Solution. Assignment 4

We consider the initial value problem
\[
\dot{y} - 2y\ddot{y} = -1, \quad y(0) = 1, \quad \dot{y}(0) = 0
\]
on \mathcal{D} = \mathbb{R} \times \mathbb{R}.

(a) On \(\mathcal{D}_+ = \mathbb{R} \times (0, \infty)\) the equation is equivalent with
\[
\ddot{y} = \frac{\dot{y} + 1}{2y},
\]
which in turn is equivalent with the first order equation
\[
\dot{u} = X(t, u) = \left(\frac{u_2}{u_2 + 1}\right)
\]
on \(\mathbb{R} \times \{u = (u_1, u_2) \in \mathbb{R}^2 | u_1 > 0\}\).

As it is \(C^1\) with respect to \(y\), the function \(X\) is locally Lipschitz. It follows from Corollary 8.5 that there exists a unique maximal solution \((I, u)\) to the first order system with initial condition \(u(0) = (1, 0)\). Then \((I, y) = (I, u_1)\) solves (1) in \(\mathcal{D}_+\) and hence also in \(\mathcal{D}\). It is the unique maximal solution in \(\mathcal{D}_+\), but not necessarily in \(\mathcal{D}\).

Let \((I', y')\) be another solution to (1) in \(\mathcal{D}\), and assume that \(y' \neq y\) on \(I' \cap I\). As \((I, y)\) is unique in \(\mathcal{D}_+\), it follows that \(y'(t) \leq 0\) for some \(t \in I \cap I'\). Assume for example that this \(t\) is \(> 0\) (the argument if \(t < 0\) is similar), and let
\[
t_1 = \inf\{t > 0, t \in I \cap I' \mid y'(t) \leq 0\}.
\]
For \(t \in I_1 = [0, t_1]\) we have \(y'(t) > 0\), and hence \((I_1, y')\) solves (1) in \(\mathcal{D}_+\). Then \(y' = y\) in \(I_1\) by the uniqueness of \(y\), and by continuity of \(y\) and \(y'\) we conclude that also at the endpoint of \(I_1\) we have equality \(y'(t_1) = y(t_1)\). Hence \(y'(t_1) > 0\). On the other hand, it follows from the definition of \(t_1\) that \(y'(t_1) \leq 0\). We have reached a contradiction.

(b) It follows from (2) that \(\ddot{y}\) is continuous and that \(\ddot{y}(0) = \frac{\dot{y}(0)(0 + 1)}{2\dot{y}(0)} = \frac{1}{2}\). Hence there exists \(\epsilon > 0\) such that \(\epsilon \in I\) and \(\ddot{y}(t) > 0\) for \(0 \leq t \leq \epsilon\). It follows that \(\dot{y}\) is strictly increasing on this interval. Since \(\dot{y}(0) = 0\) it then follows that \(\dot{y}(t) > 0\) for \(0 < t \leq \epsilon\). Hence \(y\) is strictly increasing on \([0, \epsilon]\).