Solution. Assignment 4

We consider the initial value problem

$$\dot{y} - 2y\ddot{y} = -1, \quad y(0) = 1,\, \dot{y}(0) = 0 \tag{1}$$

on $\mathcal{D} = \mathbf{R} \times \mathbf{R}$.

(a) On $\mathcal{D}_+ = \mathbf{R} \times (0, \infty)$ the equation is equivalent with

$$\ddot{y} = \frac{\dot{y} + 1}{2y}, \tag{2}$$

which in turn is equivalent with the first order equation

$$\dot{u} = X(t, u) = \begin{pmatrix} u_2 \\ \frac{u_2+1}{2u_1} \end{pmatrix}$$

on $\mathbf{R} \times \{u = (u_1, u_2) \in \mathbf{R}^2 | u_1 > 0\}$.

As it is C^1 with respect to y, the function X is locally Lipschitz. It follows from Corollary 8.5 that there exists a unique maximal solution (I, u) to the first order system with initial condition $u(0) = (1, 0)$. Then $(I, y) = (I, u_1)$ solves (1) in $\mathcal{D}_+$ and hence also in $\mathcal{D}$. It is the unique maximal solution in $\mathcal{D}_+$, but not necessarily in $\mathcal{D}$.

Let (I', y') be another solution to (1) in $\mathcal{D}$, and assume that $y' \neq y$ on $I' \cap I$. As (I, y) is unique in $\mathcal{D}_+$, it follows that $y'(t) \leq 0$ for some $t \in I \cap I'$. Assume for example that this t is > 0 (the argument if $t < 0$ is similar), and let

$$t_1 = \inf\{t > 0, t \in I \cap I' \mid y'(t) \leq 0\}.$$

For $t \in I_1 = [0, t_1)$ we have $y'(t) > 0$, and hence (I_1, y') solves (1) in $\mathcal{D}_+$. Then $y' = y$ in I_1 by the uniqueness of y, and by continuity of y and y' we conclude that also at the endpoint of I_1 we have equality $y'(t_1) = y(t_1)$. Hence $y'(t_1) > 0$. On the other hand, it follows from the definition of t_1 that $y'(t_1) \leq 0$. We have reached a contradiction.

(b) It follows from (2) that $\ddot{y}$ is continuous and that $\ddot{y}(0) = \frac{\dot{y}(0)+1}{2y(0)} = \frac{1}{2}$. Hence there exists $\epsilon > 0$ such that $\epsilon \in I$ and $\ddot{y}(t) > 0$ for $0 \leq t \leq \epsilon$. It follows that $\dot{y}$ is strictly increasing on this interval. Since $\dot{y}(0) = 0$ it then follows that $\dot{y}(t) > 0$ for $0 < t \leq \epsilon$. Hence y is strictly increasing on $[0, \epsilon]$.