Solution. Assignment 3

(a) The function \( x \mapsto |x|^{\alpha}, \quad (x \in \mathbb{R}) \), is locally Lipschitz when \( \alpha \geq 1 \). Hence so is \( X_{\alpha} \) by Lemma 7.16 (for \( \alpha > 1 \) one could also argue by the fact that it is \( C^1 \)). If \( 0 < \alpha < 1 \), then \( X_{\alpha}(t, y) \) is not locally Lipschitz at \((t_0, 1)\) for any \( t_0 \in \mathbb{R} \), since that would imply the existence of \( R, L > 0 \) such that for all \( t \) with \( |t - t_0| < R \) and \( y \) with \( |y - 1| < R \)

\[
|X_\alpha(t, y) - X_\alpha(t, 1)| = \frac{\cosh(t)|y + 1|^\alpha|y - 1|^\alpha}{y} \leq L|y - 1|.
\]

Hence for \( y \neq 1 \)

\[
cosh(t)|y + 1|^\alpha \leq L|y - 1|^{1-\alpha}y.
\]

The right hand side of this inequality tends to 0 as \( y \to 1 \), whereas the left side is \( \geq 1 \), a contradiction.

(b) \( X_{\alpha} \) is not Lipschitz for any \( \alpha > 0 \). To prove this its suffices to regard \( X_{\alpha}(0, y) \) for \( 0 < y < \frac{1}{2} \). If it were Lipschitz, there would exist \( R, L > 0 \) such that for all \( y, y' > 0 \) with \( |y' - y| < R \) we would have

\[
|X_\alpha(0, y) - X_\alpha(0, y')| \leq L|y - y'|.
\]  \quad (1)

Take \( y' = 2y \). Then

\[
X_\alpha(0, y) - X_\alpha(0, 2y) = \frac{(1 - y^2)\alpha}{y} - \frac{(1 - 4y^2)\alpha}{2y} = y^{-1} \left( (1 - y^2)\alpha - \frac{1}{2}(1 - 4y^2)\alpha \right)
\]

and

\[
(1 - y^2)\alpha - \frac{1}{2}(1 - 4y^2)\alpha \geq (1 - y^2)\alpha - \frac{1}{2}.
\]

If \( y \) is sufficiently small then \( (1 - y^2)\alpha \geq \frac{3}{4} \) and hence

\[
X_\alpha(0, y) - X_\alpha(0, 2y) \geq \frac{1}{4}y^{-1}
\]

Since we also have

\[
|y - y'| = |y - 2y| = y < R
\]

for \( y \) sufficiently small, we then obtain from (1)

\[
\frac{1}{4}y^{-1} \leq Ly,
\]

for all such \( y \), which is a contradiction.

(c) \( X_{\alpha} \) is \( C^1 \) on \( \mathbb{R} \times ((0, 1) \cup (1, \infty)) \), since \( y^2 - 1 \neq 0 \) on this domain. Hence it is locally Lipschitz.
(d) It follows from (c) and Corollary 8.5 that there exists a unique maximal solution \((I, y)\)
in \(\mathbb{R} \times ((0, 1) \cup (1, \infty))\) with \(y(t_0) = y_0\). Note that \((I, y)\) is also a solution in \(\mathbb{R} \times (0, \infty)\) (but as such it is not necessarily maximal).

(e) Let \((I, \tilde{y})\) be a solution in \(\mathbb{R} \times (0, \infty)\) with \(\tilde{y}(t_0) = y_0\) and with the same interval \(I\) as in (d). Assume \(\tilde{y} \neq y\). Then \(\tilde{y}\) must attain the value 1 somewhere, since \((I, y)\) is the unique \((0, 1) \cup (1, \infty)\)-valued solution to the initial value problem. Say for example that 1 is attained in \(I \cap (t_0, \infty)\) (the other case is similar), and let \(b = \inf\{t > t_0|\tilde{y}(t) = 1\}\). By continuity \(\tilde{y}(b) = 1\). On the other hand, since \(\tilde{y}(t) \neq 1\) for all \(t \in [t_0, b]\), we have \(\tilde{y}(t) = y(t)\) for \(t \in [t_0, b]\) by uniqueness of the \((0, 1) \cup (1, \infty)\)-valued solutions. By continuity of \(\tilde{y}\) and \(y\) we conclude \(\tilde{y}(b) = y(b) \neq 1\), a contradiction. Hence \(\tilde{y} = y\).

To answer the question we can therefore take \(U = I\).

(f) Exercise 1 corresponds to the present exercise with \(\alpha = \frac{1}{2}\). The solution denoted \(y_1\) in Exercise 1(a) corresponds to the solution \(y\) of (d) above. The last two of the three solutions in Exercise 1(b) both solve the initial value problem \(y(1) = \cosh(1)\) and they both agree with \(y_1\) on \(I = (0, \infty)\). This is in accordance with what we proved in (e) above.