $\eta \in \mathbf{R}^n$.