Solution. Assignment 6

We know that

$$t^2x'' + tx' + t^2x = 0 \qquad (t > 0) \tag{1}$$

has a power series solution of the form $J_0(t) = \sum_{k=0}^{\infty} a_k t^k$, where $a_k = 0$ for odd k and

$$k^2 a_k + a_{k-2} = 0 \tag{2}$$

for even $k \geq 2$.

(a) Let $a_0 = 1$. Then

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2} \tag{3}$$

is valid for $n = 0$. Now (3) follows from (2) by induction:

$$a_{2n} = -\frac{1}{(2n)^2} a_{2n-2} = -\frac{1}{(2n)^2} \frac{(-1)^{n-1}}{2^{2n-2}((n-1)!)^2} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

(b) Let $\mathrm{B} = t^2 \frac{d^2}{dt^2} + t\frac{d}{dt} + t^2$. Since

$$\frac{d(\ln(t)x(t))}{dt} = \ln(t)\frac{dx}{dt}(t) + \frac{1}{t}x(t)$$

and

$$\frac{d^2(\ln(t)x(t))}{dt^2} = \ln(t)\frac{d^2x}{dt^2}(t) + \frac{2}{t}\frac{dx}{dt}(t) - \frac{1}{t^2}x(t)$$

we find

$$\begin{aligned}
&\mathrm{B}(\,\ln(t)x(t)) \\
&\quad = t^2\frac{d^2(\ln(t)x(t))}{dt^2} + t\frac{d(\ln(t)x(t))}{dt} + t^2\ln(t)x(t) \\
&\quad = t^2\left(\ln(t)\frac{d^2x}{dt^2}(t) + \frac{2}{t}\frac{dx}{dt}(t) - \frac{1}{t^2}x(t)\right) + t\left(\ln(t)\frac{dx}{dt}(t) + \frac{1}{t}x(t)\right) + \ln(t)x(t) \\
&\quad = \ln(t)\left(t^2\frac{d^2x}{dt^2}(t) + t\frac{dx}{dt}(t) + t^2x(t)\right) + 2t\frac{dx}{dt}(t) \\
&\quad = \ln(t)\,\mathrm{B}(x(t)) + 2t\frac{dx}{dt}(t).
\end{aligned}$$

Hence if $Bx = 0$ and $\mathrm{B}\,y = -2t\frac{dx}{dt}$,

$$\begin{aligned}
\mathrm{B}(\ln(t)x(t) + y(t)) &= \mathrm{B}(\ln(t)x(t)) + \mathrm{B}(y(t)) \\
&= \ln(t)\,\mathrm{B}(x(t)) + 2t\frac{dx}{dt}(t) - 2t\frac{dx}{dt}(t) = 0
\end{aligned}$$

which shows that $\ln(t)x + y$ solves (1).

(c) Let $x(t) = J_0(t) = \sum_{k=0}^{\infty} a_k t^k$ and

$$y(t) = \sum_{k=1}^{\infty} b_k t^k, \qquad \mathrm{B}\, y = -2t\frac{dx}{dt}.$$

Assume that two termwise differentiations are allowed in the series for y. Then

$$\begin{aligned} \mathrm{B}\, y = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + t^2 y &= \sum_{k=1}^{\infty} \left(b_k k(k-1)t^k + b_k k t^k + b_k t^{k+2}\right) \\ &= \sum_{k=1}^{\infty} b_k k^2 t^k + \sum_{k=3}^{\infty} b_{k-2} t^k. \end{aligned}$$

Furthermore, as termwise differentiation is allowed in the series for x,

$$-2t\frac{dx}{dt} = -2\sum_{k=1}^{\infty} a_k k t^k.$$

By the identity principle for power series we conclude that

$$b_1 = -2a_1 = 0, \quad 4b_2 = -4a_2 = 1$$

and

$$k^2 b_k + b_{k-2} = -2k a_k \qquad (k \geq 3).$$

Since $b_1 = 0$ and $a_k = 0$ for k odd, we find by induction that $b_k = 0$ for all odd k.

(d) We now define b_k for $k \geq 1$ by $b_k = 0$ for all odd k, by $b_2 = \frac{1}{4}$ and, recursively, by

$$b_k = -\frac{b_{k-2} + 2k a_k}{k^2}$$

for $k \geq 4$ even.

The inequality

$$|b_{2n}| \leq \frac{1}{(n!)^2} \tag{4}$$

is valid for $n = 1$ since $b_2 = \frac{1}{4}$. For $n > 1$ we find

$$|b_{2n}| = \left|\frac{b_{2n-2} + 2(2n)a_{2n}}{(2n)^2}\right| \leq \left|\frac{b_{2n-2}}{(2n)^2}\right| + \frac{2(2n)}{2^{2n}(n!)^2(2n)^2} = \left|\frac{b_{2n-2}}{(2n)^2}\right| + \frac{1}{2^{2n}(n!)^2 n}.$$

Assuming (4) with n replaced by for $n-1$ we obtain

$$|b_{2n}| \leq \frac{1}{((n-1)!)^2(2n)^2} + \frac{1}{2^{2n}(n!)^2 n} = \frac{1}{4(n!)^2} + \frac{1}{2^{2n}(n!)^2 n} \leq \frac{1}{(n!)^2},$$

so we conclude by induction that (4) holds for all n.

It follows that the power series for y has infinite radius of convergence. Since termwise differentiations are allowed in a power series, it follows from (c) that y solves the inhomogeneous equation $\mathrm{B}\, y = -2tx'$. As seen in (b), it follows that $\ln(t)J_0 + y$ solves (1).

(e) Since $\ln(t) \to -\infty$, $J_0(t) \to J_0(0) = 1$ and $y(t) \to y(0) = 0$ for $t \to 0^+$, it follows that

$$\ln(t)J_0(t) + y(t) \to -\infty. \tag{5}$$

The functions $J_0(t)$ and $\ln(t)J_0(t) + y(t)$ are linearly independent, for if a linear combination

$$c_1 J_0(t) + c_2(\ln(t)J_0(t) + y(t)) \tag{6}$$

is zero for all $t > 0$, then

$$\lim_{t \to 0^+} c_2(\ln(t)J_0(t) + y(t)) = -\lim_{t \to 0^+} c_1 J_0(t) = 0$$

and hence $c_2 = 0$ since otherwise (5) would be contradicted. But then (6) implies $c_1 = 0$ since $J_0 \neq 0$.

Since the space of maximal solutions is known to be a two-dimensional vector space, it follows that the set of all linear combinations (6) comprise the complete solution of (1) on $(0, \infty)$

(f) The transformation $t \mapsto -t$ changes (1) to the same equation, but on $\{t \in \mathbf{R} \mid t < 0\}$. Since J_0 and y are even functions of t, the complete solution on this set is given by (6) with $\ln(t)$ replaced by $\ln|t|$.