Consider Bessel’s equation with \( p = 0 \),
\[
t^2 x'' + tx' + t^2 x = 0 \quad (t > 0).
\tag{1}
\]
We have seen that the equation has a power series solution of the form
\[
J_0(t) = \sum_{k=0}^{\infty} a_k t^k
\]
given by the recursion formula
\[
k^2 a_k + a_{k-2} = 0 \quad (k \geq 2).
\tag{2}
\]
and with \( a_k = 0 \) for odd \( k \).

Since it is a second order linear equation, the solution space is two dimensional. The purpose of this exercise is to find a solution independent of \( J_0 \).

(a) Show that with \( a_0 = 1 \) we have
\[
a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2}.
\]

(b) Let \( B \) denote the differential operator
\[
B = t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} + t^2.
\]
Show that
\[
B(\ln(t)x) = \ln(t)Bx + 2tx'
\]
for every two times differentiable function \( x = x(t) \). Conclude that if \( x \) solves the Bessel equation \( Bx = 0 \) and \( y \) is a solution to the inhomogeneous equation \( By = -2tx' \), then \( \ln(t)x + y \) also solves the Bessel equation.

(c) Let \( x(t) = J_0(t) = \sum_{k=0}^{\infty} a_k t^k \) and consider the Ansatz
\[
y(t) = \sum_{k=1}^{\infty} b_k t^k, \quad By = -2tx'.
\]
Show that if termwise differentiation is allowed in the series for \( y \), then
\[
k^2 b_k + b_{k-2} = -2ka_k \quad (k \geq 2)
\]
and \( b_1 = 0 \).
(d) Conclude that $b_k = 0$ when $k$ is odd, and show that

$$|b_{2n}| \leq \frac{1}{(n!)^2}$$

for $n = 1, 2, 3, \ldots$. Conclude that $y$ exists, that $y$ solves the inhomogeneous equation $By = -2tx'$, and finally that $\ln(t)J_0 + y$ solves the Bessel equation with $p = 0$.

(e) Determine the behaviour of $\ln(t)J_0(t) + y(t)$ for $t \to 0$, and show that as a function of $t$, it is linearly independent of $J_0$. Determine the complete solution of the Bessel equation for $p = 0$ on $(0, \infty)$.

(f) Determine also the complete solution on $(-\infty, 0)$. (Hint: apply the transformation $t \mapsto -t$).